

Factorizing Defects from Generalized Pinning Fields

Fedor K. Popov¹ and Yifan Wang²

¹*Simons Center for Geometry and Physics, SUNY, Stony Brook, New York 11794, USA*

²*Center for Cosmology and Particle Physics, New York University, New York, New York 10003, USA*



(Received 23 April 2025; revised 4 September 2025; accepted 14 October 2025; published 12 November 2025)

We introduce generalized pinning fields in conformal field theory that model a large class of critical impurities at large distance, enriching the familiar universality classes. We provide a rigorous definition of such defects as certain unbounded operators on the Hilbert space and prove that when inserted on codimension-one surfaces they factorize the spacetime into two halves. The factorization channels are further constrained by symmetries in the bulk. As a corollary, we solve such critical impurities in the 2D minimal models and establish the factorization phenomena previously observed for localized mass deformations in the 3D $O(N)$ model.

DOI: [10.1103/rh7j-yg11](https://doi.org/10.1103/rh7j-yg11)

Introduction and summary—Extended operators defined on submanifolds of the spacetime constitute a fundamental component of modern quantum field theory (QFT), shaping our understanding of symmetry principles, phases of matter, and strong coupling phenomena. When extended in time, they are also known as defects—such as impurities and boundaries—which exhibit rich dynamics, particularly in gapless bulk systems described by conformal field theories (CFTs). To this end, powerful monotonicity theorems have been established for defect renormalization group (RG) flows [1–10], ensuring that certain defects remain unscreened at large distances, while additional constraints from generalized symmetries are beginning to be explored [11–18]. Furthermore, the study of defect fusion and the Casimir effect offers new perspectives on universal defect data via the framework of effective field theory [19–22]. Nevertheless, explicit examples of nontrivial conformal defects remain scarce, and even with a well-defined short-distance description, their behavior at large distances remains unclear.

Here we introduce a large family of defects with the following simple UV definition in an arbitrary CFT in $d \geq 2$,

$$\mathcal{D}_h(\mathcal{O}) \equiv \left[e^{h\hat{\mathcal{O}}} \right]_{\text{ren}}, \quad \hat{\mathcal{O}} \equiv \int_{\Sigma_p} \mathcal{O}(x), \quad (1)$$

where $\mathcal{O}$ is a scalar operator of dimension $\Delta_{\mathcal{O}} < p$ and Σ_p is the defect world volume which we will often take to be an $\mathbb{R}^p$ hyperplane in the Euclidean spacetime. With

appropriate renormalization, this defines a p -dimensional flat defect in the CFT (similarly for $\Sigma_p = S^p$ which defines the spherical defect). Such a defect is expected to flow to a nontrivial conformal defect in the IR, which we denote as,

$$\mathcal{D}(\mathcal{O}) \equiv \lim_{h \rightarrow \infty} \mathcal{D}_h(\mathcal{O}). \quad (2)$$

This is because the monotonicity theorems requires the defect entropy (for p odd) and conformal anomaly (for p even) to decrease under defect RG, and they both vanish for the trivial defect [23]. We will thus refer to (1) and (2) as the generalized pinning field defect and its conformal fixed point. In particular, our definition includes the defect RG flows studied by [24–28] in the very same $O(N)$ CFT but triggered by certain mass deformations on a surface. There it was observed based on large N analysis and ϵ expansion that the generalized pinning field surface defect at three dimensions appears to factorize into conformal boundary conditions for two disconnected regions of the spacetime. This factorization also manifests in different ways depending on the sign of the pinning field h , revealing the intricate phase structure on the surface, which includes a extraordinary-log phase at strong surface coupling [24,29]. In this Letter, we focus on codimension-one generalized pinning field defects and establish this factorization property in great generality.

Intuitively, this is because the conformal pinning defect (2) will take the form of an un-normalized projector on the CFT rigged Hilbert space $\mathcal{H}$ [see around (7)],

$$\mathcal{D}(\mathcal{O}) = \bigoplus_{\alpha} |\alpha\rangle\langle\alpha|, \quad (3)$$

by projecting to the highest or lowest eigenvalues of the operator $\hat{\mathcal{O}}$ depending on the sign of h , where the eigenbasis is denoted by $|\alpha\rangle$. Furthermore, by conformal

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

symmetry, we expect these eigenstates to describe conformal boundary conditions of the CFT.

However, one should be cautious with the above reasoning, since the conformal boundary conditions do not define normalizable states in $\mathcal{H}$ and $\hat{\mathcal{O}}$ may not be a healthy bounded operator on $\mathcal{H}$. We will formalize these points using results from the spectral theory of unbounded operators on Hilbert spaces. This framework allows us to precisely define generalized pinning field defects and prove their IR factorizations here.

The factorization property enables us to determine the long-distance dynamics of such defects by leveraging the full spectrum of conformal boundary conditions, thereby rendering the problem significantly more tractable. For example, as we discuss in Example B, this elucidates the conjectured factorization of the 3D $O(N)$ CFT induced by the $O(N)$ -symmetric surface mass term in terms of known boundary conditions and also yields nontrivial predictions for more general surface mass deformations. In Example A, we analyze a large class of 2D pinning defects defined as in (1), demonstrating how factorization together with the monotonicity g theorem and symmetry constraints allow us to completely nail down the IR conformal defect.

As mentioned before, these generalized pinning defects are ubiquitous in CFT and thus it would be interesting to study their applications, as probes of strongly coupled systems. For instance, quantum chromodynamics (QCD) with a sufficiently large number of massless fermion flavors (within the conformal window) admit nontrivial Caswell-Banks-Zaks type fixed points [30,31]. Finally, in CFTs with holographic duals, it would be interesting to investigate the dual description of codimension-one pinning defects, which may correspond to factorizing geometries involving new end-of-the-world branes [32–36].

Factorization—The main goal of this section is to prove the factorization property for *generic* pinning flows,

$$\mathcal{D} \equiv \lim_{h \rightarrow \infty} \left[e^{h\hat{\mathcal{O}}} \right]_{\text{ren}} = |\mathcal{B}\rangle\langle\mathcal{B}|, \quad (4)$$

where $|\mathcal{B}\rangle$ is a linear combination of Ishibashi states which are in one-to-one correspondence with scalar primary operators in the CFT [37]. This linear combination is further constrained by locality, and gives rise to a Cardy state in two dimensions [38]. For this reason, we will also refer to $|\mathcal{B}\rangle$ as a Cardy state in general dimensions. As will soon become evident, the primary task lies in rigorously defining each component of (4). By *generic*, we mean the flow does not preserve any extra symmetry and towards the end of the section we will discuss cases with extra symmetry, and how (4) is modified accordingly. We emphasize that the proof relies on the operator-state correspondence and the standard resolution of identity [39]. We also assume that the CFT is not a tensor product, unless the flow couples all sectors, in which case the Ishibashi states

are unambiguously defined with respect to the diagonal conformal symmetry.

For convenience, we will work in the Euclidean flat space with coordinates $\{\vec{x}, x_d\}$ such that the defect is planar and located at $x_d = 0$. We first define the CFT Hilbert space $\mathcal{H}$ at $x_d = 0$ in the N-S quantization [41] as spanned by

$$|\phi_f\rangle = \sum_a \int_{\mathbb{R}_+^d} d^d x f_a(x) \phi_a(x) |0\rangle, \quad f_a: \mathbb{R}_+^d \rightarrow \mathbb{C}, \quad (5)$$

where $\phi_a(x)$ are normalized primary operators of dimension Δ_a with smearing factors f_a on the half space $x_d > 0$, and the index a may also include spin indices. The inner product with scalar operators ϕ_a in (5) reads

$$\langle\phi_f|\phi_g\rangle = \sum_a \int_{\mathbb{R}_+^d} d^d x d^d y \frac{f_a^*(x) g_a(y)}{(|\vec{x} - \vec{y}|^2 + (x_d + y_d)^2)^{\Delta_a}}, \quad (6)$$

and the generalization with spinning operators is obvious. The factors f_a are constrained such that $|\phi_f\rangle \in \mathcal{H}$ has a finite norm, namely, $\| |\phi_f\rangle \| \equiv \sqrt{\langle\phi_f|\phi_f\rangle} < \infty$, and the Hilbert space $\mathcal{H}$ is complete with respect to this norm.

Next we define the space to which the Ishibashi states belong, since they are not normalizable. We first consider a dense subspace $\Phi \subset \mathcal{H}$ (i.e., its closure $\bar{\Phi} = \mathcal{H}$) spanned by finite linear combinations of primary operators and their descendants. We then introduce the dual space Φ' , consisting of all continuous linear functionals on Φ and equipped with the weak topology. The full structure is captured by a Gelfand triple (also known as a rigged Hilbert space) [42,43],

$$\Phi \subset \mathcal{H} \subset \Phi'. \quad (7)$$

For instance, if $|I\rangle = |\phi\rangle$ is an Ishibashi state corresponding to a scalar primary operator ϕ ,

$$\langle\psi|\phi(x)\rangle = x_d^{-\Delta_\phi}. \quad (8)$$

We emphasize that this overlap is not well defined for general states in $\mathcal{H}$ [see Appendix].

Now we define the regularization of the defect operator in (4). The first step is to regularize the operator $\hat{\mathcal{O}}$ in the exponent so that it is self-adjoint and has well-defined spectral decomposition. Here we consider a sequence of subspaces $\mathcal{H}_\Delta \subset \mathcal{H}$ which contain all conformal families of primary operators ϕ_a with dimension $\Delta_a \leq \Delta$. We assume that $\mathcal{H}_\Delta$ is a finite sum over conformal families and a Hilbert space itself. This regularization still contains an infinite number of states but explicitly preserves the conformal symmetry of the bulk theory which will be important in the subsequent analysis.

To regularize $\hat{\mathcal{O}}$ amounts to defining its action on $\mathcal{H}_\Delta$. We define $\Phi_\Delta = \Phi \cap \mathcal{H}_\Delta$ and introduce the following sesquilinear form on Φ_Δ

$$(\phi_f, \phi_g) \equiv \langle \phi_f | \hat{\mathcal{O}} | \phi_g \rangle, \quad (9)$$

for $|\phi_f\rangle, |\phi_g\rangle \in \Phi_\Delta$ and the rh is defined by the bulk OPE. There is a unique self-adjoint operator $\hat{\mathcal{O}}_\Delta$ with a dense domain in Φ_Δ (thus also dense in $\mathcal{H}_\Delta$) whose matrix elements coincide with that of $\hat{\mathcal{O}}$ in (9) [44,45]. The operator $\hat{\mathcal{O}}_\Delta$ is self-adjoint but unbounded. We analyze in detail its spectrum in the Appendix. The key result there is that in the IR limit, the dominating eigenvector of $\hat{\mathcal{O}}_\Delta$ is an Ishibashi state $|\mathcal{B}, \Delta\rangle$.

We now deal with the regulated exponential in (4),

$$\mathcal{D} \equiv \text{w-lim}_{\Delta \rightarrow \infty} \text{w-lim}_{h \rightarrow \infty} \left[e^{h\hat{\mathcal{O}}_\Delta} \right]_{\text{ren}}, \quad (10)$$

defined by introducing a cutoff Λ on the spectrum of $\hat{\mathcal{O}}_\Delta$ and then taking weak limits. Before these limits, the renormalized defect is defined via minimal subtraction as

$$\mathcal{D}_\Delta(h, \Lambda) \equiv \mathcal{N} \int_{-\infty}^{\Lambda} d\sigma_\lambda \frac{e^{h(\lambda-\Lambda)}}{h} \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} |\lambda, \vec{p}, \Delta\rangle \langle \lambda, \vec{p}, \Delta|, \quad (11)$$

with a normalization factor $\mathcal{N}$ which we will fix later. The following weak limit clearly exists,

$$\text{w-lim}_{h \rightarrow \infty} \mathcal{D}_\Delta(h, \Lambda) = \mathcal{N} \frac{d\sigma_\lambda}{d\lambda} \Big|_{\lambda=\Lambda} \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} |\Lambda, \vec{p}, \Delta\rangle \langle \Lambda, \vec{p}, \Delta|. \quad (12)$$

Taking weak limit $\Lambda \rightarrow \infty$, using (A11) together with (A13) and then $\Delta \rightarrow \infty$, we obtain

$$\mathcal{D} = |\mathcal{B}\rangle \langle \mathcal{B}|, \quad |\mathcal{B}\rangle \propto \text{w-lim}_{\Delta \rightarrow \infty} |\mathcal{B}, \Delta\rangle, \quad (13)$$

where we have absorbed the normalization constants in (12) and the IR volume factor into $|\mathcal{B}\rangle$, which is a linear combination of Ishibashi states. By locality, we expect $|\mathcal{B}\rangle$ to be an indecomposable Cardy state of the CFT and its normalization is fixed this way.

Let us comment on the extensions to (4) when the pinning flow preserves additional global symmetries. This happens when $\hat{\mathcal{O}}$ commutes with the corresponding topological defects. Correspondingly, the spectrum of $\hat{\mathcal{O}}$ has degeneracies due to the symmetry action (fusion with the topological defects). When the symmetry preserved is finite, this leads to a finite direct sum of factorized interfaces in the IR for the pinning defect,

$$\mathcal{D} = \sum_\alpha |\mathcal{B}_\alpha\rangle \langle \mathcal{B}_\alpha|, \quad (14)$$

where the individual Cardy states $|\mathcal{B}_\alpha\rangle$ furnish a representation of the symmetry. We will see concrete examples in the Examples section. If the symmetry is continuous, we have instead an integral over the factorization channels,

$$\mathcal{D} = \int d\alpha |\mathcal{B}_\alpha\rangle \langle \mathcal{B}_\alpha|, \quad (15)$$

for a spectral measure $d\alpha$ invariant under the symmetry.

The factorized form of the pinning defect derived here makes their IR fusion product $\circ$ defined in [19] (see also Refs. [20–22,46–49]) more transparent. In particular, the defect self-fusion follows an idempotent rule,

$$\mathcal{D} \circ \mathcal{D}(\Sigma) = \mathcal{S}(\Sigma) \mathcal{D}(\Sigma), \quad (16)$$

where Σ is the defect world volume and $\mathcal{S}$ denotes the fusion coefficient theory (CFT) on Σ and $\mathcal{S}(\Sigma)$ is its partition function. Here we have used that the boundary states $|\mathcal{B}_\alpha\rangle$ in the factorization of $\mathcal{D}$ are related by symmetries and the Casimir energy densities $\mathcal{E}_{\alpha\beta}$ for a pair of such boundaries defined by

$$\langle \mathcal{B}_\alpha | e^{-zP_d} | \mathcal{B}_\beta \rangle \xrightarrow{z \rightarrow 0} e^{-\frac{A}{z^{d-1}} \mathcal{E}_{\alpha\beta}}, \quad (17)$$

with IR regulator A for the defect volume.

Symmetry constraints—The factorization channels that describe the pinning defect in the IR are further constrained by generalized global symmetries in the bulk CFT. In modern understanding, while local operators transform as ordinary charges of standard 0-form symmetry, extended defects transform in higher representations of the generalized symmetries (see, e.g., Refs. [50–56]). Intuitively, such a representation contains information about the fusion of the topological symmetry defects with the (nontopological) defects and the topological junctions between them, subject to consistencies of isotopy invariance. To make the following discussion concrete and self-contained, we will focus on two dimensions, in which case the CFT symmetry is described in part by a fusion (sub)category $\mathcal{C}$ and the symmetry properties of general line defects are captured by a bimodule category $\mathcal{M}$ over $\mathcal{C}$. Physically, the simple objects label indecomposable defects, and include their cousins from fusion with the topological defects $\mathcal{L}_i \in \mathcal{C}$. As we explain below, the symmetry properties of pinning defects are described by special bimodule categories.

We define a $\mathcal{C}$ -symmetric defect $\mathcal{D}$ to be such that it is transparent to, and thus commutes with, all topological defects $\mathcal{L}_i \in \mathcal{C}$,

$$\mathcal{L}_i \circ \mathcal{D} = \mathcal{D} \circ \mathcal{L}_i. \quad (18)$$

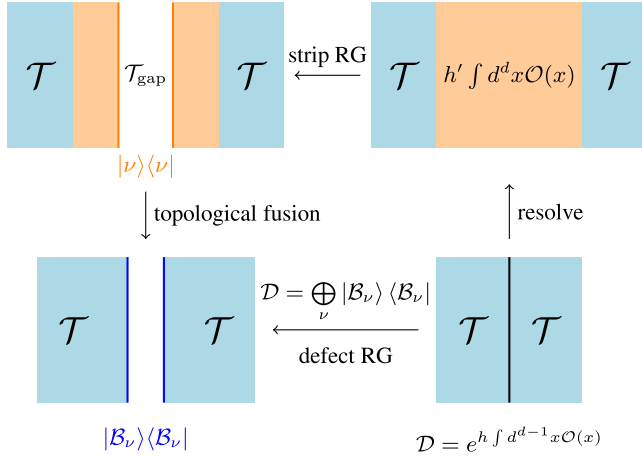


FIG. 1. IR factorization of the pinning flow via the strip flow by resolution. Here it is assumed that strip flow of the CFT $\mathcal{T}$ ends in a gapped phase (orange region) described by TQFT $\mathcal{T}_{\text{gap}}$ with Cardy branes $|\nu\rangle$.

The trivial defect is obviously $\mathcal{C}$ symmetric, and so are the generalized pinning defects (1) as long as the defining local operator $\mathcal{O}$ commutes with $\mathcal{L}_i$. As we will see, the $\mathcal{C}$ -symmetric condition produces nontrivial constraints on the possible IR behaviors of the defects, analogous to what was found for symmetric RG flows in the bulk CFT [57], such as *symmetry enforced degeneracies* in the IR factorization.

Let us offer some intuition behind this connection between defect and bulk symmetric RG flows as in Fig. 1, which also gives another way to see the factorization phenomena we have proven in the Factorization section [58]. The pinning flow defined in the UV by (1) can be alternatively described by turning on the same operator $\mathcal{O}$ on a strip of thickness δ in the limit of $\delta/L \ll 1$ where L is the typical size for a bulk observable (possibly with fine-tuning when the OPE of $\mathcal{O}$ generate more relevant operators). If we follow the bulk RG flow on the strip (namely, taking $\mu\delta \gg 1$ with μ as the scale of the flow) and suppose this flow gaps the bulk CFT, we conclude the strip region is effectively described by a TQFT $\mathcal{T}_{\text{gap}}$. A TQFT can be cut open by inserting a complete basis of Cardy branes $|\nu\rangle$ and further fusing each of these topological boundaries onto the interface between $\mathcal{T}$ and $\mathcal{T}_{\text{gap}}$ produce a simple boundary condition $|\mathcal{B}_{\nu}\rangle$ for the CFT $\mathcal{T}$. Assuming that the orders of the limits commute, we conclude that the pinning defect in the IR is described by the factorized product

$$\mathcal{D}(\mathcal{O}) = \bigoplus_{\nu} |\mathcal{B}_{\nu}\rangle \langle \mathcal{B}_{\nu}|, \quad (19)$$

where ν labels the Cardy branes in the TQFT $\mathcal{T}_{\text{gap}}$. The Cardy branes are in one-to-one correspondence with the vacua of the TQFT (that obey cluster decomposition) [59,60], and these vacuum degeneracies are consequences

of generalized anomalies for $\mathcal{C}$. Here we see how such constraints translate to the pinning defects straightforwardly (in terms of the factorization channels).

Note that in (19) we have not specified the conformal boundaries $|\mathcal{B}_{\nu}\rangle$, which depend on the interface between $\mathcal{T}$ and $\mathcal{T}_{\text{gap}}$. These are known as RG interfaces and have been determined explicitly in the 2D Ising and tricritical Ising CFTs [61–64]. We will see how to bootstrap this information efficiently in Example A with general criteria from monotonicity theorems and the $\mathcal{C}$ -symmetric condition.

As a sanity check, the indecomposable $\mathcal{C}$ -bimodule category $\mathcal{M}$ labeling the conformal pinning defect should be orientation-reversal invariant

$$\mathcal{M} \cong \bar{\mathcal{M}}, \quad (20)$$

and compatible with the idempotent fusion rule (16), naturally in terms of the relative Deligne tensor product over $\mathcal{C}$ [65,66],

$$\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{M} \cong n\mathcal{M}, \quad (21)$$

where $n \in \mathbb{Z}_+$ is the fusion multiplicity. The physical meaning of n is better understood in the case of topological fusion as capturing an emergent 1d TQFT, but it is somewhat mysterious in the non-topological fusion considered here. One way to connect the two (and give a physical definition of n in the present case) is by considering a bulk $\mathcal{C}$ -symmetric RG flow to a gapped phase. Note that the symmetry category $\mathcal{C}$ itself has a canonical $\mathcal{C}$ -bimodule category structure and is known as the canonical (trivial) $\mathcal{C}$ -bimodule category because it is the identity with respect to the relative tensor product. In particular it trivially solves (21) with $n = 1$ as expected because it labels the trivial (transparent) defect.

The indecomposable $\mathcal{C}$ -symmetric TQFTs, describing $\mathcal{T}_{\text{gap}}$ above, are in one-to-one correspondence with indecomposable $\mathcal{C}$ -module categories $\mathcal{M}_{\text{gap}}$ [60,67]. The simple objects of $\mathcal{M}_{\text{gap}}$ are precisely the Cardy branes $|\nu\rangle$. In fact, $\mathcal{M}_{\text{gap}}$ is a $(\mathcal{C}, \mathcal{C}_{\text{dual}})$ -bimodule category where $\mathcal{C}_{\text{dual}}$ is the dual symmetry under a generalized gauging in $\mathcal{C}$ [60,67]. Naively, one may expect that the defect $\mathcal{C}$ -bimodule category $\mathcal{M}$ is given by the relative tensor product of $\mathcal{M}_{\text{gap}}$ and its orientation reversal $\bar{\mathcal{M}}_{\text{gap}}$ over $\mathcal{C}_{\text{dual}}$,

$$\mathcal{M}_{\text{gap}} \boxtimes_{\mathcal{C}_{\text{dual}}} \bar{\mathcal{M}}_{\text{gap}} \cong \mathcal{C}, \quad (22)$$

which gives the trivial bimodule category $\mathcal{C}$, shared by the trivial defect as mentioned above [65]. However, in general, $\mathcal{M}$ is given by a quotient of $\mathcal{C}$, realized by a surjective $\mathcal{C}$ -bimodule functor $\mathcal{C} \rightarrow \mathcal{M}$, reflecting the fact that some topological defects in $\mathcal{C}$ may be absorbed by the defect,

$$\mathcal{L}_i \circ \mathcal{D} = \mathcal{D} \circ \mathcal{L}_i = \mathcal{D}. \quad (23)$$

Finally, so far we have focused on the symmetries that commute with the defect as in (18). There are further constraints from the broken symmetries (knowing exactly how the symmetry is broken). For example, when the symmetry defect $\mathcal{L}$ anticommutes with $\mathcal{O}$, one can infer the following fusion rule between the pinning defects (with opposite signs of h) and $\mathcal{L}$,

$$\mathcal{L} \circ \mathcal{D}_{\pm}(\mathcal{O}) = \mathcal{D}_{\mp}(\mathcal{O}) \circ \mathcal{L}, \quad (24)$$

which further implies that the IR g function is independent of the sign of the pinning flow [11]. In the special case of invertible $\mathcal{L}$, namely, its orientation reversal $\tilde{\mathcal{L}}$ defines the fusion inverse,

$$\mathcal{L} \circ \tilde{\mathcal{L}} = 1 \Rightarrow \mathcal{D}_{\pm}(\mathcal{O}) = \mathcal{L} \circ \mathcal{D}_{\mp}(\mathcal{O}) \circ \mathcal{L}. \quad (25)$$

Examples—

Pinning flows in 2D Virasoro minimal model CFT: While boundary conditions of Virasoro minimal model CFTs are well-known and in one-to-one correspondence with scalar primaries, much less is known about conformal defects, besides the exception for the Ising CFT [68]. Nonetheless, with the general results presented here, we will have definite predictions for the IR behaviors of generalized pinning defects in $\mathcal{T}$.

As a warm up, let us consider the Ising CFT, where we consider the spin operator σ of dimension $\Delta = \frac{1}{8}$. This operator breaks all the symmetries in the Ising CFT and produces a trivially gapped phase when turned on in the bulk. Consequently, we conclude that the IR defect takes the form of (19) with a single factorization channel $\mathcal{D}(\sigma) = |\mathcal{B}\rangle\langle\mathcal{B}|$, into a fixed boundary condition $|\mathcal{B}\rangle$ for the Ising CFT. Furthermore, the g theorem requires the corresponding boundary to have g function $g_{\mathcal{B}} < 1$. This fixes uniquely the conformal pinning defect in the Ising CFT to be

$$\mathcal{D}_{\pm}(\sigma) = |\pm\rangle\langle\pm|, \quad g_{\pm} = \frac{1}{\sqrt{2}} \quad (26)$$

depending on the sign of the pinning field h . As a consistency check, since the operator σ is odd under the

$\mathbb{Z}_2$ global symmetry, the two flows in (26) must be related by conjugation (25) with the symmetry defect η ,

$$D_+(\sigma) = \eta \circ D_-(\sigma) \circ \eta, \quad (27)$$

which is indeed satisfied since $\eta|\pm\rangle = |\mp\rangle$ under defect-boundary fusion.

As a more interesting application, we now consider the tricritical Ising CFT which contains three operators of dimension $\Delta < 1$: ϵ , σ , and σ' . The global symmetries of the theory is described by

$$\begin{aligned} \mathcal{C} &= \text{Ising} \boxtimes \text{Fib}, & \text{Irr}(\text{Ising}) &= \{1, \eta, \mathcal{N}\}, \\ \text{Irr}(\text{Fib}) &= \{1, W\}, \end{aligned} \quad (28)$$

where $\mathcal{N}$ is the duality defect and W is the nontrivial Fibonacci symmetry defect and they satisfy the topological fusion rules

$$\begin{aligned} \mathcal{N} \circ \mathcal{N} &= 1 \oplus \eta, & \mathcal{N} \circ \eta &= \eta \circ \mathcal{N} = \mathcal{N}, \\ W \circ W &= 1 \oplus W. \end{aligned} \quad (29)$$

The tricritical Ising CFT has 6 indecomposable boundary conditions

$$\begin{aligned} |+\rangle, |-\rangle &= \eta|+\rangle, & |0\rangle &= \mathcal{N}|+\rangle, \\ |d\rangle &= W|0\rangle, & |0-\rangle &= W|+\rangle, & | + 0 \rangle &= \eta|0-\rangle. \end{aligned} \quad (30)$$

In particular, the most stable boundary conditions $|\pm\rangle$ have g function $g_{|\pm\rangle} = \sqrt{[\sin(\pi/5)/5]}$.

It is well known that the bulk flows with respect to the three operators $\mathcal{O} = \epsilon, \sigma, \sigma'$ are massive [69]. We summarize the symmetries $\mathcal{C}$ preserved by such flows and the corresponding IR gapped phases in Table I. The corresponding conformal pinning defects are then fixed in terms of conformal boundaries (30) by the factorization channels in (19), together with the g theorem [70]. The results are listed in Table I where we also include the bimodule categories $\mathcal{M}$ labeling the conformal defects in the IR. There Vec denotes symmetry-absorbing cases (23) and all other cases coincide with the canonical (trivial) bimodule category $\mathcal{C}$. The boundary states that participate in the

TABLE I. The phase structure in the bulk and on the pinning defect depending on the deforming operator $\mathcal{O}$. We use SSB_k to denote a symmetry breaking phase with k degenerate ground states.

Operator $\mathcal{O}$	Symmetry $\mathcal{C}$	Bulk flow		Defect flow		Bimodule category $\mathcal{M}$	
		$h > 0$	$h < 0$	$h > 0$	$h < 0$	$h > 0$	$h < 0$
ϵ	$\text{Vec}_{\mathbb{Z}_2}$	SSB_2	Trivial	$ +\rangle\langle+ \oplus -\rangle\langle- $	$ 0\rangle\langle 0 $	$\text{Vec}_{\mathbb{Z}_2}$	Vec
σ	Vec	Trivial	Trivial	$ +\rangle\langle+ $	$ -\rangle\langle- $	Vec	Vec
σ'	Fib	SSB_2	SSB_2	$ +\rangle\langle+ \oplus 0-\rangle\langle-0 $	$ -\rangle\langle- \oplus + 0 \rangle\langle + 0 $	Fib	Fib

factorization channels also agree with the RG interfaces identified in [62–64].

It is straightforward to extend this analysis to other minimal models [44].

Pinning flows in 3D $O(N)$ CFT: In the 3D $O(N)$ CFT, there are two simple kinds of surface pinning flows, by considering either the fundamental scalar operator ϕ_I or the mass operators $M_{IJ}\phi_I\phi_J$ [either an $O(N)$ scalar or tensor], which all have dimension $\Delta < 2$.

From our general result in the Factorization section, the corresponding IR pinning defect will factorize into conformal boundary conditions of the $O(N)$ CFT. In particular, the $O(N)$ symmetric boundary conditions are classified into the ordinary, the special and the extraordinary-log classes [29,71]. The first two cases can be thought of as interacting versions of the Dirichlet and Neumann conformal boundary conditions for free theory.

Let us first consider the case of $O(N)$ symmetric pinning flows, namely, $\mathcal{O} = (\phi^2)_S$. Based on large N analysis, it was proposed in [24] that for $h < 0$, the pinning defect factorizes into the ordinary boundary conditions,

$$D_+((\phi^2)_S) = |\text{Ord}\rangle\langle\text{Ord}|. \quad (31)$$

This is also supported by resumming perturbative results for scaling dimensions on the surface defect in $d = 4 - \epsilon$ dimensions [28] and comparing to numerical results for $|\text{Ord}\rangle$ [72,73]. This factorization is in agreement with our general result since the positive mass ($h < 0$) deformation gaps the theory to a single ground state (thus one factorization channel) when turned on in the bulk. The case with $h > 0$ is more interesting, as the negative bulk mass deformation leads to a gapless phase with $N - 1$ Goldstone modes, and correspondingly, the proposal of [24] suggests that the defect is described by factorized normal boundary conditions $|\vec{n}\rangle\langle\vec{n}|$ weakly coupled to $N - 1$ Goldstone modes. Effectively, defined as an operator on the CFT Hilbert space, in the IR limit, we have

$$D_-((\phi^2)_S) \propto \int_{S^{n-1}} d\vec{n} |\vec{n}\rangle\langle\vec{n}|, \quad (32)$$

which again agrees with our general results.

For $O(N)$ breaking pinning flows, such as that by the fundamental scalar, the natural expectation is a factorization into normal boundary conditions,

$$D_\pm(\vec{n} \cdot \phi) = |\pm \vec{n}\rangle\langle\pm \vec{n}|, \quad (33)$$

preserving the residual $O(N - 1)$ symmetry. For $O(N)$ breaking surface mass deformations, the factorization property, together with previous work for such surface defects in $d = 4 - \epsilon$ dimensions [25,26] suggest new boundary universality classes in the $O(N)$ CFT. It would

be interesting to study them from complementary methods such as the fuzzy sphere approach [73,74].

Acknowledgments—We thank Gabriel Cuomo, Zohar Komargodski, Justin Kulp, Brandon Rayhaun, Max Metlitski, Yu Nakayama, Ananda Roy, Ingo Runkel, and Slava Rychkov for helpful questions and discussions. Y. W. thanks Kyushu University Institute for Advanced Study and RIKEN Interdisciplinary Theoretical and Mathematical Sciences Program for hospitality and discussions at the “Kyushu IAS-iTHEMS workshop: Non-perturbative methods in QFT” during the completion of this work. The work of Y. W. was supported in part by the NSF Grant No. PHY-2210420 and by the Simons Junior Faculty Fellows program.

Data availability—No data or figures were created; all results are purely theoretical.

-
- [1] I. Affleck and A. W. W. Ludwig, *Phys. Rev. Lett.* **67**, 161 (1991).
 - [2] D. Friedan and A. Konechny, *Phys. Rev. Lett.* **93**, 030402 (2004).
 - [3] K. Jensen and A. O’Bannon, *Phys. Rev. Lett.* **116**, 091601 (2016).
 - [4] H. Casini, I. Salazar Landea, and G. Torroba, *J. High Energy Phys.* **04** (2019) 166.
 - [5] N. Kobayashi, T. Nishioka, Y. Sato, and K. Watanabe, *J. High Energy Phys.* **01** (2019) 039.
 - [6] Y. Wang, *J. High Energy Phys.* **11** (2021) 122.
 - [7] Y. Wang, *J. High Energy Phys.* **02** (2022) 061.
 - [8] G. Cuomo, Z. Komargodski, and A. Raviv-Moshe, *Phys. Rev. Lett.* **128**, 021603 (2022).
 - [9] T. Shachar, R. Sinha, and M. Smolkin, *SciPost Phys.* **15**, 240 (2023).
 - [10] H. Casini, I. Salazar Landea, and G. Torroba, *J. High Energy Phys.* **07** (2023) 004.
 - [11] A. Konechny, *J. Phys. A* **53**, 155401 (2020).
 - [12] R. Thorngren and Y. Wang, *J. High Energy Phys.* **09** (2021) 017.
 - [13] O. Aharony, G. Cuomo, Z. Komargodski, M. Mezei, and A. Raviv-Moshe, *Phys. Rev. Lett.* **130**, 151601 (2023).
 - [14] Y. Choi, B. C. Rayhaun, Y. Sanghavi, and S.-H. Shao, *Phys. Rev. D* **108**, 125005 (2023).
 - [15] O. Aharony, G. Cuomo, Z. Komargodski, M. Mezei, and A. Raviv-Moshe, *J. High Energy Phys.* **12** (2023) 183.
 - [16] L. Bhardwaj, C. Copetti, D. Pajer, and S. Schafer-Nameki, *SciPost Phys.* **19**, 061 (2025).
 - [17] Y. Choi, B. C. Rayhaun, and Y. Zheng, *arXiv:2409.02159*.
 - [18] A. Antinucci, C. Copetti, G. Galati, and G. Rizi, *Phys. Rev. D* **111**, 065025 (2025).
 - [19] O. Diatlyk, H. Khanchandani, F. K. Popov, and Y. Wang, *J. High Energy Phys.* **09** (2024) 006.
 - [20] O. Diatlyk, H. Khanchandani, F. K. Popov, and Y. Wang, *Phys. Rev. Lett.* **133**, 261601 (2024).
 - [21] P. Kravchuk, A. Radcliffe, and R. Sinha, *arXiv:2406.04561*.
 - [22] G. Cuomo, Y.-C. He, and Z. Komargodski, *J. High Energy Phys.* **11** (2024) 061.

- [23] For general topological defects, the defect entropy is non-negative (for p odd) and the defect conformal anomaly always vanishes (for p even).
- [24] A. Krishnan and M. A. Metlitski, *SciPost Phys.* **15**, 090 (2023).
- [25] M. Trépanier, *J. High Energy Phys.* 09 (2023) 074.
- [26] A. Raviv-Moshe and S. Zhong, *J. High Energy Phys.* 08 (2023) 143.
- [27] S. Giombi and B. Liu, *J. High Energy Phys.* 12 (2023) 004.
- [28] O. Diatlyk, Z. Sun, and Y. Wang, *J. High Energy Phys.* 06 (2025) 131.
- [29] M. A. Metlitski, *SciPost Phys.* **12**, 131 (2022).
- [30] W. E. Caswell, *Phys. Rev. Lett.* **33**, 244 (1974).
- [31] T. Banks and A. Zaks, *Nucl. Phys.* **B196**, 189 (1982).
- [32] A. Karch and L. Randall, *J. High Energy Phys.* 05 (2001) 008.
- [33] A. Karch and L. Randall, *J. High Energy Phys.* 06 (2001) 063.
- [34] O. DeWolfe, D. Z. Freedman, and H. Ooguri, *Phys. Rev. D* **66**, 025009 (2002).
- [35] T. Takayanagi, *Phys. Rev. Lett.* **107**, 101602 (2011).
- [36] M. Fujita, T. Takayanagi, and E. Tonni, *J. High Energy Phys.* 11 (2011) 043.
- [37] Y. Nakayama and H. Ooguri, *J. High Energy Phys.* 10 (2015) 114.
- [38] J. L. Cardy, *Nucl. Phys.* **B324**, 581 (1989).
- [39] For instance, in the case of nonlocal theories (e.g., generalized free fields) this assumption fails and factorization does not hold [40].
- [40] D. Ge and Y. Nakayama, [arXiv:2505.15018](https://arxiv.org/abs/2505.15018).
- [41] S. Rychkov, *EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions*, SpringerBriefs in Physics (2016), [arXiv:1601.05000](https://arxiv.org/abs/1601.05000).
- [42] I. M. Gel'fand and N. Y. Vilenkin, *Generalized Functions: Applications of Harmonic Analysis* (Academic Press, New York, 2014), Vol. 4.
- [43] I. Gel'fand and G. E. Šilov, *Generalized Functions: Theory of Differential Equations* (Academic Press, New York, 1967).
- [44] See SM A at <http://link.aps.org/supplemental/10.1103/rh7j-yg11> for analysis of conformal line defects in the other minimal models. Supplemental Material B for the proof of the existence of self-adjoint operator $\hat{\mathcal{O}}_\Delta$. See SM E for an example where this limit does not exist. See SM C for the proof of this result. One can show under the weak limit that $\lim_{b \rightarrow \infty} U_b |\psi\rangle = 0$ for any $|\psi\rangle \in \mathcal{H}$ (see Sec. D for details).
- [45] Note that the matrix element as defined in B.1 [YW: fix] is IR finite only when $\Delta > [(d-1)/2]$, a condition stemming from the use of the planar setup for the defect. However these divergences are naturally regularized in the spherical conformal frame, which is related to the planar frame by a conformal transformation. For the sake of conciseness, we choose to work in the planar frame with this understanding.
- [46] C. Bachas and I. Brunner, *J. High Energy Phys.* 02 (2008) 085.
- [47] C. Bachas, I. Brunner, and D. Roggenkamp, *J. Stat. Mech.* (2013) P08008.
- [48] A. Söderberg, *J. High Energy Phys.* 04 (2021) 087.
- [49] A. Söderberg Rousu, *J. High Energy Phys.* 10 (2023) 183.
- [50] D. G. Delmastro, J. Gomis, P.-S. Hsin, and Z. Komargodski, *SciPost Phys.* **15**, 079 (2023).
- [51] T. D. Brennan, C. Cordova, and T. T. Dumitrescu, [arXiv:2206.15401](https://arxiv.org/abs/2206.15401).
- [52] T. Bartsch, M. Bullimore, A. E. V. Ferrari, and J. Pearson, *SciPost Phys.* **17**, 015 (2024).
- [53] T. Bartsch, M. Bullimore, A. E. V. Ferrari, and J. Pearson, *SciPost Phys.* **17**, 067 (2024).
- [54] L. Bhardwaj and S. Schafer-Nameki, *SciPost Phys.* **16**, 093 (2024).
- [55] L. Bhardwaj and S. Schafer-Nameki, *SciPost Phys.* **19**, 098 (2025).
- [56] C. Copetti, [arXiv:2408.01490](https://arxiv.org/abs/2408.01490).
- [57] C.-M. Chang, Y.-H. Lin, S.-H. Shao, Y. Wang, and X. Yin, *J. High Energy Phys.* 01 (2019) 026.
- [58] We emphasize that the general factorization property proved in the Factorization section does not rely on the additional assumptions introduced here, which are included purely for an intuitive connection to bulk RG flows, as illustrated in Fig. 1.
- [59] Z. Komargodski, K. Ohmori, K. Roumpedakis, and S. Seifnashri, *J. High Energy Phys.* 03 (2021) 103.
- [60] T.-C. Huang, Y.-H. Lin, and S. Seifnashri, *J. High Energy Phys.* 12 (2021) 028.
- [61] A. Konechny, *J. Phys. A* **50**, 145403 (2017).
- [62] J. Cardy, *SciPost Phys.* **3**, 011 (2017).
- [63] M. Lencses, J. Viti, and G. Takacs, *J. High Energy Phys.* 01 (2019) 177.
- [64] A. Konechny, *J. High Energy Phys.* 11 (2023) 004.
- [65] P. Etingof, D. Nikshych, and V. Ostrik, *Quantum Topol.* **1**, 209 (2010).
- [66] C. L. Douglas, C. Schommer-Pries, and N. Snyder, *Kyoto J. Math.* **59**, 167 (2019).
- [67] R. Thorngren and Y. Wang, *J. High Energy Phys.* 04 (2024) 132.
- [68] M. Oshikawa and I. Affleck, *Nucl. Phys.* **B495**, 533 (1997).
- [69] M. Lässig, G. Mussardo, and J. L. Cardy, *Nucl. Phys.* **B348**, 591 (1991).
- [70] For the pinning flow with $\mathcal{O} = \sigma$, to completely fix the answer, we have additionally assumed that the unique factorization channel is via a stable boundary state. This also agrees with the findings of [62–64].
- [71] H. W. Diehl, *Int. J. Mod. Phys. B* **11**, 3503 (1997).
- [72] F. P. Toldin, *Phys. Rev. B* **108**, L020404 (2023).
- [73] Z. Zhou and Y. Zou, *SciPost Phys.* **18**, 031 (2025).
- [74] M. Dedushenko, [arXiv:2407.15948](https://arxiv.org/abs/2407.15948).
- [75] A. N. Kolmogorov and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis* (Courier Corporation, Moscow, 1957), Vol. 1.
- [76] If $\hat{\mathcal{O}}_\Delta$ also respects additional symmetries, these eigenstates are further diagonalized with respect to the corresponding charges.
- [77] N. Marzari, A. A. Mostofi, J. R. Yates, I. Souza, and D. Vanderbilt, *Rev. Mod. Phys.* **84**, 1419 (2012).

End Matter

Appendix: Spectral decomposition and limit states of $\mathcal{O}_\Delta$ —In this section we will analyze the properties of spectrum of eigenvectors of $\mathcal{O}_\Delta$. For that let us note how the conformal generators preserving the quantization surface are dilation D , translations P_i , special conformal transformations K_i , and $\text{SO}(d-1)$ rotations M_{ij} act on the states (5). Acting on the Hilbert space $\mathcal{H}$, we have explicitly

$$\begin{aligned} D|\phi_f\rangle &= \sum_a \int_{\mathbb{R}_+^d} d^d x (x^\mu \partial_\mu + d - \Delta_a) f_a(x) \phi_a(x) |0\rangle, \\ P_i|\phi_f\rangle &= \sum_a \int_{\mathbb{R}_+^d} d^d x \partial_i f_a(x) \phi_a(x) |0\rangle. \end{aligned} \quad (\text{A1})$$

Importantly, the unitary operator $U_b = e^{bD}$ implies,

$$\begin{aligned} \langle U_b \phi_f | U_b \phi_g \rangle &= \langle \phi_f | \phi_g \rangle, \\ \langle U_b \phi_f | \hat{\mathcal{O}}_\Delta | U_b \phi_g \rangle &= e^{b(d-1-\Delta_O)} \langle \phi_f | \hat{\mathcal{O}}_\Delta | \phi_g \rangle. \end{aligned} \quad (\text{A2})$$

The operator $\hat{\mathcal{O}}_\Delta$ is thus clearly unbounded, since U_b can be used to indefinitely increase its matrix elements between normalizable states [44]. By applying the Hellinger-Toeplitz theorem [75], we deduce that the domain of $\hat{\mathcal{O}}_\Delta$ is a proper subspace of the Hilbert space, i.e., $\Phi_\Delta \subset \text{dom}(\hat{\mathcal{O}}_\Delta) \subsetneq \mathcal{H}_\Delta$. From (A2) we also conclude $\hat{\mathcal{O}}_\Delta$ does not possess nontrivial eigenvectors in $\mathcal{H}_\Delta$.

Nevertheless, *generalized eigenvectors* of $\hat{\mathcal{O}}_\Delta$ do exist in Φ'_Δ [as in the Gelfand triple in (7)] by invoking the Gelfand-Maurin theorem [42,43]. Specifically, we represent the self-adjoint operator $\hat{\mathcal{O}}_\Delta$ as

$$\hat{\mathcal{O}}_\Delta = \int_{\mathbb{R}} \mu dE_{\hat{\mathcal{O}}_\Delta}(\mu), \quad E_{\hat{\mathcal{O}}_\Delta}(\mu) E_{\hat{\mathcal{O}}_\Delta}(\lambda) = E_{\hat{\mathcal{O}}_\Delta}[\min(\mu, \lambda)].$$

Here $E_{\hat{\mathcal{O}}_\Delta}(\mu)$, known as projection measure, denotes the projection operator onto the closed subspace of $\mathcal{H}_\Delta$ where the expectation value of $\hat{\mathcal{O}}_\Delta$ is less than or equal to μ . Then the Gelfand-Maurin theorem (see Chap. 4 of [43]) assures the existence of the following derivative (almost everywhere) for any unit vector $|e\rangle \in \mathcal{H}_\Delta$ with spectral density σ_λ defined below,

$$\begin{aligned} |\lambda\rangle &= \frac{dE_{\hat{\mathcal{O}}_\Delta}(\lambda)}{d\sigma_\lambda} |e\rangle \in \Phi'_\Delta, \quad d\sigma_\lambda \equiv \langle e | dE_{\hat{\mathcal{O}}_\Delta} | e \rangle, \\ \text{s.t. } \forall |\psi\rangle \in \Phi_\Delta, \quad \langle \lambda | \psi \rangle &= \frac{d}{d\sigma_\lambda} \langle e | E_{\hat{\mathcal{O}}_\Delta}(\lambda) | \psi \rangle, \end{aligned} \quad (\text{A3})$$

and $|\lambda\rangle$ are the (generalized) eigenvectors of $\hat{\mathcal{O}}_\Delta$ in Φ'_Δ .

With subspaces $\mathcal{H}_\Delta(e) \equiv \text{span}(\{E_{\hat{\mathcal{O}}_\Delta}(\lambda)e\}) \subseteq \mathcal{H}_\Delta$, the Gelfand-Maurin theorem states that if the Hilbert space decomposes as an orthogonal sum $\mathcal{H}_\Delta = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{H}_\Delta(e_\alpha)$ for a set of vectors $\{e_\alpha\}_{\alpha \in \mathcal{A}}$, we can define a spectral measure $d\sigma_{\lambda,\alpha}$ and the following decomposition holds:

$$\hat{\mathcal{O}}_\Delta = \int_{\mathbb{R}} \lambda d\sigma_{\lambda,\alpha} |\alpha, \lambda, \Delta\rangle \langle \alpha, \lambda, \Delta|. \quad (\text{A4})$$

Physically, α keeps track of extra symmetry charges that commute with $\hat{\mathcal{O}}_\Delta$.

For the case of interest, $\hat{\mathcal{O}}_\Delta$ commutes with the momentum operators P_i and $\text{SO}(d-1)$ rotations, which do not mutually commute and we will choose to diagonalize P_i below [76]. The generic condition stated below (4) corresponds to assuming the existence of a cyclic vector $|e\rangle$ such that all linear combinations of the form

$$E_{\mathcal{O}_\Delta}(\lambda) \prod_{i=1}^{d-1} E_{P_i}(p_i) |e\rangle \quad (\text{A5})$$

span the Hilbert space $\mathcal{H}_\Delta$, where E_{P_i} denotes the projection measure for P_i ; we can then decompose $\hat{\mathcal{O}}_\Delta$ in a basis of simultaneous eigenstates of $\hat{\mathcal{O}}_\Delta$ and P_i as,

$$\begin{aligned} P_i |\lambda, \vec{p}, \Delta\rangle &= p_i |\lambda, \vec{p}, \Delta\rangle, \quad \hat{\mathcal{O}} |\lambda, \vec{p}, \Delta\rangle = \lambda |\lambda, \vec{p}, \Delta\rangle, \\ \hat{\mathcal{O}}_\Delta &= \int_{\mathbb{R}} \lambda d\sigma_\lambda \int \frac{d^{d-1} \vec{p}}{(2\pi)^{d-1}} |\vec{p}, \lambda\rangle \langle \vec{p}, \lambda|, \end{aligned} \quad (\text{A6})$$

This basis is complete, allowing us to express inner products of states in $\mathcal{H}$ explicitly as

$$\langle \psi | \chi \rangle = \int d\sigma_\lambda \int \frac{d^{d-1} \vec{p}}{(2\pi)^{d-1}} \langle \psi | \vec{p}, \lambda, \Delta \rangle \langle \vec{p}, \lambda, \Delta | \chi \rangle. \quad (\text{A7})$$

For convenience, we introduce Wannier wave functions [77],

$$|\lambda, \vec{x}, \Delta\rangle \equiv \int \frac{d^{d-1} \vec{p}}{(2\pi)^{d-1}} e^{-i\vec{p} \cdot \vec{x}} |\lambda, \vec{p}, \Delta\rangle. \quad (\text{A8})$$

In this representation, the operators D and P_i act as

$$\begin{aligned} D |\lambda, \vec{x}, \Delta\rangle &= (x^\mu \partial_\mu + (d-1-\Delta_O) \lambda \partial_\lambda) |\lambda, \vec{x}, \Delta\rangle, \\ P_i |\lambda, \vec{x}, \Delta\rangle &= \partial_i |\lambda, \vec{x}, \Delta\rangle. \end{aligned} \quad (\text{A9})$$

This allows us to determine the action of K_i from the conformal algebra,

$$K_j|\lambda, \vec{x}, \Delta\rangle = (2(d-1-\Delta_{\mathcal{O}})x_i\lambda\partial_i + (2x_i(x\cdot\partial) - x^2\partial_i))|\lambda, \vec{x}, \Delta\rangle. \quad (\text{A10})$$

Now we consider the dense subset $\mathcal{H}_{\mathcal{D}_\Delta} \subset \mathcal{H}_\Delta$, where for any $|f\rangle \in \mathcal{H}_{\mathcal{D}_\Delta}$ the following limit exists:

$$\exists \lim_{\lambda \rightarrow \infty} \langle \lambda, \vec{x}, \Delta | f \rangle \Rightarrow |\vec{x}, \Delta\rangle = \text{w-lim}_{\lambda \rightarrow \infty} |\lambda, \vec{x}, \Delta\rangle. \quad (\text{A11})$$

In general this weak limit might not exist or could simply be zero [44]. Physically, this amounts to demanding a well-defined IR limit of correlation functions of $\hat{\mathcal{O}}$ (and the defect) with bulk local operators. It is straightforward to check that the limit states $|\vec{x}, \Delta\rangle$ form a representation

of the algebra $\{P_i, D, K_j, M_{ij}\}$. For instance, for any state $|f\rangle \in \mathcal{H}_{\mathcal{D}_\Delta}$

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \langle f | e^{bD} | \lambda, \vec{x} \rangle &= \lim_{\lambda \rightarrow \infty} \langle f | \lambda e^{b(d-1-\Delta)}, e^b \vec{x} \rangle \\ &= \langle f | e^b \vec{x} \rangle \Rightarrow e^{bD} |\vec{x}\rangle = |e^b \vec{x}\rangle, \end{aligned} \quad (\text{A12})$$

where Δ dependence is implicit. We show that for all $\vec{x}$, the limit is the same in (A11),

$$|\vec{x}, \Delta\rangle = |\mathcal{B}, \Delta\rangle, \quad (\text{A13})$$

which is a linear combination of Ishibashi states corresponding to scalar primaries [44].