

Exponential $U(1)$ Symmetry-Breaking Phase as a Disorder-Free Quantum Glass

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We study the phase diagram of a one-dimensional spin quantum breakdown model, which has an exponential $U(1)$ symmetry with charge unit decaying as 2^{-j} with site position j . By exact diagonalization and density matrix renormalization group, we show that the model with spin $S \geq 2$ exhibits an exponential $U(1)$ spontaneous symmetry-breaking (SSB) phase dubbed a quantum breakdown condensate. It exhibits a bulk gap violating the Goldstone theorem and an edge mode only on the left edge if in open boundary condition. In a length L lattice, the condensate has $\mathcal{O}(2^L)$ number of SSB ground states originating from the $\mathcal{O}(2^L)$ number of exponential $U(1)$ charge sectors, leading to a finite entropy density $\ln 2$. This enforces a first-order SSB phase transition into this phase, as observed numerically and verified in the large S limit on an exactly solvable Rokhsar-Kivelson line. The condensate has an SSB order parameter being the local in-plane spin, which points in angles related by the chaotic Bernoulli (dyadic) map and thus is effectively random. Moreover, we show the condensate exhibits nondecaying local autocorrelations and does not have an off-diagonal long-range order. The quantum breakdown condensate thus behaves as a disorder-free quantum glass and is beyond the existing classifications of phases of matter.

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Symmetries and spontaneous symmetry breaking (SSB) play a unifying role in the classification and characterization of quantum phases of matter. Recently, generalized symmetries [1,2] have attracted extensive interests. On lattices, a class of on-site symmetries is the modulated symmetry [3] with symmetry charge $\hat{Q} = \sum_j f_j \hat{n}_j$, where the unit of charge f_j depends on site j , and $\hat{n}_j$ is the particle number. Examples include dipole [4–12], multipole [13,14], subsystem symmetries [15–23], and more recently exponential symmetries [24–34]. For instance, in one dimension (1D), the k th multipole $U(1)$ symmetry has $f_j = j^k$, and the exponential $U(1)$ symmetry with base $q \neq 1$ has $f_j = f_0 q^{-j}$, for which translationally invariant models can be written down [35].

Particularly, as found in a recent mean-field study [27], the SSB ground state of a boson quantum breakdown model with exponential $U(1)$ symmetry, dubbed the quantum breakdown condensate, exhibits a gapped excitation spectrum violating the Goldstone theorem. This

motivates us to uncover more unconventional characters of this state and to explore its realization in more generic models.

In this Letter, we propose and study the SSB of a 1D spin quantum breakdown model [Eq. (1)], which has an exponential $U(1)$ symmetry with base $q = 2$. Compared to the boson model [27], the spin model has a finite on-site Hilbert space and is more experimentally feasible. Employing exact diagonalization (ED) and density matrix renormalization group (DMRG), we find the model with spin $S \geq 2$ generically undergoes a phase transition from a symmetric paramagnetic (PM) phase to the exponential $U(1)$ SSB quantum breakdown condensate. We verify that the condensate has a bulk gap as found in [27], and we further reveal an edge mode only existing on the left edge which alters the ED spectrum for open boundary condition. For a lattice of length L , we find the condensate has $\mathcal{O}(2^L)$ SSB ground states originating from $\mathcal{O}(2^L)$ number of charge sectors, which span a finite total bandwidth, in contrast to typical SSB. Also, both [27] and our work find the SSB phase transition being first order. We further identify a Rokhsar-Kivelson line in the large S limit to be an exactly solvable first-order phase boundary. We argue that this SSB phase transition has to be first order, as enforced by a finite entropy density $\ln 2$ from $\mathcal{O}(2^L)$ ground states in the condensate. Moreover, we show the condensate has a local SSB order parameter giving in-plane spins pointing in angles related by the chaotic Bernoulli (dyadic) map [36] from site to site, which typically look random. We also find

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it exhibits nondecaying local autocorrelations and has no off-diagonal long-range order. These facts indicate the quantum breakdown condensate behaves as a disorder-free quantum glass and lies beyond any known classifications of phases of matter.

The model—In a spin chain of length L with a spin $S \geq 1$ at each site, we define the 1D spin quantum breakdown model Hamiltonian [Fig. 1(a)],

$$\hat{H} = -\frac{J}{(2S)^{3/2}} \sum_{j=1}^{L_{P/O}} \left[\hat{S}_j^- (\hat{S}_{j+1}^+)^2 + \hat{S}_j^+ (\hat{S}_{j+1}^-)^2 \right] - h \sum_{j=1}^L \left[\lambda \hat{S}_j^z + (1-\lambda) \frac{4S^2}{\pi^2} \sin\left(\frac{\pi}{2S} \hat{S}_j^z\right) \right], \quad (1)$$

where $\hat{S}_j^\nu$ ($\nu = x, y, z$) are the spin operators on site j , and $\hat{S}_j^\pm = \hat{S}_j^x \pm i\hat{S}_j^y$. The first term with coefficient $-J/(2S)^{3/2}$ is the inversion asymmetric breakdown interaction [24], in which the sum is up to $L_O = L - 1$ for the open boundary condition (OBC) and up to $L_P = L$ for the periodic boundary condition (PBC) with site $L + j$ identified with site j . The second term is a nonlinear coupling to a z -direction magnetic field h with $\lambda \in [0, 1]$, which reduces to linear when $\lambda = 1$.

The model in Eq. (1) has an exponential symmetry $\hat{U}(\varphi) = \exp(i\varphi \sum_{j=1}^L 2^{L-j} \hat{S}_j^z)$, which transforms the spin on site j as $\hat{U}(\varphi)(\hat{S}_j^\pm) \hat{U}(\varphi)^\dagger = (\hat{S}_j^\pm, e^{\pm i2^{L-j}\varphi} \hat{S}_j^\pm)$. For OBC, any angle $\varphi \in [0, 2\pi)$ is allowed, which gives a $U(1)$ exponential symmetry. For PBC, the transformation on site $j = L$ should be identical to $j = 0$, which enforces $(2^L - 1)\varphi = 0 \pmod{2\pi}$, leading to a reduction from $U(1)$ to $\mathbb{Z}_{2^L-1}$ exponential symmetry depending on system size L . We can rewrite $\hat{U}(\varphi) = e^{i\varphi \hat{Q}}$ as generated by an exponential charge $\hat{Q}$,

$$\hat{Q} = \begin{cases} \sum_{j=1}^L 2^{L-j} \hat{S}_j^z, & \text{(OBC)} \\ \sum_{j=1}^L 2^{L-j} \hat{S}_j^z \pmod{2^L - 1}. & \text{(PBC)} \end{cases} \quad (2)$$

The number of charge sectors is $N_Q = 2S(2^L - 1) + 1$ for OBC and $N_Q = 2^L - 1$ for PBC, respectively. As $L \rightarrow \infty$, we shall simply call $\hat{U}(\varphi)$ an exponential $U(1)$ symmetry regardless of boundary conditions. For PBC, there is additionally a translation symmetry $\hat{T}$ sending site j to $j + 1$, which does not commute with the exponential symmetry: $\hat{T} \hat{U}(\varphi) \hat{T}^{-1} = \hat{U}(2\varphi)$.

Large spin limit and Rokhsar-Kivelson line—We first analyze the phases of the model in the large spin limit. For $h > 0$, by a Holstein-Primakoff transformation $\hat{S}_j^z = S - \hat{n}_j$, $\hat{S}_j^+ = (\hat{S}_j^-)^\dagger = \sqrt{2S - \hat{n}_j} \hat{a}_j$ with boson operators $\hat{a}_j$, $\hat{a}_j^\dagger$ and $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$, and taking $S \rightarrow \infty$, one can map

the spin model in Eq. (1) to the boson quantum breakdown model [27],

$$\hat{H} = -J \sum_{j=1}^{L_{P/O}} \left[\hat{a}_j (\hat{a}_{j+1}^\dagger)^2 + \hat{a}_j^\dagger (\hat{a}_{j+1})^2 \right] + \sum_{j=1}^L \left[-\mu \hat{n}_j + \frac{U}{2} \hat{n}_j (\hat{n}_j - 1) \right], \quad (3)$$

with a Hubbard interaction $U = (1 - \lambda)h$ and chemical potential $\mu = -(1 + \lambda)h/2$. As numerically revealed in [27], as J/U increases, the ground state of boson model in Eq. (3) undergoes a phase transition from a symmetric Mott insulator to a quantum breakdown condensate with the exponential symmetry spontaneously broken.

Here we present an analytical understanding for this SSB near a Rokhsar-Kivelson (RK) line, which is defined by $\mu U = -2J^2$ with $U > 0$. On this RK line, by defining a real parameter $\alpha = 2J/U$, we can rewrite the boson Hamiltonian $\hat{H}$ in Eq. (3) (exactly for PBC and up to boundary terms for OBC) into a semipositive definite form,

$$\hat{H}_{\text{RK}} = \frac{U}{2} \sum_j \hat{O}_j^\dagger \hat{O}_j, \quad \hat{O}_j = \hat{a}_{j+1}^2 - \alpha \hat{a}_j. \quad (4)$$

Apparently, a state $|\Psi\rangle$ satisfying $\hat{O}_j |\Psi\rangle = 0$ for all j gives an exact ground state of $\hat{H}_{\text{RK}}$ with energy $E_{\text{gs}} = 0$, which has two families of solutions (unnormalized),

$$|\Psi_{\text{sym}}\rangle = |0\rangle, \quad |\Psi_{\text{SSB}}(\theta)\rangle = \prod_{j=1}^L \exp(\alpha e^{i2^{L-j}\theta} \hat{a}_j^\dagger) |0\rangle, \quad (5)$$

where $\theta \in 2\pi\mathbb{R}/\mathbb{Z}$ for OBC, and $\theta \in [2\pi/(2^L - 1)]\mathbb{Z}_{2^L-1}$ for PBC, while $|0\rangle$ is the boson vacuum. Thus, $\hat{H}_{\text{RK}}$ in Eq. (4) is a frustration-free Hamiltonian with solvable ground states in Eq. (5) [37,38]. $|\Psi_{\text{sym}}\rangle$ is a symmetric Mott insulator state with zero boson per site. The coherent state $|\Psi_{\text{SSB}}(\theta)\rangle$ gives an SSB quantum breakdown condensate, which has a local order parameter $\langle \hat{a}_j \rangle = \alpha e^{i2^{L-j}\theta}$ breaking the exponential $U(1)$ or $\mathbb{Z}_{2^L-1}$ symmetry [27]. The coherent states $|\Psi_{\text{SSB}}(\theta)\rangle$ can be further superposed into the charge Q eigenbasis, which gives exactly one ground state $|\Psi_Q\rangle$ per charge Q sector.

Similar to the RK model [37,38], the fact that both $|\Psi_{\text{sym}}\rangle$ and $|\Psi_{\text{SSB}}(\theta)\rangle$ are ground states of $\hat{H}_{\text{RK}}$ implies that the RK line $\mu U = -2J^2$ is the phase boundary of the zero-temperature SSB transition. When J is perturbatively increased (decreased) away from the RK line, the state $|\Psi_{\text{SSB}}(\theta)\rangle$ ($|\Psi_{\text{sym}}\rangle$) will have a lower energy, leading to the SSB (symmetric) phase. This is an energy level crossing transition with the order parameter $|\langle \hat{a}_j \rangle|$ jumping discontinuously from 0 to $|\alpha|$, which indicates that the RK line is a first-order phase boundary. This quantitatively matches the numerical results in [27]. In fact, we will argue later that the

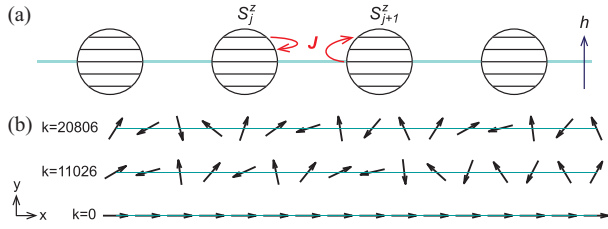


FIG. 1. (a) Illustration of the spin breakdown model in Eq. (1). (b) The in-plane spin directions on different sites for the quantum breakdown condensate under PBC with order parameter $\langle \hat{S}_j^+ \rangle = \alpha_S e^{i2^{L-j}\theta}$. We take $L = 16$ and $\theta = [2\pi/(2^L - 1)]k$ with representative k values labeled in the plot.

SSB phase transition of exponential $U(1)$ symmetry is generically first order.

In the spin picture, the SSB order parameter $\langle \hat{a}_j \rangle$ becomes $\langle \hat{S}_j^+ \rangle = \alpha_S e^{i2^{L-j}\theta}$, with $\alpha_S \neq 0$. Interestingly, this points the in-plane spin components ($\langle \hat{S}_j^x \rangle$, $\langle \hat{S}_j^y \rangle$) on site j in an angle $\theta_j = 2^{L-j}\theta \pmod{2\pi}$, which is typically a random looking glasslike configuration for generic θ [Fig. 1(b)]. In fact, the angles follow a recursion relation $\theta_j = 2\theta_{j+1} \pmod{2\pi}$, which is exactly the ergodic and chaotic Bernoulli (dyadic) map [36] describing the Lorenz strange attractor [39–41]. The SSB state is thus analogous to a spin glass, with more evidence to be shown later.

Finite spin—For finite spin S , we study the Hamiltonian in Eq. (1) employing ED below and DMRG in the End Matter. We find the SSB phase transition of exponential $U(1)$ symmetry robustly occurs for $S \geq 2$, with characters qualitatively agreeing with the above large S limit analysis (see End Matter and Supplemental Material [42]). Hereafter, we set $\lambda = 1$, $h = 1$ and focus on the example of $S = 5$, for which the ground-state SSB phase transition happens at $J = J_0 \approx 0.52$.

For $J < J_0$, the model is in the symmetric phase, denoted as the paramagnetic phase in the spin picture. Figures 2(a) and 2(b) show the ED low-energy spectra at $J = 0.1$ for $L = 6$ with PBC and OBC, respectively, with the horizontal axis labeling the charge sector. The PBC and OBC spectra both have a unique gapped PM ground state and are similar by noting that the PBC charge sectors can be viewed as the OBC charge sectors folded into the range $Q/(2^L - 1) \in [-1/2, 1/2]$ [folding boundaries marked by dashed lines in Fig. 2(b)]. As shown by the triangle marker line in Figs. 2(e) and 2(f), the PM ground state has spin $\langle \hat{S}_j^z \rangle = S$ maximally polarized.

In contrast, the ED spectra for $J > J_0$ in the SSB quantum breakdown condensate phase are notably different between PBC and OBC, as shown in Figs. 2(c) and 2(d) calculated at $J = 2$ for $L = 6$. For PBC, each charge Q sector contributes a ground state, the energy of which we denote as $E_{\text{gs}}(Q)$. These $2^L - 1$ states $E_{\text{gs}}(Q)$ form a nearly degenerate SSB ground-state manifold spanning an energy

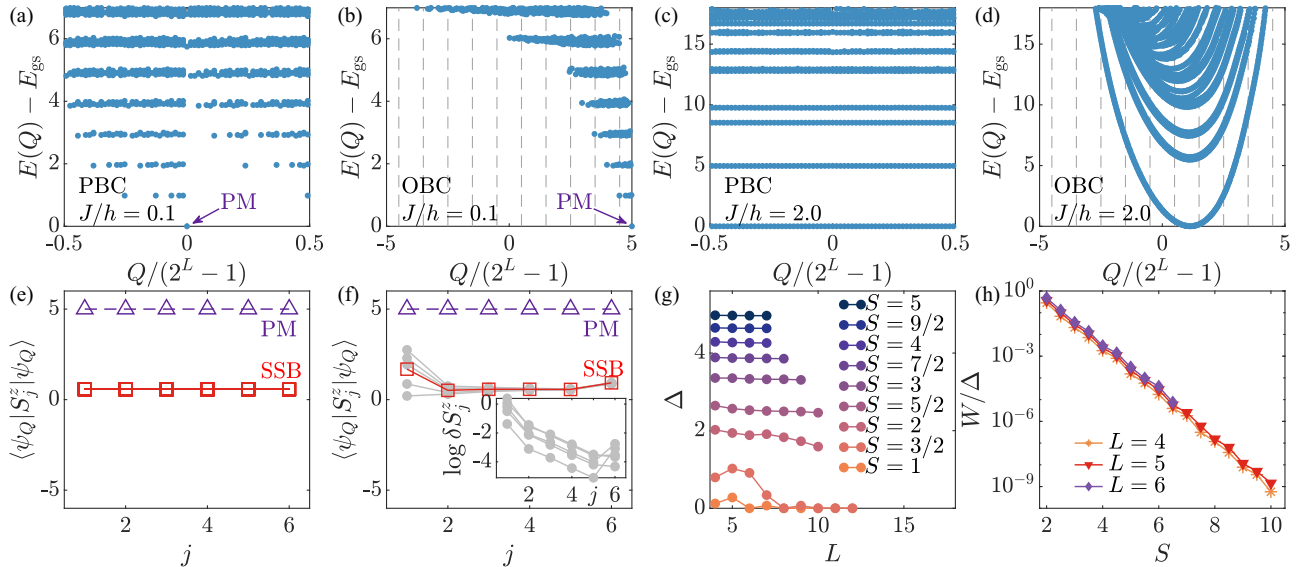


FIG. 2. ED calculations for $\lambda = h = 1$. (a)–(d) The ED low-energy spectra (subtracting the global ground-state energy E_{gs}) versus charge sector Q for $S = 5$, $L = 6$, in the PM phase ($J = 0.1$) in (a),(b) and the SSB quantum breakdown condensate phase ($J = 2$) in (c),(d). (a) and (c) are calculated with PBC; (b) and (d) are calculated with OBC. (e),(f) The expectation $\langle \hat{S}_j^z \rangle$ versus site j for (e) PBC and (f) OBC ground states in (a)–(d). The triangular markers in (e),(f) represent the PM ground state. The square markers in (e) plot the PBC SSB ground states in all Q sectors, the differences among which are negligible. The square markers in (f) are for the global SSB ground state $|\psi_{Q_0}\rangle$ which is in the $Q_0 = 71$ sector, while dotted markers in (f) are for SSB ground states $|\psi_Q\rangle$ in several other charge sectors $Q \in \{19, 42, 80, 93, 110\}$, and the (f) inset shows $\delta S_j^z \equiv \langle \psi_Q | \hat{S}_j^z | \psi_Q \rangle - \langle \psi_{Q_0} | \hat{S}_j^z | \psi_{Q_0} \rangle$ which decay exponentially in j . (g) The excitation gap Δ and (h) bandwidth W of the SSB ground-state manifold with PBC for different S and L , where we fix $J = 2$ and $h = \lambda = 1$.

bandwidth $W = E_{2^L-1} - E_1$ and are separated with higher-energy states by a finite gap $\Delta = E_{2^L} - E_{2^L-1}$, where E_n is the n th global energy level in ascending order. Figure 2(g) shows the gap Δ is stably finite as L increases for $S \geq 2$ in the SSB regime. Figure 2(h) implies that W decays exponentially in S , but is nearly L independent. These finite-size ED results suggest that W is finite in the thermodynamic limit, which remains to be verified more rigorously. For OBC, as Fig. 2(d) shows, while each charge sector Q still has a gap (roughly equal to the PBC gap Δ), the full spectrum is gapless. The ground-state energy $E_{\text{gs}}(Q)$ of charge sector Q is well fitted by $E_{\text{gs}}(Q) - E_{\text{gs}}(Q_0) \approx D[(Q - Q_0)/2^L]^2$ with D independent of L , where Q_0 is the charge sector of the global ground state.

To understand the unconventional ED energy spectra of the SSB quantum breakdown condensate, we first note that it has no gapless Goldstone modes as shown by [27]. To see this, consider perturbing the SSB order parameter into $\langle \hat{S}_j^z \rangle = \alpha_S e^{i2^{-L-j}\theta} e^{i\delta\theta_j}$ with a small phase $\delta\theta_j$, which yields a Goldstone boson Lagrangian $\mathcal{L} = (g_0/2) \sum_j [(\partial_t \delta\theta_j)^2 - m_b^2 (\delta\theta_j - 2\delta\theta_{j+1})^2]$ as enforced by the symmetry. In the bulk, it can be rewritten as $\mathcal{L} = (g_0/2) \sum_j [(\partial_t \delta\theta_j)^2 - 2m_b^2 (\delta\theta_j - \delta\theta_{j+1})^2 - m_b^2 \delta\theta_j^2] \approx (g_0/2) \int dx [(\partial_t \delta\theta)^2 - v_b^2 (\partial_x \delta\theta)^2 - m_b^2 \delta\theta^2]$, which gives a bulk gap m_b matching well with the ED gap Δ in Fig. 2(g). For OBC, there is an extra left-edge boson zero mode with $\delta\theta_j = 2^{-j} \delta\theta_{\text{edge}}$ [27], which simply generates the exponential $U(1)$ transformation $e^{i2^{-L} \delta\theta_{\text{edge}} \hat{Q}}$. Upon quantizing $\delta\theta_{\text{edge}}$, this gives a canonical commutation relation $[\delta\hat{\theta}_{\text{edge}}, 2^{-L} \hat{Q}] = i$, which indicates $\partial_t \delta\hat{\theta}_{\text{edge}} \propto [(\hat{Q} - Q_0)/2^L]$ with some number Q_0 . The Lagrangian $\mathcal{L}$ then yields an edge mode Hamiltonian $\mathcal{H}_{\text{edge}} \propto (g_0/2) (\partial_t \delta\hat{\theta}_{\text{edge}})^2 \propto [(\hat{Q} - Q_0)/2^L]^2$, which exactly accounts for the ground-state energy difference $E_{\text{gs}}(Q) - E_{\text{gs}}(Q_0)$ in the OBC spectrum in Fig. 2(d). This edge mode can be further seen in Figs. 2(e) and 2(f). For PBC shown in Fig. 2(e), the ground state $\langle \hat{S}_j^z \rangle$ of every charge sector is spatially uniform (the line with square markers). For OBC shown in Fig. 2(f), the ground state $\langle \hat{S}_j^z \rangle$ of generic charge Q sectors (lines with solid circle markers) differs from that of the charge Q_0 sector (square markers) near site 1, and their differences δS_j^z decay exponentially with site j [Fig. 2(f) inset], indicating an edge mode (see detailed analysis in Supplemental Material [42]). Thus, for OBC, the bulk spectrum is still gapped. We note that the sensitivity of energy spectrum and δS_j^z to boundary conditions is interestingly akin to the skin effect of non-Hermitian systems [48–50].

Phase diagram—Since the SSB quantum breakdown condensate at $J > J_0$ has a bulk gap $\Delta > 0$ above the ground-state manifold, the Mermin-Wagner theorem [51,52] does not apply, and the SSB is stable in 1D at temperature $T = 0$ [27].

We now argue that the $T = 0$ phase boundary between the PM phase and SSB phase is generically first order. The key observation is that the SSB phase at $J > J_0$ has $\mathcal{O}(2^L)$ number of $(2^L - 1)$ for PBC) nearly degenerate ground states distributed in a finite energy window, which gives a finite entropy density $S/L = \ln 2$ at zero temperature, violating the third law of thermodynamics. Intrinsically, this is due to the existence of $N_Q \simeq \mathcal{O}(2^L)$ exponential $U(1)$ charge sectors (the number of SSB ground states is generically polynomial in the number of charge sectors [53,54]). In contrast, the PM phase has a unique gapped ground state and thus an entropy density $S/L = 0$ at zero temperature. Explicitly, upon ignoring the energy levels above the bulk gap, we first take the $L \rightarrow \infty$ limit at temperature $T > 0$, which yields a thermal entropy density regardless of the boundary condition (see Supplemental Material [42])

$$\frac{S}{L} = \begin{cases} 0, & (J < J_T) \\ \ln 2, & (J \geq J_T) \end{cases}, \quad J_T = J_0 - v_{\text{gs}}^{-1} T \ln 2, \quad (6)$$

with some $v_{\text{gs}} > 0$. We then take the $T \rightarrow 0$ limit, which reveals that the entropy density S/L jumps discontinuously by $\ln 2$ at $J = J_0$. This enforces the $T = 0$ phase boundary between PM and SSB to be first order. This is further verified by the level crossings numerically observed in the End Matter, Fig. 5, and Supplemental Material [42].

For 1D Gibbs states (with finite local Hilbert space) at temperatures $T > 0$, it is known that fluctuations forbid any phase transition and SSB [55]. Nevertheless, Eq. (6) still implies a sharp crossover at $J = J_T$ at low temperatures $T > 0$, as illustrated by the Fig. 3(a) phase diagram.

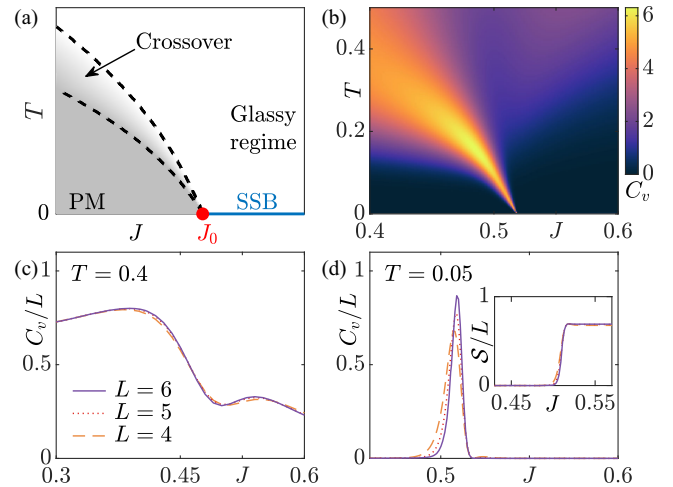


FIG. 3. (a) The phase diagram for Gibbs states. (b) The heat capacity C_v for $\lambda = h = 1$, $S = 5$, $L = 6$ with PBC. (c),(d) The specific heat C_v/L versus J at temperature (c) $T = 0.4$ and (d) $T = 0.05$, for parameters in (b). The inset of (d) shows the entropy density S/L at $T = 0.05$. The legend in (c) labels L for (c),(d).

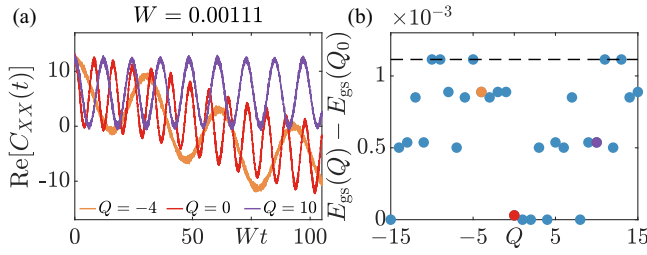


FIG. 4. (a) Autocorrelation $C_{XX}(t) = \langle \psi_Q | \hat{S}_L^x(t) \hat{S}_L^x(0) | \psi_Q \rangle$ for PBC ground state $|\psi_Q\rangle$ of three representative charge sectors Q , where $J = 2$, $\lambda = h = 1$, $S = 5$, and $L = 5$. These states are highlighted in (b), which plots the energy levels in the PBC ground-state manifold (the dashed line indicates the bandwidth $W = 0.00111$).

Figure 3(b) shows the color map of heat capacity $C_v = T(\partial S / \partial T)$ from ED for $L = 6$, which is indeed peaked around $J = J_T$. The specific heat C_v/L as a function of J is smooth at high T [Fig. 3(c)] and shows a peak at low T [Fig. 3(d)]. The inset of Fig. 3(d) further shows that the entropy density S/L increases sharply by approximately $\ln 2$ (sharper for the larger L calculated). Note that, at $J < J_0$, the system crosses over from a more symmetric low-entropy regime to a less symmetric high-entropy regime as T increases, realizing a “Pomeranchuk effect” [56,57].

Glassy features—The fact that the quantum breakdown condensate possesses exponentially many low-energy eigenstates and finite entropy density $S/L = \ln 2$ suggests its analogy to a glass state. A glass state is hallmarked by exponentially many equilibria and slow equilibration after a local perturbation, which are characters typically associated with quenched disorders. To examine whether the quantum breakdown condensate fulfills the defining features of a glass, we calculate the autocorrelation function $C_{XX}(t) = \langle \psi_Q | \hat{S}_j^x(t) \hat{S}_j^x(0) | \psi_Q \rangle$ of local operator $\hat{S}_j^x$ for the charge Q sector ground state $|\psi_Q\rangle$ at $J > J_0$. Figure 4 plots $C_{XX}(t)$ at $j = L$ for PBC in randomly selected charge sectors, all of which show nondecaying oscillations with periods on the order of $2\pi/W$, indicating glassy quantum dynamics (see also the End Matter). This is due to the fact that all local operators have a highly restricted connectivity only among the ground states of $\mathcal{O}(1)$ number of charge sectors—for instance, $\hat{S}_j^x$ only hops from charge sectors Q to $Q \pm 2^{L-j}$.

Moreover, there is no off-diagonal long-range order (ODLRO) [58] two-point function associated with the exponential $U(1)$ SSB (see proof in the End Matter). The simplest symmetric two-point function formed by local order parameters is $\langle \hat{S}_j^+ (\hat{S}_{j+l}^-)^{2^l} \rangle$, which vanishes identically as $l \rightarrow \infty$, since $(\hat{S}_{j+l}^-)^{2^l} = 0$ for $2^l > 2S$. One can go beyond the conventional description to consider a string correlator such as $\langle \hat{S}_j^+ (\prod_{k=1}^{l-1} \hat{S}_{j+k}^-) (\hat{S}_{j+l}^-)^2 \rangle$, which, however, always

scales (grows or decays) exponentially in l [27]. Therefore, the order parameter $\langle \hat{S}_j^+ \rangle$ is effectively short-range correlated and is rather disordered, as depicted in Fig. 1(b), making the exponential $U(1)$ SSB drastically different from the SSB of conventional (including multipole) symmetries. These observations suggest that the quantum breakdown condensate can be viewed as a quantum glass, which interestingly arises from a disorder-free quantum Hamiltonian, in contrast to most known glass models. We note that a few disorder-free quantum models with glassy dynamics exist in higher dimensions [59–61].

Discussion—The quantum breakdown condensate from SSB of the exponential $U(1)$ symmetry is exotic in its e^{cL} ($c > 0$) number of SSB ground states in an $\mathcal{O}(1)$ energy window from $\mathcal{O}(e^{cL})$ number of charge sectors and its absence of gapless Goldstone mode and ODLRO. This is different from the SSB of any other known symmetries, which at most have $\mathcal{O}(L^p)$ ($p \geq 0$) number of charge sectors. Note that the exponential $\mathbb{Z}_N$ symmetry (N independent of L) is very different, which has at most N SSB ground states [29–32]. The quantum breakdown condensate thus acts as a disorder-free quantum glass, which raises questions on its equilibration to Gibbs ensembles and its classification as a quantum phase of matter.

The exponentially many charge sectors closely resemble the Hilbert space fragmentation [8,9,62–64]. Each charge sector has no further fragmentation except for rare cases in PBC (see Supplemental Material [42]). Moreover, a local operator can only sparsely connect an $\mathcal{O}(1)$ number of charge sectors, leading to nondecaying autocorrelations (Fig. 4). This provides robustness against a class of errors in light of potentially applying the quantum breakdown condensate to store and process quantum information. In PBC, the charge of the $2^L - 1$ SSB ground states naturally maps to bit strings, making it possible to store information employing distinct ground states far separated in Hamming distance, for which the damage from a local perturbation is highly constrained and correctable. It would be interesting to explore experimental realizations of the spin quantum breakdown model in platforms such as Rydberg atoms.

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Data availability—The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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End Matter

In this End Matter, we use ITensor [65] to perform density matrix renormalization group simulations for larger system sizes, providing large-scale evidence for the zero-temperature first-order phase transition, slow relaxation dynamics, and spontaneous breaking of the exponential symmetry. In all DMRG-related simulations, we set the bond dimension $\chi = 500$, which already provides converged results. We also prove that there is no ODLRO characterizing the SSB in our model.

DMRG for the ground-state phase transition—Focusing on $S = 5$, we compute the ground-state energy E under both periodic and open boundary conditions and analyze its derivative with respect to the breakdown interaction J , namely, dE/dJ , for larger system sizes, as shown in Fig. 5. Under PBC, the energy derivatives for different system sizes exhibit a clear discontinuous jump, while the inset

shows a good collapse of $dE/(LdJ)$ [Fig. 5(a)]. These results identify a first-order transition at $J_c \simeq 0.5186$ with negligible finite-size effects. On the other hand, the OBC transition point extracted from the discontinuous jump strongly depends on system size [Fig. 5(b)], but systematically approaches the PBC value as L increases. As shown in Fig. 5(c), the finite-size scaling up to $L = 40$ is well fitted by $J_c \simeq 0.91L^{-1.70} + 0.52$, converging to the PBC critical point $J_c \simeq 0.5186$ in the thermodynamic limit. These large-scale DMRG results provide strong evidence for the existence of the first-order phase transition in large systems, with OBC finite-size effects well controlled and smoothly connected to the bulk transition identified under PBC.

PBC autocorrelation functions—We now discuss the dynamics of the PBC autocorrelation functions on site L

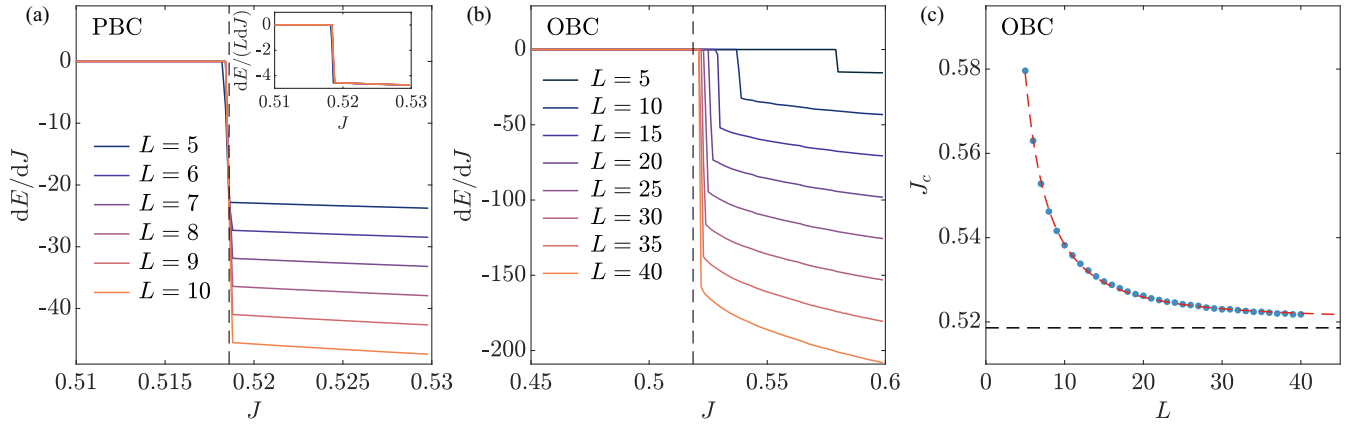


FIG. 5. The derivative (dE/dJ) of the ground-state energy E (obtained by DMRG) with respect to the breakdown interaction J for PBC in (a) and OBC in (b). We set $h = \lambda = 1$ and $S = 5$ for different system sizes L . The inset of (a) shows that (dE/LdJ) collapse to the same curve. The dashed black line marks $J_c = 0.5186$, which is the PBC phase boundary with negligible finite-size effects. (c) The finite-size OBC phase boundary J_c identified from the discontinuous jumps in (b). The red dashed line shows a fitting formula $J_c \approx 0.91L^{-1.70} + 0.52$, which approaches the PBC phase boundary $J_c = 0.5186$.

(all sites are equivalent for PBC),

$$\begin{aligned} C_{ZZ}(t) &= \langle \psi_Q | \hat{S}_L^z(t) \hat{S}_L^z(0) | \psi_Q \rangle - \langle \hat{S}_L^z \rangle_Q^2, \\ C_{XX}(t) &= \langle \psi_Q | \hat{S}_L^x(t) \hat{S}_L^x(0) | \psi_Q \rangle, \end{aligned} \quad (\text{A1})$$

where $|\psi_Q\rangle$ is the ground state in the PBC charge sector Q , and $\langle \hat{S}_L^z \rangle_Q = \langle \psi_Q | \hat{S}_L^z(t) | \psi_Q \rangle$ is independent of t . Also, $\langle \hat{S}_L^x(t) \rangle_Q = \langle \psi_Q | \hat{S}_L^x(t) | \psi_Q \rangle \equiv 0$. We denote the n th ($n \geq 1$, from low to high) eigenvalue and eigenstate of the Hamiltonian in charge sector Q as $E_n^{(Q)}$ and $|\phi_n^{(Q)}\rangle$, so that $|\psi_Q\rangle \equiv |\phi_1^{(Q)}\rangle$ is the charge sector Q ground state. Accordingly, $C_{XX}(t) = \sum_{Q'} \sum_{n_{Q'}} e^{i(E_1^{(Q)} - E_{n_{Q'}}^{(Q')})t} |\langle \phi_{n_{Q'}}^{(Q')} | \hat{S}_L^x | \psi_Q \rangle|^2$, where Q' is the label of the charge sectors that are connected to the sector Q via $\hat{S}_L^x$. As shown in Fig. 2(c), in the SSB phase under PBC, all the excited states ($n_{Q'} \geq 2$) have $E_{n_{Q'}}^{(Q')} - E_1^{(Q)} \geq \Delta$ and contribute fast oscillations. In contrast, all the ground states have $|E_1^{(Q)} - E_1^{(Q')}| \leq W$ and have $\langle \psi_{Q'} | \hat{S}_L^x | \psi_Q \rangle \neq 0$ only if $Q' = Q \pm 1$. This leads to a slow nondecaying oscillation of timescale $\tau_X \propto 1/W$, as shown in Fig. 4(a) calculated by ED.

As for $C_{ZZ}(t)$, we note that $\hat{S}_L^z$ acting on $|\psi_Q\rangle$ does not change the charge sector Q . This gives $C_{ZZ}(t) = \sum_{n_{Q \geq 2}} e^{i(E_1^{(Q)} - E_{n_{Q \geq 2}}^{(Q)})t} |\langle \phi_{n_{Q \geq 2}}^{(Q)} | \hat{S}_L^z | \psi_Q \rangle|^2$. Since $E_{n_{Q \geq 2}}^{(Q)} - E_1^{(Q)} > \Delta$, we expect $C_{ZZ}(t)$ to have only a fast dynamics of timescale $\tau_Z \propto 1/\Delta$. The ED results of $C_{ZZ}(t)$ for small system sizes can be found in Supplemental Material [42].

We further examine the autocorrelation functions in larger system sizes by employing the charge-conserving DMRG. For instance, for the $Q = 0$ sector ground state

$|\psi_{Q=0}\rangle$, we perform the time-dependent variational principle to simulate the real-time evolution. As shown in Fig. 6, we find $C_{ZZ}(t)$ rapidly oscillating around zero with a timescale $\tau_Z \sim 1/\Delta_{Q=0}$, where $\Delta_{Q=0}$ is the $\mathcal{O}(1)$ excitation gap in the sector $Q = 0$. In contrast, $C_{XX}(t)$ shown in Fig. 6 stays around a finite value within the accessible time window. The expected oscillation timescale for $C_{XX}(t)$ is $\tau_X \gtrsim 10^3$, which far exceeds the accessible time window.

Symmetry-breaking perturbations—To demonstrate SSB in the thermodynamic limit, we numerically investigate the following Hamiltonian with an exponential $U(1)$ symmetry-breaking perturbation $h_x > 0$:

$$\hat{H}_{\text{pert}} = \hat{H} - h_x \sum_{j=1}^L \hat{S}_j^x, \quad (\text{A2})$$

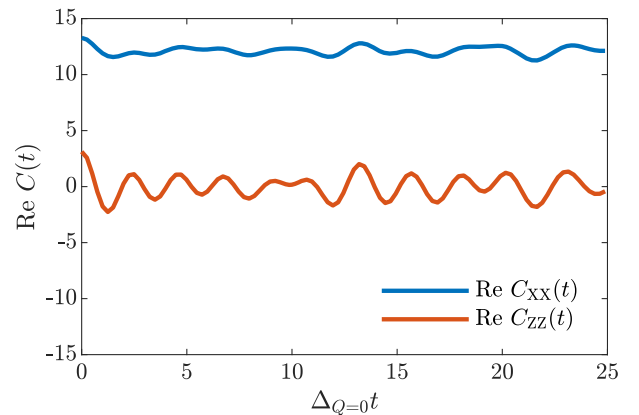


FIG. 6. The $L = 10$ DMRG results of the PBC autocorrelation functions $C_{XX}(t)$ and $C_{ZZ}(t)$ with the initial state $|\psi_Q\rangle$ being the ground state in the PBC charge sector $Q = 0$. We set $S = 5$, $J = 2$, and $h = \lambda = 1$. $\Delta_{Q=0} \approx 4.9774$ is the energy gap between the ground state and first excited state in the $Q = 0$ charge sector.

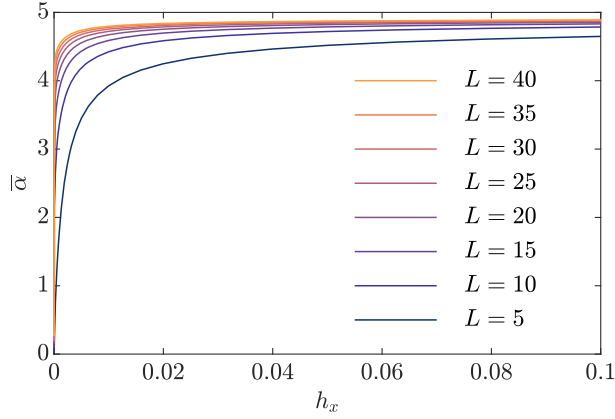


FIG. 7. The average order parameter $\bar{\alpha}$ [Eq. (A3)] for different system sizes L and perturbations h_x under OBC, obtained by large-scale DMRG. The other parameters are set to $S = 5$, $J = 2$, and $h = \lambda = 1$.

where $\hat{H}$ is the quantum breakdown Hamiltonian in Eq. (1). Taking OBC, we employ the DMRG to find the global ground state $|\Psi(h_x)\rangle$ of $\hat{H}_{\text{pert}}$ and calculate the spin order parameter $\langle \Psi(h_x) | \hat{S}_j^+ | \Psi(h_x) \rangle \equiv \alpha_j e^{i\theta_j}$ on site j , where α_j is the amplitude and θ_j is the phase angle. The perturbation $h_x > 0$ pins a specific symmetry-breaking direction with all $\theta_j = 0$. We define the average magnitude of order parameter

$$\bar{\alpha} \equiv \frac{1}{L} \sum_{j=1}^L \alpha_j, \quad (\text{A3})$$

the value of which from large-scale DMRG for different system sizes L are shown in Fig. 7. The numerical results clearly indicate that

$$\lim_{h_x \rightarrow 0^+} \lim_{L \rightarrow \infty} \bar{\alpha} \neq 0. \quad (\text{A4})$$

Therefore, the SSB of exponential $U(1)$ symmetry in the thermodynamic limit can be detected in the response to an infinitesimal symmetry-breaking field. Additional ED numerical results for symmetry-breaking perturbations are presented in Supplemental Material [42].

Proof of the absence of ODLRO—We prove that there are no ODLRO two-point functions of local operators characterizing the exponential $U(1)$ SSB in our spin model. Assume there exists an ODLRO two-point function $\langle \hat{A}_j \hat{B}_{j+l} \rangle$ nonvanishing as $|l| \rightarrow \infty$, where $\hat{A}_j$ and $\hat{B}_{j+l}$ are two local operators supported in the intervals of sites $[j-w, j+w]$ and $[j+l-w, j+l+w]$, respectively, with w finite. As a valid ODLRO, one requires the two operators to carry nonzero exponential $U(1)$ charges q_{AB} and $-q_{AB}$, respectively, i.e., $[\hat{Q}, \hat{A}_j] = q_{AB} \hat{A}_j$ and $[\hat{Q}, \hat{B}_{j+l}] = -q_{AB} \hat{B}_{j+l}$. In a spin S model, the on-site Hilbert space dimension $d = 2S + 1$ is finite. Accordingly, the smallest (largest) exponential $U(1)$ charge that can be changed by an operator on site j is 2^{L-j} ($2^{L-j}d$). Therefore, the nonzero charge q_{AB} carried by operator A_j must satisfy

$$2^{L-j-w} \leq |q_{AB}| \leq \sum_{k=-w}^w 2^{L-j-k}d < 2^{L-j+w+1}d. \quad (\text{A5})$$

Similarly, the nonzero charge $-q_{AB}$ carried by operator B_{j+l} must satisfy

$$2^{L-j-l-w} \leq |q_{AB}| < 2^{L-j-l+w+1}d. \quad (\text{A6})$$

As $|l| \rightarrow \infty$, Eqs. (A5) and (A6) necessarily contradict each other. Thus, such an ODLRO cannot exist.