

General Class of Functionals for Certifying Quantum Incompatibility

Kuan-Yi Lee^{1,2,*}, Jhen-Dong Lin^{2,†}, Adam Miranowicz³, and Yueh-Nan Chen^{2,4,‡}

¹*Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy*

²*Department of Physics and Center for Quantum Frontiers of Research and Technology (QFort), National Cheng Kung University, Tainan 701, Taiwan*

³*Institute of Spintronics and Quantum Information, Faculty of Physics and Astronomy, Adam Mickiewicz University, 61-614 Poznań, Poland*

⁴*Physics Division, National Center for Theoretical Sciences, Taipei 106319, Taiwan*



(Received 16 October 2025; revised 5 January 2026; accepted 23 February 2026; published 24 March 2026)

Quantum steering, measurement incompatibility, and instrument incompatibility have recently been recognized as unified manifestations of quantum incompatibility. Building on this perspective, we develop a general framework for constructing optimization-free, nonlinear incompatibility witnesses based on convex functionals, valid in arbitrary dimensions. We prove that these witnesses are nontrivial precisely when the underlying functional is nonaffine on extremal points (e.g., pure states for ensembles). For pure bipartite states, the witnesses yield lower bounds on entanglement measures, thereby outperforming most linear steering inequalities in the pure-state regime. Moreover, the construction extends in full generality to certify measurement and instrument incompatibility, where the witnesses act as genuine incompatibility monotones. We demonstrate the versatility of our approach with two operationally relevant functionals: the Wigner-Yanase skew information and an ℓ_2 -type coherence functional.

DOI: [10.1103/rsx1-zvbp](https://doi.org/10.1103/rsx1-zvbp)

Introduction—Incompatibility is a foundational quantum resource. In classical physics, any prescribed family of operations can be reproduced by a single device together with classical postprocessing [1,2]. By contrast, many quantum families admit no such joint simulation, i.e., compatibility fails. This structural constraint, already exemplified by the position-momentum trade-off, powers steering [3–5], Bell nonlocality [6,7], cryptography [8,9], and precision advantages in estimation and discrimination [10–16].

An increasing number of studies emphasize a unified compatibility viewpoint for states, measurements, and instruments [2,17]. In this formulation, quantum steering is the failure of a local-hidden-state model [18] (the compatibility of state assemblage), its link to measurement incompatibility ties (un)steerability to (joint) measurability [19–22]. Related developments extend the same framework to temporal and channel scenarios [23–28], allowing assemblages across settings to be analyzed on comparable footing.

For certification, two approaches dominate: first, optimization-based measures (e.g., robustness and compatibility via semidefinite programs) are informative but computationally demanding [29,30], and, second, linear or nonlinear witnesses (e.g., moment-matrix criteria [31,32], uncertainty-based inequalities [33–35], and tests based on the Clauser-Horne-Shimony-Holt inequality [36,37]) are easy to implement yet sometimes weak or lacking a clear task-related interpretation. This gap motivates simple, model-independent tests that avoid optimization while retaining operational meaning.

In this Letter, we develop a convex-analytic framework to assemblages and show that *any* convex functional yields a class of optimization-free, nonlinear, and dimension-independent incompatibility witnesses. We prove a necessary and sufficient criterion: a witness of quantum incompatibility is nontrivial if and only if the functional is nonaffine on extremal points. The framework delivers two concrete, operationally meaningful witnesses—instantiated with the Wigner-Yanase skew information (WYSI) [38,39] and an ℓ_2 -type coherence functional [40]—and encompasses earlier metrology- [41] and coherence-based [42] tasks within a single framework.

For pure states, the witness lower-bounds an entanglement measure and its algebraic maximum is saturated only by entangled states. Via the steering-equivalent-observable map and minimal Stinespring dilation [17], the construction applies uniformly to measurements and instruments,

*Contact author: kuanyi.lee@sns.it

†Contact author: jhendonglin@gmail.com

‡Contact author: yuehnan@mail.ncku.edu.tw

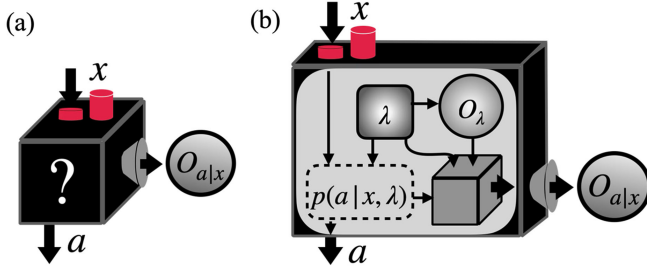


FIG. 1. Schematic illustration of compatibility tasks. (a) Schematic of the device/assemblage scenario (classical input x , classical output a , and quantum object $O_{a|x}$). (b) Hidden-object model that simulates a compatible assemblage via a parent object O_λ drawn from $p(\lambda)$ and classical postprocessing $p(a|x, \lambda)$.

recovering the known joint-measurability thresholds [43] and illustrating distinctive channel behavior.

A unified framework of quantum incompatibility test—It is known that quantum steering, measurement incompatibility, and instrument incompatibility can be investigated within a single unified framework in semi-device-independent settings [2,5,17]. We refer to this framework as “quantum incompatibility.” Specifically, as depicted in Fig. 1(a), we consider a quantum device with multiple classical inputs labeled by x , each corresponding to a button. When a particular input is chosen, the device stochastically produces a classical output a with probability $p(a|x)$, along with a corresponding quantum object $O_{a|x}$. We note that this quantum object can generally represent a quantum state, a quantum measurement, or a quantum instrument with corresponding notations in Table I. We temporally leave it unspecified to maintain the generality.

This device can now be fully characterized by an assemblage $\mathcal{O} = \{p(a|x)O_{a|x}\}_{a,x}$, which allows us to introduce the notion of quantum incompatibility. We regroup the assemblage according to each input x , defining $\mathcal{O}_x = \{p(a|x)O_{a|x}\}_a$, and ask whether $\mathcal{O}_x$ are mutually compatible. If this is the case, the assemblage can be classically simulated by a hidden-object (HO) model as depicted in Fig. 1(b). This model consists of predetermined hidden parent objects $\{O_\lambda\}_\lambda$ drawn from a distribution p_λ and a classical stochastic map $p(a|x, \lambda)$ such that

$$p(a|x)O_{a|x} = \sum_\lambda p(\lambda)p(a|x, \lambda)O_\lambda. \quad (1)$$

An assemblage is quantum incompatible if it cannot be simulated by the HO model.

TABLE I. Notations of assemblages and witnesses.

Spaces	Assemblages	Witnesses
States	σ	$\mathcal{S}_g(\sigma)$
Measurements	$\mathbf{M}$	$\mathcal{M}_g(\mathbf{M})$
Instruments	Λ	$\mathcal{I}_g(\Lambda)$

We are now in position to present our main result, in which we propose a framework comprising a general class of information tasks to certify quantum incompatibility. The central ingredient of this approach relies on the concept of real-valued convex functionals of quantum objects. Specifically, let $\mathcal{O}$ be the space of all quantum objects. A convex functional $g: \mathcal{O} \rightarrow \mathbb{R}^+$ satisfies $\sum_i p_i g(O_i) \geq g(\sum_i p_i O_i)$ with $p_i \geq 0$ and $\sum_i p_i = 1$. Operationally, most resource quantifiers satisfy convexity [44], which implies that classical randomness and mixture cannot create nonclassical resources. For every setting x , we define a functional of $\mathcal{O}_x$ as

$$g^{\text{as}}(\mathcal{O}_x) := \sum_a p(a|x)g(O_{a|x}), \quad (2)$$

where superscript “as” refers to a functional of assemblage. In addition, we define the roof extension of g as $F_g(\mathcal{O}) := \sup_{\{p_i, \psi_i\}} \sum_i p_i g(\psi_i)$ that satisfies $\sum_i p_i \psi_i = \mathcal{O}$ and $\{\psi_i\}_i \subset \text{ext}(\mathcal{O})$, where $\text{ext}(\mathcal{O})$ denotes the set of extreme points of $\mathcal{O}$, e.g., pure states for a state space. Note that the roof extension is achievable provided the supremum is attained by a physically realizable decomposition. Because of the convexity of g in $\mathcal{O}$, the roof extension serves as an upper bound of g . Similarly, we define $F_g^{\text{as}}(\mathcal{O}_x) := \sum_a p(a|x)F_g(O_{a|x})$. Following the standard convex-theoretic analysis, we obtain the following general criterion for certifying quantum incompatibility.

Result 1: If an assemblage $\mathcal{O}$ admits the HO model, then

$$g^{\text{as}}(\mathcal{O}_x) \leq F_g^{\text{as}}(\mathcal{O}_{x'}) \quad \forall x, x'. \quad (3)$$

The details of our proof can be found in Supplemental Material [45], where the central idea relies on the convexity (concavity) of g (F_g) by showing that $g^{\text{as}}(\mathcal{O}_x) \leq \sum_\lambda p(\lambda)g(O_\lambda)$ [$F_g^{\text{as}}(\mathcal{O}_x) \geq \sum_\lambda p(\lambda)F_g(O_\lambda)$], respectively.

If the HO model holds, each conditional object $O_{a|x}$ is generated from the same parent ensemble $\{O_\lambda\}_\lambda$ by classical data processing. Hence, the observed average over any setting $g^{\text{as}}(\mathcal{O}_x)$ cannot exceed the value obtainable from $\sum_\lambda p(\lambda)g(O_\lambda)$. On the other hand, $F_g^{\text{as}}(\mathcal{O}_x)$ is the maximal average value of g achievable from the same $\{O_\lambda\}_\lambda$ when one has full classical information about the hidden variable λ and is free to optimally postprocess λ with the outcome labels (a, x) . As such, Eq. (3) states that, when the assemblage can be generated by data processing of a single parent object $\{O_\lambda\}_\lambda$, the resource observed in a specific task can never surpass the information-assisted formation limit based on the same ingredients. A violation therefore certifies the impossibility of such data processing, and its amount indicates the extra resource enabled by the incompatibility. We note that an entirely parallel construction holds for any concave function f and its convex roof G_f such that $f \geq G_f$. For conciseness, in the following, we only discuss the convex function g .

One can ask two natural questions. First, can the inequality be violated at all, i.e., is the inequality nontrivial? Second, how can one determine from g whether the resulting inequality is trivial or nontrivial? To address these questions, we establish the necessary and sufficient condition for Eq. (3) to be nontrivial and provide explicit examples illustrating both the trivial and nontrivial cases. For convenience, we define the violation degree of Eq. (3) as a witness,

$$\mathcal{V}_g(\mathcal{O}) := \max \left\{ \max_x g^{\text{as}}(\mathcal{O}_x) - \min_x F_g^{\text{as}}(\mathcal{O}_x), 0 \right\}, \quad (4)$$

which vanishes for every HO assemblage by Result 1.

Below we give a necessary and sufficient condition for $\mathcal{V}_g$ to be nontrivial—that is, to attain strictly positive values. Trivial choices of g follow immediately from this statement, followed by two nontrivial examples.

Result 2: $\mathcal{V}_g$ is nontrivial, i.e., $\exists \mathcal{O}'$ such that $\mathcal{V}_g(\mathcal{O}') > 0$, if and only if $g|_{\text{ext}(\mathcal{O})}$ is not affine.

The details are presented in Supplemental Material [45]. An affine functional is simultaneously convex and concave, hence invariant under both data processing and convex decomposition. Consequently, Eq. (3) holds with equality if g is affine on the whole $\mathcal{O}$, rendering the criterion trivial. However, nonaffineness in $\mathcal{O}$ alone is not sufficient for being nontrivial: if the restriction $g|_{\text{ext}(\mathcal{O})}$ is affine (e.g., constant along extremal segments), the inequality remains trivial. An example is ergotropy [46] defined as

$$\mathcal{E}(\rho) := \max_U \text{Tr } H(\rho - U\rho U^\dagger), \quad (5)$$

where H is a fixed system Hamiltonian and U ranges over all unitaries. Ergotropy is convex on the state space, yet for any pure input state Eq. (5) reduces to an affine form $F_{\mathcal{E}}(\rho) = \text{Tr} \rho H - h_{\min}$, where $h_{\min}$ is the smallest eigenvalue of H , so that $F_{\mathcal{E}} = F_{\mathcal{E}}^{\text{as}}$ uniformly upper-bounds every $\mathcal{E}^{\text{as}}$ [cf. Eq. (2)]. Consequently, the HO inequality collapses to an identity and becomes trivial.

Here, we provide two convex functionals as examples of constructing nontrivial inequalities. The first one is the WYSI $I(\rho, H) = -\frac{1}{2} \text{Tr}[\sqrt{\rho}, H]^2$, which serves as a measure of quantum uncertainty associated with an observable H . Unlike the total variance, WYSI excludes contributions arising from classical mixing of the state [47,48]. In addition, when H is interpreted as the generator of time evolution, the WYSI quantifies the asymmetry of the state ρ , bounding the intrinsic speed of quantum evolution [49–52]. Note that the task of such an incompatibility scenario can be described as follows. After Bob receives the state assemblage, he can either measure the asymmetry by evaluating the WYSI on each conditional state $\rho_{a|x}$ and averaging, obtaining I^{as} by Eq. (2), or estimate an upper bound using only classical postprocessing via the concave-roof envelope F_I^{as} . In such a scenario, if Bob discovers that there exists a pair (x, x') such that the measured asymmetry surpasses the estimated upper bound in Eq. (3), the

assemblage must be incompatible. Similar scenarios can also be applied for quantum Fisher information and distillable coherence (cf. Refs. [41,42]). A systemic way of constructing the witness by $I(\rho, H)$ is given in Supplemental Material [45], followed by the second example: ℓ_2 -type of coherence $N(\rho, \Pi) := \sum_{i \neq j} |\langle i|\rho|j \rangle|^2$, which directly measures the off-diagonal quantum coherence within the fixed reference basis $\Pi := \{|i\rangle\langle i|\}_i$ [40,53]. This quantity bounds the energy storage in quantum batteries [54] and detects localization transitions [55].

State assemblage and steering scenario—In this section we apply our convex-analytic framework for the case where the object space $\mathcal{O}$ is the set of quantum states. Under this identification, the task of assemblage incompatibility becomes the familiar task of Einstein-Podolsky-Rosen (EPR) steering [3,5]. We begin by recalling the standard steering protocol and then show how its conditional states form a state assemblage to which our criterion directly applies.

In a typical EPR steering experiment, Alice (A) and Bob (B) share a bipartite state ρ^{AB} . In each round Alice chooses a measurement setting x and performs the positive-operator valued measure $\mathbf{M}_x = \{M_{a|x}\}_a$ with outcomes a , where $M_{a|x} \geq 0$ and $\sum_a M_{a|x} = \mathbb{1}^A$. After obtaining an outcome a , she sends the pair (a, x) to Bob. Bob’s subnormalized conditional state is $\sigma_{a|x} := \text{Tr}_A[(M_{a|x} \otimes \mathbb{1}^B)\rho^{\text{AB}}]$. The collection $\sigma := \{\sigma_{a|x} = p(a|x)\rho_{a|x}\}_{a,x}$ is called a “state assemblage,” satisfying $\sum_a p(a|x)\rho_{a|x} = \rho^B = \text{Tr}_A \rho^{\text{AB}}$ for all x . An assemblage is said to be “unsteerable” if it admits a local-hidden-state model [18],

$$p(a|x)\rho_{a|x} = \sum_{\lambda} p(\lambda)p(a|x, \lambda)\rho_{\lambda}, \quad (6)$$

where $\{\rho_{\lambda}\}_{\lambda}$ are preexisting states on Bob’s side and $p(a|x, \lambda)$ is a classical response function. Otherwise the assemblage (and hence the underlying bipartite state) is “steerable.” Consequently, given any functional of state that is convex and nonaffine under pure states, i.e., extreme points of the state space, Eq. (4) becomes a nontrivial steering witness; we denote the violation degree as the witness $\mathcal{S}_g(\sigma)$.

Moreover, one can notice that the witness $\mathcal{S}_g$ is not only determined by the underlying correlations of the shared state ρ^{AB} , but it also depends on Alice’s measurement $M_{a|x}$. In fact, inappropriate choices on Alice’s side may hide nonclassical correlations altogether, yielding little or no violation even when ρ^{AB} is highly nonclassical [19,20]. In the following, we temporarily set aside the problem of measurement design (which is analyzed in the next section) and focus on how much violation degree can be attributed to the state itself.

To factor out the measurement dependence, we henceforth optimize over Alice’s measurements. For pure ρ^{AB} , every ensemble of ρ^B is pure by performing rank-1 projective measurements by Alice [56], so the supremum of g coincides with the concave-roof value. In such a case,

the steering violation achieves the maximum value without measurement dependency (see Supplemental Material [45] for the detailed proof) and possesses the following property.

Result 3: If g is convex and symmetric, $\mathcal{S}_g$ lower bounds a full entanglement measure when evaluating pure bipartite states, and the maximum violation degree $\mathcal{S}_g^{\max} = [F_g - g](\mathbb{1}_d)$ is achieved if and only if the shared states ρ^{AB} are maximally entangled.

We prove this property by showing that when ρ^{AB} is pure entangled, $\mathcal{S}_g$ is upper-bounded by a closed form $[F_g - g](\rho^{\text{B}})$, which is concave and symmetric [57]; the maximum violation is directly obtained by Schur concavity. Notably, the bound is saturated when the roof extensions are achievable, i.e., the optimal decompositions always exist. In such case, the $\mathcal{S}_g$ becomes a faithful entanglement measure for pure bipartite states. A detailed proof can be found in Supplemental Material [45]. We remark that Result 3 shows that the algebraic maximum of $\mathcal{S}_g^{\max}$ is achieved only by maximally entangled states. This certifies the underlying states up to local isometries, but, since our figure of merit is optimized over Alice's measurements, it does not fix a unique measurement realization. From the perspective of self-testing [58,59], this corresponds to a *weak form* of self-testing, where a state is identified but the measurements remain unidentified [60].

Measurement assemblage and steering-induced incompatibility—Here, we restrict the object space to measurements with the corresponding assemblage $\mathbf{M} := \{M_{a|x}\}_{a,x}$. An assemblage of measurements is said to be “incompatible” when the corresponding observables cannot be implemented simultaneously—an intrinsically quantum feature that underpins contextuality [61,62] and uncertainty relations [1,63], and serves as a necessary resource to demonstrate EPR steering [5]. Specifically, a measurement assemblage $\mathbf{M}$ is “compatible” if there exists a single measurement $G_\lambda \geq 0$ such that

$$M_{a|x} = \sum_{\lambda} p(a|x, \lambda) G_{\lambda} \quad \forall a, \quad x, \quad (7)$$

where $p(a|x, \lambda)$ is a conditional probability distribution. Operationally, Eq. (7) means that the statistics generated by $\mathbf{M}$ can be simulated by performing a single measurement $\{G_{\lambda}\}_{\lambda}$ followed by postprocessing. Following Result 1, we can conclude that, for any convex g of a measurement $\mathbf{M}$, we can construct a witness of measurement incompatibility, denoted as $\mathcal{M}_g(\mathbf{M})$. To adopt the same functionals, i.e., $I(\rho, H)$ and $N(\rho, \Pi)$, we demonstrate below a method for constructing a valid incompatibility measure through convex functionals on the state space.

The structure of Eq. (7) mirrors the local-hidden-state model in Eq. (6). Indeed, Refs. [19–21] establish a one-to-one correspondence between the measurement assemblage on Alice's side and the state assemblage on Bob's side, testing incompatibility structures in the steering scenario by

applying pure entangled states [64]. Specifically, Alice performs $\mathbf{M}$ and introduces state assemblage σ on Bob, one can define the steering-equivalent observables (SEOs), $\mathbf{B} := \{B_{a|x} = \rho^{\text{B}-1/2} \sigma_{a|x} \rho^{\text{B}-1/2}\}_{a,x}$. Then $\mathbf{M}$ is compatible if and only if the corresponding σ is unsteerable (equivalently, if and only if $\mathbf{B}$ is compatible). By doing so, one can map the steering witness $\mathcal{S}_g(\sigma)$ to a measurement-incompatibility witness by setting $\sigma \rightarrow \sqrt{\rho^{\text{B}}} \mathbf{B} \sqrt{\rho^{\text{B}}} = \sqrt{\rho^{\text{B}}} \mathbf{M} \sqrt{\rho^{\text{B}}}$.

In addition, to eliminate the explicit dependence on the choice of state, we consider the scenario of a steering-induced incompatibility measure [22,65] by optimizing ρ^{B} , and we have

$$\mathcal{M}_g(\mathbf{M}) = \sup_{\rho^{\text{B}}} \mathcal{S}_g(\sqrt{\rho^{\text{B}}} \mathbf{M} \sqrt{\rho^{\text{B}}}), \quad (8)$$

where the supremum is taking over all full-rank ρ^{B} [21]. In other words, given a convex $g: \mathcal{D}(\mathcal{H}) \rightarrow \mathbb{R}^+$, we can extend it into a functional that maps $\mathcal{L}(\mathcal{H}) \rightarrow \mathbb{R}^+$. Furthermore, we show that $\mathcal{M}_g(\mathbf{M})$ serves as a valid incompatibility monotone [13], as presented in the following result.

Result 4: The steering-induced incompatibility $\mathcal{M}_g(\mathbf{M})$ quantifies incompatibility of $\mathbf{M}$ on Alice's side.

We prove this result by demonstrating that $\mathcal{M}_g(\mathbf{M})$ satisfies (i) $\mathcal{M}_g(\mathbf{M}) = 0$ if $\mathbf{M}$ is compatible, (ii) $\mathcal{M}_g(\mathbf{M})$ is convex, and (iii) $\mathcal{M}_g(\mathbf{M})$ is nonincreasing under postprocessing. The details are given in Supplemental Material [45].

To demonstrate our result, we consider a set of two noisy qubit Pauli measurements defined as

$$M_{a|x}^w = (1 - w) \Pi_{a|x} + w \frac{\mathbb{1}}{2}, \quad (9)$$

where $x \in \{0, 1\}$ represents Pauli Z and X measurement with its corresponding eigen-projectors $\Pi_{a|x}$. These measurements are incompatible for the mixing parameter $w \leq 1 - 1/\sqrt{2} \approx 0.29$ [43]. We plot the value of $\mathcal{S}_g(\sqrt{\rho^{\text{B}}} \mathbf{M} \sqrt{\rho^{\text{B}}})$ with the pure entangled input $|\varphi^{\text{AB}}(\theta)\rangle = \sin \theta |00\rangle + \cos \theta |11\rangle$ in Figs. 2(a) and 2(b). The black-dashed curves represent the boundary of zero violation degrees. Taking $\theta = \pi/4$ (corresponding to optimizing ρ^{B} and marking as the blue curves), we obtain the largest violation degree for both examples. For such slices, $\mathcal{S}_I(w)|_{\theta=\pi/4}$ and $\mathcal{S}_N(w)|_{\theta=\pi/4}$ reduce to $\mathcal{M}_I(w)$ and $\mathcal{M}_N(w)$. One can observe that, while the $\mathcal{M}_I(w)$ is more sensitive to noise (vanishes when $w \approx 0.21$), the $\mathcal{M}_N(w)$ is a tighter quantifier as it meets the threshold of $w \approx 0.29$ (the analytical results are shown in Supplemental Material [45]).

Instrument assemblage and channel steering—Now, we consider the object assemblage to instrument assemblage $\Lambda = \{\Lambda_{a|x}\}_{a,x}$, where each $\Lambda_{a|x}$ is a completely-positive trace-nonincreasing map. For a fixed setting x , the family $\Lambda_x = \{\Lambda_{a|x}\}_a$ constitutes an instrument satisfying the same marginal channel $\Lambda^{\text{m}} = \sum_a \Lambda_{a|x} \quad \forall x$. Physically, each

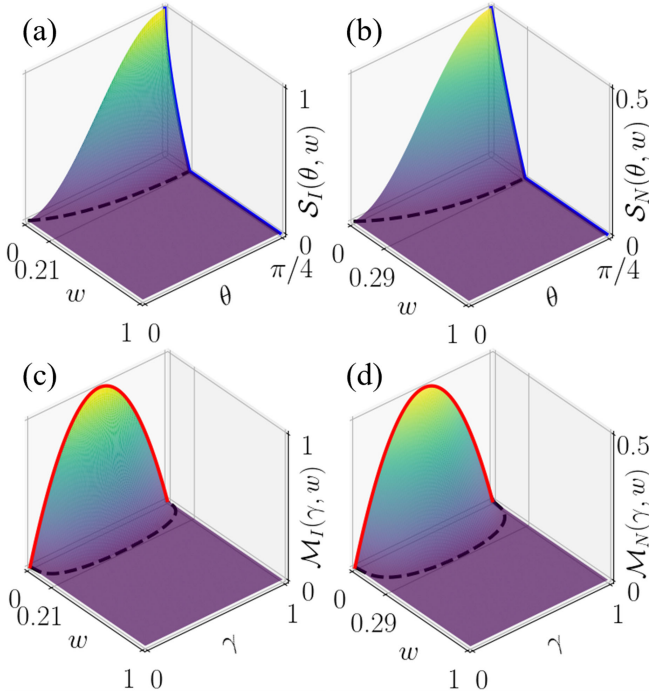


FIG. 2. The violation degrees of (a) $S_I(\theta, w)$ and (b) $S_N(\theta, w)$, where the black-dashed curves represent the boundaries of zero violation. The largest violations appear at $\theta = \pi/4$ are marked as blue curves; at these slices, $S_I|_{\theta=\pi/4} = \mathcal{M}_I(w)$ and $S_N|_{\theta=\pi/4} = \mathcal{M}_N(w)$ and both monotonically decrease when w increases. One can observe that the $\mathcal{M}_I$ is more sensitive to noise and vanishes as $w \approx 0.21$, while the $\mathcal{M}_N$ meets the threshold of compatible measurement, i.e., $w = 1 - 1/\sqrt{2} \approx 0.29$. The violation degrees of (c) $\mathcal{M}_I(\gamma, w)$ and (d) $\mathcal{M}_N(\gamma, w)$. Analogously, when $w = 0$, $\mathcal{M}_I|_{w=0}$ and $\mathcal{M}_N|_{w=0}$ reduce to $\mathcal{I}_I(\gamma)$ and $\mathcal{I}_N(\gamma)$, respectively, indicating the instrument incompatibility (see the red curves in each figures). Additionally, let $\gamma = 0.5$, the minimal dilation maps the input state $|1\rangle$ to a maximally entangled state, reflecting the measurement incompatibility $\mathcal{M}_I(\mathbf{M})$ and $\mathcal{M}_N(\mathbf{M})$.

element in an instrument $\{\Lambda_{a|x}\}_a$ acts as a general quantum measurement: it first postselects the input state ρ with a success probability $\text{Tr} \Lambda_{a|x}(\rho)$, and then outputs the post-measurement state $\Lambda_{a|x}(\rho)/\text{Tr} \Lambda_{a|x}(\rho)$.

A natural question arises whether the statistics generated by an instrument assemblage can be reproduced by a single instrument $\{\mathcal{G}_\lambda\}_\lambda$ followed by classical postprocessing. Concretely, Λ is compatible if

$$\Lambda_{a|x} = \sum_\lambda p(a|x, \lambda) \mathcal{G}_\lambda \quad \forall a, x, \quad (10)$$

and incompatible otherwise (cf. the channel-assemblage definition and unsteerability criterion given in Ref. [66]). By applying Result 1, we label the instrument incompatibility witness $\mathcal{I}_g(\Lambda)$, as shown in Eq. (4) [67–69].

To demonstrate the violation degree of instrument incompatibility, let us consider the channel steering scenario [66]: Charlie (C) prepares an input ρ^C and sends it to Bob by a fixed total channel $\Lambda^{C \rightarrow B}$ that might be correlated

with an ancillary system of, say, Alice. The task is to decide if the correlation between Alice and Bob allows Alice to steer Bob's outputs of the channel. In such a scenario, the instrument assemblage can be defined through an extended channel $\Lambda^{C \rightarrow A \otimes B}$, which reads

$$\Lambda_{a|x}(\rho^C) = \text{Tr}_A(\mathbf{M}_{a|x} \otimes \mathbb{1}) \Lambda^{C \rightarrow A \otimes B}(\rho^C), \quad (11)$$

with $\sum_a \Lambda_{a|x} = \Lambda^{C \rightarrow B}$. Following Ref. [17], every channel $\Lambda^{C \rightarrow B}$ admits a Stinespring isometry $V: \mathcal{H}^C \rightarrow \mathcal{H}^A \otimes \mathcal{H}^B$. A minimal dilation $V^{\min}$ is obtained by choosing the ancilla dimension equal to the Kraus (Choi) rank r of $\Lambda^{C \rightarrow B}$, namely $V^{\min}|\psi\rangle = \sum_{i=0}^{r-1} |i\rangle \otimes K_i|\psi\rangle$ with a linearly independent set of Kraus operators $\{K_i\}_{i=0}^{r-1}$ and $\sum_i K_i^\dagger K_i = \mathbb{1}$. In this setting, any instrument assemblage Λ on C to B can be represented via a measurement assemblage $\mathbf{M}$ on the ancilla (A) as in Eq. (11). The minimal dilation ensures a compatibility equivalence: Λ is unsteerable (i.e., compatible) if and only if the associated $\mathbf{M}$ is compatible. Consequently, instrument-level incompatibility can be certified by optimizing any measurement-level witness over all minimal dilations. Writing $\mathbf{M}(\Lambda, V^{\min}, \rho^C)$ for the measurement assemblage associated with Λ through $V^{\min}$ and ρ , we define

$$\mathcal{I}_g(\Lambda) := \sup_{\rho^C, \mathbf{M}} \mathcal{M}_g[\mathbf{M}(\Lambda, V^{\min}, \rho^C)], \quad (12)$$

which is vanishing for compatible instruments, convex, and postprocessing monotonic. By optimizing the measurement assemblage $\mathbf{M}$, the correlations between A and B certify instrument incompatibility—and, whenever such a violation occurs, also certify that $\Lambda^{C \rightarrow B}$ is not incompatibility-breaking [70].

As an illustrative example, we consider the total channel be a qubit-amplitude-damping channel $\Lambda^{C \rightarrow B} \rightarrow \Lambda_\gamma^{\text{amp}}$ with visibility γ , and we set the input state $\rho^C \rightarrow |1\rangle$. In such an example, $V_\gamma^{\min}|1\rangle = \sum_{i=0}^1 |i\rangle \otimes \sqrt{\alpha_i}|i \oplus 1\rangle$, where $\alpha_0 = 1 - \gamma$ and $\alpha_1 = \gamma$. Together with the set of two noisy qubit Pauli measurements defined in Eq. (9) (see Supplemental Material [45] for details). The results of $\mathcal{M}_I(\gamma, w)$ and $\mathcal{M}_N(\gamma, w)$ are shown in Figs. 2(c) and 2(d), respectively. In both figures, the violation degrees disappear when $\gamma = \{0, 1\}$ due to the minimal dilation of the channel $\Lambda_{\gamma=0,1}^{\text{amp}}$, which is a single unitary, i.e., one-dimensional, such that $V^{\min}|\psi\rangle = |0\rangle \otimes U|\psi\rangle$ is separable. When $w = 0$ (the red curves shown in each figures), both $\mathcal{M}_I|_{w=0}$ and $\mathcal{M}_N|_{w=0}$ reduce to instrument incompatibility parameters $\mathcal{I}_I(\gamma)$ and $\mathcal{I}_N(\gamma)$, indicating the correlation between Alice and Bob generated by the extended channel $\Lambda^{C \rightarrow A \otimes B}$. Note that, when $\gamma = 0.5$, the minimal dilation maps the input state $|1\rangle$ to maximally entangled state, representing measurement incompatibility $\mathcal{M}_g(\mathbf{M})$.

Conclusion—We introduce a framework based on quantum operational information tasks to certify incompatibility and, from it, an optimization-free recipe: any convex

functional g on the relevant object space induces a compatibility bound, whose violation signals quantum incompatibility. We establish a necessary and sufficient criterion for nontriviality: witnesses built from g are non-trivial iff g is nonaffine on extremal points, underscoring the generality of the convex-analytic approach. As concrete examples, we present functionals based on the Wigner-Yanase skew information and ℓ_2 -type coherence that efficiently certify incompatibility in a dimension-independent manner.

For pure bipartite states, any violation lower-bounds the corresponding entanglement measure, and achievability of the functional's roof extension ensures faithfulness. Characterizing sufficient conditions for quantifying entanglement of general mixed states, however, remains an open problem, which we leave as a future work.

Beyond states, the same recipe defines a steering-induced incompatibility monotone for measurements and instruments, and any observed violation can be read as preservation of incompatibility—in particular, a certificate that the channel is *not* incompatibility-breaking [70].

Acknowledgments—We thank Ludovico Lami for useful conversations. K.-Y.L. acknowledges financial support from the European Union (ERC StG ETQO, Grant Agreement No. 101165230). This work is supported by the National Center for Theoretical Sciences and National Science and Technology Council, Taiwan, Grant No. NSTC 114-2112-M-006 -015 -MY3. A.M. was supported by the Polish National Science Centre (NCN) under the Maestro Grant No. DEC-2019/34/A/ST2/00081.

Data availability—The data that support the findings of this article are openly available [71]; embargo periods may apply.

[1] T. Heinosaari, T. Miyadera, and M. Ziman, An invitation to quantum incompatibility, *J. Phys. A* **49**, 123001 (2016).
 [2] O. Gühne, E. Haapasalo, T. Kraft, J.-P. Pellonpää, and R. Uola, Colloquium: Incompatible measurements in quantum information science, *Rev. Mod. Phys.* **95**, 011003 (2023).
 [3] R. Gallego and L. Aolita, Resource theory of steering, *Phys. Rev. X* **5**, 041008 (2015).
 [4] Y. Xiang, S. Cheng, Q. Gong, Z. Ficek, and Q. He, Quantum steering: Practical challenges and future directions, *PRX Quantum* **3**, 030102 (2022).
 [5] O. Gühne, E. Haapasalo, T. Kraft, J.-P. Pellonpää, and R. Uola, Colloquium: Incompatible measurements in quantum information science, *Rev. Mod. Phys.* **95**, 011003 (2023).
 [6] F. Buscemi, All entangled quantum states are nonlocal, *Phys. Rev. Lett.* **108**, 200401 (2012).
 [7] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, *Rev. Mod. Phys.* **86**, 419 (2014).
 [8] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Quantum cryptography, *Rev. Mod. Phys.* **74**, 145 (2002).

[9] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering, *Phys. Rev. A* **85**, 010301(R) (2012).
 [10] M. Piani and J. Watrous, Necessary and sufficient quantum information characterization of Einstein-Podolsky-Rosen steering, *Phys. Rev. Lett.* **114**, 060404 (2015).
 [11] R. Takagi, B. Regula, K. Bu, Z.-W. Liu, and G. Adesso, Operational advantage of quantum resources in subchannel discrimination, *Phys. Rev. Lett.* **122**, 140402 (2019).
 [12] P. Skrzypczyk and N. Linden, Robustness of measurement, discrimination games, and accessible information, *Phys. Rev. Lett.* **122**, 140403 (2019).
 [13] P. Skrzypczyk, I. Šupić, and D. Cavalcanti, All sets of incompatible measurements give an advantage in quantum state discrimination, *Phys. Rev. Lett.* **122**, 130403 (2019).
 [14] K. C. Tan, V. Narasimhachar, and B. Regula, Fisher information universally identifies quantum resources, *Phys. Rev. Lett.* **127**, 200402 (2021).
 [15] K. Flatt, H. Lee, C. R. I. Carceller, J. B. Brask, and J. Bae, Contextual advantages and certification for maximum-confidence discrimination, *PRX Quantum* **3**, 030337 (2022).
 [16] K.-Y. Lee, J.-D. Lin, A. Miranowicz, F. Nori, H.-Y. Ku, and Y.-N. Chen, Steering-enhanced quantum metrology using superpositions of noisy phase shifts, *Phys. Rev. Res.* **5**, 013103 (2023).
 [17] R. Uola, F. Lever, O. Gühne, and J.-P. Pellonpää, Unified picture for spatial, temporal, and channel steering, *Phys. Rev. A* **97**, 032301 (2018).
 [18] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Steering, entanglement, nonlocality, and the Einstein-Podolsky-Rosen paradox, *Phys. Rev. Lett.* **98**, 140402 (2007).
 [19] M. T. Quintino, T. Vértesi, and N. Brunner, Joint measurability, Einstein-Podolsky-Rosen steering, and Bell nonlocality, *Phys. Rev. Lett.* **113**, 160402 (2014).
 [20] R. Uola, T. Moroder, and O. Gühne, Joint measurability of generalized measurements implies classicality, *Phys. Rev. Lett.* **113**, 160403 (2014).
 [21] R. Uola, C. Budroni, O. Gühne, and J.-P. Pellonpää, One-to-one mapping between steering and joint measurability problems, *Phys. Rev. Lett.* **115**, 230402 (2015).
 [22] H.-Y. Ku, C.-Y. Hsieh, S.-L. Chen, Y.-N. Chen, and C. Budroni, Complete classification of steerability under local filters and its relation with measurement incompatibility, *Nat. Commun.* **13**, 4973 (2022).
 [23] Y.-N. Chen, C.-M. Li, N. Lambert, S.-L. Chen, Y. Ota, G.-Y. Chen, and F. Nori, Temporal steering inequality, *Phys. Rev. A* **89**, 032112 (2014).
 [24] K. Bartkiewicz, A. Černoč, K. Lemr, A. Miranowicz, and F. Nori, Temporal steering and security of quantum key distribution with mutually unbiased bases against individual attacks, *Phys. Rev. A* **93**, 062345 (2016).
 [25] S.-L. Chen, N. Lambert, C.-M. Li, A. Miranowicz, Y.-N. Chen, and F. Nori, Quantifying non-Markovianity with temporal steering, *Phys. Rev. Lett.* **116**, 020503 (2016).
 [26] F. Buscemi, E. Chitambar, and W. Zhou, Complete resource theory of quantum incompatibility as quantum programmability, *Phys. Rev. Lett.* **124**, 120401 (2020).
 [27] L. Leppäjärvi and M. Sedláč, Incompatibility of quantum instruments, *Quantum* **8**, 1246 (2024).

- [28] K. Ji and E. Chitambar, Incompatibility as a resource for programmable quantum instruments, *PRX Quantum* **5**, 010340 (2024).
- [29] D. Cavalcanti and P. Skrzypczyk, Quantum steering: A review with focus on semidefinite programming, *Rep. Prog. Phys.* **80**, 024001 (2016).
- [30] S. Designolle, M. Farkas, and J. Kaniewski, Incompatibility robustness of quantum measurements: A unified framework, *New J. Phys.* **21**, 113053 (2019).
- [31] I. Kogias, P. Skrzypczyk, D. Cavalcanti, A. Acín, and G. Adesso, Hierarchy of steering criteria based on moments for all bipartite quantum systems, *Phys. Rev. Lett.* **115**, 210401 (2015).
- [32] S.-L. Chen, C. Budroni, Y.-C. Liang, and Y.-N. Chen, Natural framework for device-independent quantification of quantum steerability, measurement incompatibility, and self-testing, *Phys. Rev. Lett.* **116**, 240401 (2016).
- [33] T. Pramanik, M. Kaplan, and A. S. Majumdar, Fine-grained Einstein-Podolsky-Rosen-steering inequalities, *Phys. Rev. A* **90**, 050305(R) (2014).
- [34] T. Kriváchy, F. Fröwis, and N. Brunner, Tight steering inequalities from generalized entropic uncertainty relations, *Phys. Rev. A* **98**, 062111 (2018).
- [35] Ana C. S. Costa, R. Uola, and O. Gühne, Steering criteria from general entropic uncertainty relations, *Phys. Rev. A* **98**, 050104(R) (2018).
- [36] M. M. Wolf, D. Perez-Garcia, and C. Fernandez, Measurements incompatible in quantum theory cannot be measured jointly in any other no-signaling theory, *Phys. Rev. Lett.* **103**, 230402 (2009).
- [37] E. G. Cavalcanti, C. J. Foster, M. Fuwa, and H. M. Wiseman, Analog of the Clauser-Horne-Shimony-Holt inequality for steering, *J. Opt. Soc. Am. B* **32**, A74 (2015).
- [38] E. P. Wigner and M. M. Yanase, Information contents of distributions, *Proc. Natl. Acad. Sci. U.S.A.* **49**, 910 (1963).
- [39] S. Luo, Wigner-Yanase skew information and uncertainty relations, *Phys. Rev. Lett.* **91**, 180403 (2003).
- [40] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying coherence, *Phys. Rev. Lett.* **113**, 140401 (2014).
- [41] B. Yadin, M. Fadel, and M. Gessner, Metrological complementarity reveals the Einstein-Podolsky-Rosen paradox, *Nat. Commun.* **12**, 2410 (2021).
- [42] K.-Y. Lee, J.-D. Lin, K. Lemr, A. Černoč, A. Miranowicz, F. Nori, H.-Y. Ku, and Y.-N. Chen, Unveiling quantum steering by quantum-classical uncertainty complementarity, *npj Quantum Inf.* **11**, 72 (2025).
- [43] T. Heinosaari, D. Reitzner, and P. Stano, Notes on joint measurability of quantum observables, *Found. Phys.* **38**, 1133–1147 (2008).
- [44] B. Regula, Convex geometry of quantum resource quantification, *J. Phys. A: Math. Theor.* **51**, 045303 (2018).
- [45] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/rsx1-zvbp> for additional proofs and examples.
- [46] R. Alicki and M. Fannes, Entanglement boost for extractable work from ensembles of quantum batteries, *Phys. Rev. E* **87**, 042123 (2013).
- [47] D. Girolami, T. Tufarelli, and G. Adesso, Characterizing nonclassical correlations via local quantum uncertainty, *Phys. Rev. Lett.* **110**, 240402 (2013).
- [48] D. Girolami, Observable measure of quantum coherence in finite dimensional systems, *Phys. Rev. Lett.* **113**, 170401 (2014).
- [49] I. Marvian, R. W. Spekkens, and P. Zanardi, Quantum speed limits, coherence, and asymmetry, *Phys. Rev. A* **93**, 052331 (2016).
- [50] D. P. Pires, M. Cianciaruso, L. C. Céleri, G. Adesso, and D. O. Soares-Pinto, Generalized geometric quantum speed limits, *Phys. Rev. X* **6**, 021031 (2016).
- [51] R. Takagi, Skew informations from an operational view via resource theory of asymmetry, *Sci. Rep.* **9**, 14562 (2019).
- [52] K. Yamaguchi and H. Tajima, Beyond i.i.d. in the resource theory of asymmetry: An information-spectrum approach for quantum Fisher information, *Phys. Rev. Lett.* **131**, 200203 (2023).
- [53] A. Streltsov, G. Adesso, and M. B. Plenio, Colloquium: Quantum coherence as a resource, *Rev. Mod. Phys.* **89**, 041003 (2017).
- [54] F. Caravelli, B. Yan, L. P. García-Pintos, and A. Hamma, Energy storage and coherence in closed and open quantum batteries, *Quantum* **5**, 505 (2021).
- [55] G. Styliaris, N. Anand, L. Campos Venuti, and P. Zanardi, Quantum coherence and the localization transition, *Phys. Rev. B* **100**, 224204 (2019).
- [56] L. P. Hughston, R. Jozsa, and W. K. Wootters, A complete classification of quantum ensembles having a given density matrix, *Phys. Lett. A* **183**, 14 (1993).
- [57] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, *Rev. Mod. Phys.* **81**, 865 (2009).
- [58] I. Šupić and J. Bowles, Self-testing of quantum systems: A review, *Quantum* **4**, 337 (2020).
- [59] S. Sarkar, D. Saha, J. Kaniewski, and R. Augusiak, Self-testing quantum systems of arbitrary local dimension with minimal number of measurements, *npj Quantum Inf.* **7**, 151 (2021).
- [60] J. Kaniewski, Weak form of self-testing, *Phys. Rev. Res.* **2**, 033420 (2020).
- [61] A. Tavakoli and R. Uola, Measurement incompatibility and steering are necessary and sufficient for operational contextuality, *Phys. Rev. Res.* **2**, 013011 (2020).
- [62] C. Budroni, A. Cabello, O. Gühne, M. Kleinmann, and J.-A. Larsson, Kochen-Specker contextuality, *Rev. Mod. Phys.* **94**, 045007 (2022).
- [63] Y.-L. Mao, H. Chen, C. Niu, Z.-D. Li, S. Yu, and J. Fan, Testing Heisenberg-type measurement uncertainty relations of three observables, *Phys. Rev. Lett.* **131**, 150203 (2023).
- [64] M. T. Quintino, C. Budroni, E. Woodhead, A. Cabello, and D. Cavalcanti, Device-independent tests of structures of measurement incompatibility, *Phys. Rev. Lett.* **123**, 180401 (2019).
- [65] C.-Y. Hsieh, H.-Y. Ku, and C. Budroni, Characterisation and fundamental limitations of irreversible stochastic steering distillation, [arXiv:2309.06191](https://arxiv.org/abs/2309.06191).

- [66] M. Piani, Channel steering, *J. Opt. Soc. Am. B* **32**, A1 (2015).
- [67] A. Mitra *et al.*, Compatibility of quantum instruments, *Phys. Rev. A* **105**, 052202 (2022).
- [68] A. Mitra *et al.*, Characterizing and quantifying the incompatibility of quantum instruments, *Phys. Rev. A* **107**, 032217 (2023).
- [69] J. Ghai *et al.*, Instrument-based quantum resources: Quantification, hierarchies and towards constructing resource theories, [arXiv:2508.09134](https://arxiv.org/abs/2508.09134).
- [70] H.-Y. Ku, J. Kadlec, A. Černoch, M. T. Quintino, W. Zhou, K. Lemr, N. Lambert, A. Miranowicz, S.-L. Chen, F. Nori, and Y.-N. Chen, Quantifying quantumness of channels without entanglement, *PRX Quantum* **3**, 020338 (2022).
- [71] K.-Y. Lee, Supporting code for “General class of functionals for certifying quantum incompatibility,” [10.5281/zenodo.18761270](https://zenodo.org/record/105281).