

Pion β Decay and $\tau \rightarrow \pi\pi\nu_\tau$ beyond Leading Logarithms

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The consistent matching of short-distance contributions and hadronic matrix elements is crucial for precise predictions of weak processes involving hadrons. In this Letter, we address this point for charged-current processes involving two pions—pion β decay $\pi^\pm \rightarrow \pi^0 e^\pm \nu_e$ and hadronic τ decays $\tau^\pm \rightarrow \pi^\pm \pi^0 \nu_\tau$ —whose decay rates depend on the so-called γW box correction. Using recent results from lattice QCD, we show how to formulate the matching beyond leading-logarithmic accuracy, in particular, how to cancel the dependence on the scheme choice for evanescent operators. As main results, we obtain a prediction for the decay rate of pion β decay with theory uncertainties improved by a factor of 3, which renders theory uncertainties negligible for future determinations of V_{ud} even beyond the reach of the PIONEER experiment, and an evaluation of isospin-breaking corrections to $\tau \rightarrow \pi\pi\nu_\tau$ with negligible uncertainty from the short-distance matching, as necessary for a future τ -based determination of the hadronic-vacuum-polarization contribution to the anomalous magnetic moment of the muon.

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Introduction—A precision determination of the Cabibbo-Kobayashi-Maskawa (CKM) [1,2] matrix element V_{ud} , either from neutron decay or superallowed nuclear β decays, crucially relies on a careful consideration of radiative corrections [3], which thus play a key role in the first-row unitarity test of the CKM matrix. Despite a long history in their evaluation [4–18], it has proven challenging to quantify and ideally remove the remaining model dependence in these corrections, and substantial effort has been invested in recent years to address this issue using dispersion relations, lattice QCD, effective field theory (EFT), and *ab initio* nuclear-structure methods [19–35]. Such robust uncertainty quantification is critical to either corroborate current hints for a unitarity deficit or impose stringent constraints on potential physics beyond the standard model (SM) [36–54].

Already in the single-nucleon system, this program requires the calculation of perturbative corrections and their resummation [26,31,35], but also the incorporation of nonperturbative matrix elements, most notably the so-called γW box corrections [17,19,20,23,55–59]. Extending previous analyses in the meson sector [60] to the nucleon case, in Ref. [26] it was shown how to formulate

the traditional master formula for the neutron decay rate based on an EFT approach, allowing for the consistent resummation of large logarithms. This approach requires the matching of low-energy EFT (LEFT) to chiral perturbation theory (ChPT), to separate low-energy constants (LECs) into short-distance components and nonperturbative matrix elements. As a key feature, the scheme dependence due to the choice of evanescent operators manifestly cancels between LEFT Wilson coefficients and the matrix elements, leading to a result for the decay rate in which the scheme dependence as well as the dependence on the LEFT scale μ and the chiral scale μ_χ cancel up to the order in the fine-structure constant α and the strong coupling α_s to which the perturbative calculation has been performed.

While not competitive at present, an alternative avenue toward the determination of V_{ud} proceeds via pion β decay, $\pi^\pm \rightarrow \pi^0 e^\pm \nu_e$ [61–63], which is sensitive to the same LEFT operator as the β decays involving nucleons, in such a way that the same challenges arise in its theoretical description. In this case, input from lattice QCD [64–66] is available to determine the nonperturbative parts, and given ongoing efforts to improve the experimental measurement of the branching fraction at PIONEER [67,68] it is timely to work out the matching beyond leading-logarithmic (LL) accuracy for this process as well, by adapting the methods from Refs. [26,60] accordingly.

Similarly, by crossing lepton and charged pion, it is evident that the exact same procedure also determines the

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short-distance contribution to hadronic τ decays $\tau^\pm \rightarrow \pi^\pm \pi^0 \nu_\tau$, a crucial ingredient for the evaluation of isospin-breaking (IB) corrections in the context of the hadronic-vacuum-polarization (HVP) contribution to the anomalous magnetic moment of the muon a_μ [69–78]. While a τ -based estimate is provided in Ref. [77], based on input from Refs. [73,75,79–81], several key short-comings need to be addressed to allow for a full uncertainty quantification of the IB corrections. One of them concerns the scheme dependence of the short-distance contribution, which boils down to the exact same LECs required for pion β decay, as a direct consequence of lepton flavor universality.

Accordingly, we first provide a general discussion of the pion matrix element, matching relations, and renormalization-group (RG) resummation, highlighting the universal form of the resulting correction to the decay rate, and then discuss the phenomenological applications to pion β decay and hadronic τ decays. To obtain a complete result for the decay rates, we also need a calculation of the corresponding long-range radiative corrections, which due to the small phase space for pion β decay can proceed in ChPT [62], while for τ decays a full calculation of the structure-dependent corrections has only recently become available [82,83].

Renormalization group and matrix elements—We use EFT methods to efficiently compute β decay rates in the SM beyond the LL approximation. Integrating out the weak gauge bosons, the Higgs boson, and the top quark, we write the pertinent part of the LEFT Lagrangian with n_f active quark flavors in the convention

$$\begin{aligned} \mathcal{L}_{\text{LEFT}} = & -2\sqrt{2}G_F \bar{e}_L \gamma_\rho \mu_L \bar{\nu}_\mu \gamma^\rho \nu_{eL} \\ & - 2\sqrt{2}G_F V_{ud} C_\beta^{(f)}(\mu, a) \bar{e}_L \gamma_\rho \nu_{eL} \bar{u}_L \gamma^\rho d_L + \text{H.c.}, \end{aligned} \quad (1)$$

i.e., the Fermi constant G_F is defined in muon decay [84], and a Wilson coefficient $C_\beta^{(f)}(\mu, a) = 1 + \mathcal{O}(\alpha)$ describes the semi-leptonic operator in the n_f -flavor theory. It depends on the LEFT scale μ in the $\overline{\text{MS}}$ scheme as well as the scheme chosen for the evanescent operator, defined by $E(a)$

$$\gamma^\alpha \gamma^\rho \gamma^\beta P_L \otimes \gamma_\alpha \gamma_\rho \gamma_\beta P_L = 4(4 - a\epsilon) \gamma^\rho P_L \otimes \gamma_\rho P_L + E(a), \quad (2)$$

where $d = 4 - 2\epsilon$ is the dimensional regulator, and naive dimensional regularization for γ_5 is used. (We use here the evanescent scheme parameter a as defined in Ref. [35], which is related to the conventions of Ref. [26] (CDMT) by $a = 2a^{\text{CDMT}} + 3$.) In these conventions one finds for the Wilson coefficient in the five-flavor theory at $\mu \simeq M_{W,Z}$ [10,26,35,85–87]

$$\begin{aligned} C_\beta^{(5)}(\mu, a) = & 1 + \frac{\alpha}{\pi} \log \frac{M_Z}{\mu} + \frac{\alpha}{4\pi} B(a) - \frac{\alpha\alpha_s}{4\pi^2} \log \frac{M_Z}{\mu} \\ & + \frac{\alpha\alpha_s}{(4\pi)^2} B_s(a) - \frac{\alpha\alpha_s}{4\pi^2} \frac{1}{s_W^2} \left(\frac{c_W^2}{s_W^2} \log \frac{M_W}{M_Z} + 1 \right) \\ & + \mathcal{O}(\alpha^2, \alpha\alpha_s^2), \end{aligned} \quad (3)$$

where $B(a) = (a/3) - 4$ and $B_s(a) = (41/6) - (14/9)a$ parametrize the scheme dependence, and s_W , c_W are the sine and cosine of the weak mixing angle. The effective coupling $C_\beta^{(f)}(\mu, a)$ can be evolved down to the hadronic scale using the RG equations (RGE). The solution is known to next-to-leading-logarithmic (NLL) $\{\mathcal{O}[\alpha^{n+1} \log^n(M_Z/\mu)]\}$ [26] and NLL_s $\{\mathcal{O}[\alpha\alpha_s^{n+1} \log^n(M_Z/\mu)]\}$ [35] accuracy [88]. The final result for the decay amplitude is written in the three-flavor theory in terms of a scheme-independent Wilson coefficient $\bar{C}_\beta^{(f)}(\mu) \equiv C_\beta^{(f)}(\mu, a)/J_f(\mu, a)$ defined in terms of the quantity $J_f(\mu, a)$ that naturally appears in the solution of the RGE to NLL [88].

Matching to ChPT—Starting from Eq. (1), to achieve NLL (and NLL_s) accuracy one needs to compute the matrix element of $O_\beta \equiv \bar{e}_L \gamma_\rho \nu_{eL} \bar{u}_L \gamma^\rho d_L$ between the appropriate external states to $\mathcal{O}(\alpha)$ [and $\mathcal{O}(\alpha\alpha_s)$], including finite, nonlogarithmically enhanced terms, whose scheme dependence will cancel the one in $C_\beta^{(f)}(\mu, a)$. In the EFT approach, one computes $\langle f | O_\beta | i \rangle$ in ChPT, the low-energy EFT in which the degrees of freedom are mesons and baryons, along with light particles involved in the electroweak interactions (photons, charged leptons, and neutrinos). Through tree-level and loop amplitudes, ChPT captures the infrared (IR) physics to a given order in the low-energy expansion, while encoding the short-distance and nonperturbative effects of hard virtual photons in appropriate LECs. We next provide a representation of the LECs contributing to pion β decay and $\tau \rightarrow \pi \pi \nu_\tau$, capturing the short-distance contributions to NLL and NLL_s accuracy as well as nonperturbative effects originating at the hadronic scale. Next, we present the main results, while more details of the derivation are discussed in the End Matter.

In the spirit of EFT, we factorize the dependence on the LEFT Wilson coefficient from the chiral LECs [26], as a result of which these LECs carry a dependence on both scales μ and μ_χ , and on the evanescent scheme parameter a . The relevant linear combination of the X_i [113] and K_i [114] reads,

$$X_\ell(\mu_\chi, \mu, a) \equiv \left(X_6^r - 4K_{12}^r + \frac{4}{3}X_1^r \right)(\mu_\chi, \mu, a). \quad (4)$$

Matching LEFT with three quark flavors and ChPT, and using the results from Refs. [26,35] for the $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha\alpha_s)$ terms, respectively, we find

$$\begin{aligned}
 -2\pi\alpha_\chi(\mu_\chi)X_\ell(\mu_\chi, \mu, a) &\equiv \bar{\square}_\pi^{V(3)}(\mu_0) - \frac{\alpha^{(3)}(\mu)}{4\pi} \left(\frac{5}{4} + \frac{3}{2} \log \frac{\mu_\chi^2}{\mu_0^2} \right. \\
 &\quad \left. + B(a) + 2 \log \frac{\mu_0^2}{\mu^2} \right) + \frac{\alpha\alpha_s^{(3)}(\mu)}{(4\pi)^2} \\
 &\quad \times \left(-B_s(a) + 2 - 2 \log \frac{\mu^2}{\mu_0^2} \right), \quad (5)
 \end{aligned}$$

where $\bar{\square}_\pi^{V(3)}(\mu_0)$ represents the nonperturbative matrix element, $\alpha_\chi(\mu_\chi)$ denotes the fine-structure constant in ChPT [26], while the couplings on the right-hand side refer to $\overline{\text{MS}}$. [The scale in $\alpha_s^{(3)}$ is chosen following Ref. [35]. Replacing $\alpha_s^{(3)}(\mu) \rightarrow \alpha_s^{(3)}(\mu_0)$ would ensure exact analytic cancellation of μ_0 [26].] Combining Wilson coefficient and matrix element, we define the scheme-independent short-distance correction to the vector (Fermi) transitions

$$\begin{aligned}
 g_V^\pi(\mu_\chi) &\equiv 1 + \Delta\tilde{S}_{\text{EW}}(\mu_\chi) \\
 &\equiv \tilde{C}_\beta^{(3)}(\mu) \left[1 + \bar{\square}_\pi^{V(3)}(\mu_0) \right. \\
 &\quad \left. + \frac{\alpha^{(3)}(\mu)}{4\pi} \left(\frac{17}{6} - \frac{3}{2} \log \frac{\mu_\chi^2}{\mu_0^2} + 2 \log \frac{\mu^2}{\mu_0^2} \right) \right. \\
 &\quad \left. + \frac{\alpha\alpha_s^{(3)}(\mu)}{(4\pi)^2} \left(-\frac{130}{81} - 2 \log \frac{\mu^2}{\mu_0^2} \right) \right]. \quad (6)
 \end{aligned}$$

[The usual S_{EW} correction corresponds to $1 + 2\Delta\tilde{S}_{\text{EW}}(\mu_\chi)$. The tilde is introduced to indicate that now we include nonperturbative contributions.] With appropriate replacement of the external states in the transition-dependent nonperturbative matrix element $\bar{\square}_\pi^V(\mu_0)$, this correction universally applies to the Fermi component of all β decays. The dependence on μ_0 cancels between $\bar{\square}_\pi^V(\mu_0)$ and the explicit logarithms, up to terms beyond NLL_s accuracy; the dependence on μ cancels between the scheme-independent Wilson coefficient $\tilde{C}_\beta^{(3)}(\mu)$ [88] and the explicit logarithms; finally, the dependence on μ_χ cancels once one combines $g_V^\pi(\mu_\chi)$ with the chiral loops. Beyond one-loop order, RG corrections need to be included; see End Matter for details.

Nonperturbative matrix element—Following Ref. [26] and taking into account the factorization of short-distance QCD corrections from Ref. [35], the nonperturbative input is collected in

$$\begin{aligned}
 \bar{\square}_\pi^V(\mu_0) &= -ie^2 \int \frac{d^4q}{(2\pi)^4} \frac{\nu^2 + Q^2}{Q^4} \\
 &\quad \times \left[\frac{T_3^\pi(\nu, Q^2)}{2M_\pi\nu} - \frac{2}{3} \frac{1 - \frac{\alpha_s(\mu_0)}{\pi}}{Q^2 + \mu_0^2} \right], \quad (7)
 \end{aligned}$$

where μ_0 is a factorization scale that allows one to subtract the asymptotic part of the VA correlator, known from the operator product expansion (OPE). Its scalar projection is

written as in Ref. [26], translating to the conventions of Ref. [64] via

$$\frac{1}{\sqrt{2}} T_3(-i\nu, Q^2) = \frac{T_3^\pi(\nu, Q^2)}{4M_\pi\nu}, \quad (8)$$

where $\nu = q_0$, $Q^2 = -q^2$. Accordingly, we can express $\bar{\square}_\pi^V(\mu_0)$ in terms of the integrand $M_\pi(Q^2)$ defined in Ref. [64]:

$$\bar{\square}_\pi^V(\mu_0) = \frac{3\alpha}{2\pi} \int_0^\infty \frac{dQ^2}{Q^2} \left[M_\pi(Q^2) - \frac{1}{12} \frac{Q^2(1 - \frac{\alpha_s(\mu_0)}{\pi})}{Q^2 + \mu_0^2} \right]. \quad (9)$$

For sufficiently small values of $Q^2 \leq Q_0^2$, $M_\pi(Q^2)$ is known from lattice QCD [64,66], while asymptotically its form is determined by perturbative QCD (pQCD),

$$\begin{aligned}
 M_\pi(Q^2) &= \frac{C_d(Q^2)}{12}, \\
 C_d(Q^2) &= 1 - \frac{\alpha_s}{\pi} + \sum_{n \geq 2} c_n \left(\frac{\alpha_s}{\pi} \right)^n, \quad (10)
 \end{aligned}$$

with coefficients c_n known up to $n = 4$ [115,116]. The subtraction in Eq. (9), however, is only accurate at $\mathcal{O}(\alpha_s)$, so that, asymptotically, one needs to assume this form in $M_\pi(Q^2)$ starting at some value Q_1^2 . Accordingly, Eq. (9) takes the form

$$\bar{\square}_\pi^V(\mu_0) = \frac{3\alpha}{2\pi} \int_0^{Q_1^2} \frac{dQ^2}{Q^2} M_\pi(Q^2) + \frac{\alpha(1 - \frac{\alpha_s(\mu_0)}{\pi})}{8\pi} \log \frac{\mu_0^2}{Q_1^2}. \quad (11)$$

To improve the matching between lattice QCD and the $\mathcal{O}(\alpha_s)$ pQCD asymptotics, we take Q_1 larger than Q_0 and use the full four-loop pQCD result for $Q_0 \leq Q \leq Q_1$. The subtraction scale μ_0 is identified with Q_0 , while Q_1 is determined as the point at which $C_d(Q_1^2) = 1 - [\alpha_s(\mu_0)/\pi]$ to ensure a smooth transition to the asymptotic form. We observe that the convergence of the pQCD expansion prefers a slightly higher matching point than $Q_0^2 = 2 \text{ GeV}^2$ [64], at the expense of slightly larger uncertainties from lattice artifacts. We therefore choose $\mu_0 = Q_0 = 1.6 \text{ GeV}$, and evolve α_s down from $\alpha_s^{(5)}(M_Z)$ [117,118], at the loop order that corresponds to the accuracy of $C_d(Q^2)$ used. Given that the calculations in Refs. [64,66] are performed with $2 + 1$ and $2 + 1 + 1$ active quark flavors, respectively, we repeat the analysis for $n_f = 3$ and $n_f = 4$, leading to $Q_1^{(3)} = 4.46 \text{ GeV}$ and $Q_1^{(4)} = 4.53 \text{ GeV}$. Altogether, we obtain for Refs. [64,66], respectively

$$\begin{aligned}
 10^3 \times \bar{\square}_\pi^{V(3)}(\mu_0) &= 0.714(12)(28) - 0.023(6) \\
 &= 0.691(31), \\
 10^3 \times \bar{\square}_\pi^{V(4)}(\mu_0) &= 0.709(27) - 0.020(4) \\
 &= 0.690(28),
 \end{aligned} \tag{12}$$

where the two terms refer to the contribution of the integral below Q_0 and the remainder, respectively. For the latter, the uncertainty derives from the difference to the three-loop pQCD result, while for the integral the uncertainties are propagated from the lattice-QCD calculations. In the case of Ref. [64], this error is split into statistical and systematic components, where the dominant systematic effect concerns lattice artifacts, estimated as the difference of the two discretizations considered. (We define the central value for Ref. [64] by the average of the two discretizations therein. The systematic error given in Ref. [64] also accounts for higher-twist effects, which are found to be subleading compared to the discretization uncertainties. For increased Q_0 , the impact of such power corrections should be reduced further.) The corrections in Eq. (12) that arise from the improved matching between the lattice-QCD results and the $\mathcal{O}(\alpha_s)$ pQCD asymptotics are of the same size as the lattice-QCD errors, and thus control over the $\mathcal{O}(\alpha_s^2)$ terms in the OPE subtraction in Eq. (9) would be required to further improve the precision. The resulting integrand $M_\pi(Q^2)$ and its subtracted form according to Eq. (9) are shown in Fig. 1.

Pion β decay—The master formula for pion β decay reads [62,63],

$$\begin{aligned}
 \Gamma[\pi^+ \rightarrow \pi^0 e^+ \nu_e(\gamma)] &= \frac{G_F^2 |V_{ud}|^2 M_\pi^5 |f_+^\pi(0)|^2}{64\pi^3} \\
 &\times (1 + \Delta_{\text{RC}}^{\pi\ell}) I_{\pi\ell},
 \end{aligned} \tag{13}$$

where M_π denotes the mass of the charged pion. [G_F can be identified with its value extracted from muon decay $G_F = 1.1663785(6) \times 10^{-5} \text{ GeV}^{-2}$ [84,119], since the only process-dependent corrections not accounted for via the conventions (1) concern mass corrections to the W propagator $2\Delta G_F^W \simeq 3m_\mu^2/(5M_W^2) \simeq 1 \times 10^{-6}$ [120,121].] The form factor $f_+^\pi(0) \simeq 1-7 \times 10^{-6}$ deviates from unity by a negligible amount due to the Behrends-Sirlin-Ademollo-Gatto theorem [122–124], which ensures that IB corrections can only enter quadratically. The phase-space factor evaluates to

$$I_{\pi\ell} = 7.3767(41) \times 10^{-8}, \tag{14}$$

where the uncertainty derives from the pion mass difference, $M_\pi - M_{\pi^0} = 4.59364(48) \text{ MeV}$ [125]. [This value includes corrections from the slope parameter $\lambda_+ = 0.036(3)$ [124], with uncertainties derived from higher-order effects in the LEC $L_6^r(M_\rho) = 6.6(6) \times 10^{-3}$ [126–128].] Radiative corrections are included in $\Delta_{\text{RC}}^{\pi\ell}$, which decomposes as

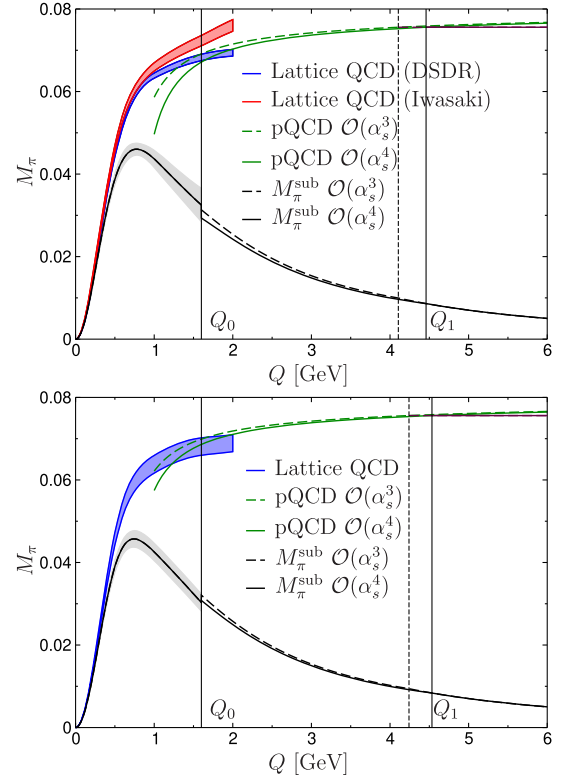


FIG. 1. Resulting integrand $M_\pi(Q^2)$ for the lattice-QCD inputs from Refs. [64] (upper, $n_f = 3$) and [66] (lower, $n_f = 4$). In both cases, we show the lattice-QCD results up to 2 GeV, the pQCD curves above 1 GeV (green dashed and solid for three- and four-loop order), and the constant one-loop result above the respective values of Q_1 (maroon dashed and solid). The resulting subtracted integrand as it enters in Eq. (9) is represented by the black curves (dashed and solid for three- and four-loop pQCD between Q_0 and Q_1). The dashed vertical line indicates the value of Q_1 for the three-loop case.

$$\frac{1 + \Delta_{\text{RC}}^{\pi\ell}}{[1 + \Delta_{\text{EW}}^{\pi\ell}(\mu_\chi)]^2} = 1 + \frac{3\alpha_\chi(\mu_\chi)}{2\pi} \log \frac{\mu_\chi}{2E_0} + \Delta I_{\pi\ell}, \tag{15}$$

where $\Delta I_{\pi\ell} = -0.00378(8)$ [62], including an estimate of higher chiral orders $\simeq (\alpha/\pi)(M_\pi/M_\rho)^2 \simeq 7.5 \times 10^{-5}$. In this form, we isolated the only large logarithm that remains in the phase-space correction $\Delta I_{\pi\ell}$ by virtue of the Kinoshita-Lee-Nauenberg theorem [129–131], sensitive to the endpoint energy $E_0 = (M_\pi^2 - M_{\pi^0}^2 + m_e^2)/(2M_\pi) = 4.51894(48) \text{ MeV}$. While we use the full relativistic calculation from Ref. [62], the value of $\Delta I_{\pi\ell}$ can be understood from the integral over the Sirlin function; see End Matter. Using Eq. (6), we find for the effective $\pi\pi$ vector coupling

$$g_V^\pi(M_\rho) = 1.01084(2) \bar{\square}_\pi(3)_\mu [4]_{\text{tot}}, \tag{16}$$

where the first uncertainty is propagated from Eq. (12) and the second one indicates the residual perturbative uncertainty estimated by varying the decoupling and matching scales as in Ref. [35]. [For Ref. [66], we use the four-flavor

analog of Eq. (6) [88], yielding a $g_V^\pi(M_\rho)$ that only differs by 3×10^{-6} from the three-flavor result derived from Ref. [64]. Accordingly, we quote the average as our final value.] For the radiative-correction factor $\Delta_{\text{RC}}^{\pi\ell}$ we further need to consider the evolution to $\mu_\chi \simeq 2E_0$, which gives

$$\Delta_{\text{RC}}^{\pi\ell} = 0.03403(4)_{\overline{\text{MS}}}(5)_{\mu}(5)_{\mu_\chi}(8)_{\text{ChPT}}[11]_{\text{tot}}, \quad (17)$$

where the uncertainty denoted by μ_χ comprises a variation of μ_χ within $[2E_0/\sqrt{2}, \sqrt{2}2E_0]$ and of the matching scale within $[M_\rho/\sqrt{2}, \sqrt{2}M_\rho]$, dominated by the latter.

Our result (17) differs substantially from $\Delta_{\text{RC}}^{\pi\ell}|_{[64]} = 0.0332(1)_{\overline{\text{MS}}}(3)_{\text{QED}}$, primarily due to the low-energy RG corrections and the quadratic terms in Eq. (15). In a fully linearized form, the direct application of Eq. (16) would give $\Delta_{\text{RC}}^{\pi\ell} = 0.0335$, which then increases by

$$\begin{aligned} \Delta_{\text{RC}}^{\pi\ell}|_{\text{LL+quad}} &\simeq \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{17}{16} \log^2 \frac{M_\rho}{2E_0} \right. \\ &\quad \left. + \left(\log \frac{M_Z}{M_\rho} + \frac{3}{4} \log \frac{M_\rho}{2E_0} \right)^2 \right] \simeq 4.7 \times 10^{-4} \end{aligned} \quad (18)$$

due to LL and quadratic terms. The final result in Eq. (17) is obtained from the full RG solution, including another $\simeq 0.6 \times 10^{-4}$ from NLL.

Our improved evaluation of $\Delta_{\text{RC}}^{\pi\ell}$ given in Eq. (17) implies

$$V_{ud}^\pi = 0.97346(281)_{\text{Br}}(9)_{\tau_\pi}(5)_{\Delta_{\text{RC}}^{\pi\ell}}(27)_{I_{\pi\ell}}[283]_{\text{tot}}, \quad (19)$$

where the uncertainties refer to the branching fraction $\text{Br}[\pi^\pm \rightarrow \pi^0 e^\pm \nu_e(\gamma)] = 1.038(6) \times 10^{-8}$ [61] [slightly adjusted to the current normalization from $\text{Br}[\pi^\pm \rightarrow e^\pm \nu_e(\gamma)] = 1.2326(23) \times 10^{-4}$ [119,132–134]], the pion lifetime τ_π [119,135,136], $\Delta_{\text{RC}}^{\pi\ell}$, and the pion mass difference (14), respectively. Accordingly, the theory uncertainty in the extraction of V_{ud} due to $\Delta_{\text{RC}}^{\pi\ell}$ is reduced by a factor of three, while all other sources of uncertainty are of experimental origin. The PIONEER experiment aims at improving the uncertainty in the branching fraction by an order of magnitude, at which level the uncertainty in the pion mass difference becomes the limiting factor.

$\tau \rightarrow \pi\pi\nu_\tau$ —The photon-inclusive $\tau \rightarrow \pi\pi\nu_\tau$ decay rate is expressed as

$$\frac{d\Gamma[\tau^\pm \rightarrow \pi^\pm \pi^0 \nu_\tau(\gamma)]}{K_\Gamma(s)ds} = [\beta_{\pi\pi^0}(s)]^3 |f_+(s)|^2 S_{\text{EW}}^{\pi\pi} G_{\text{EM}}(s), \quad (20)$$

where

$$K_\Gamma(s) = \frac{\Gamma_e |V_{ud}|^2}{2m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right), \quad (21)$$

$\Gamma_e \equiv \Gamma[\tau \rightarrow e\nu_\tau \bar{\nu}_e]$, $\beta_{\pi\pi^0}$ denotes the phase-space factor

$$\begin{aligned} \beta_{\pi\pi^0}(s) &= \lambda^{1/2} \left(1, \frac{M_\pi^2}{s}, \frac{M_{\pi^0}^2}{s}\right), \\ \lambda(a, b, c) &= a^2 + b^2 + c^2 - 2(ab + ac + bc), \end{aligned} \quad (22)$$

and $S_{\text{EW}}^{\pi\pi}$ [10,137–142] includes short-distance effects. The rate is normalized to the leptonic decay width Γ_e , which requires that the respective radiative corrections $\Delta S_{\text{EW}}^e = (\alpha/2\pi)[(25/4) - \pi^2]$ that connect it to G_F be removed:

$$S_{\text{EW}}^{\pi\pi} = [1 + \Delta \tilde{S}_{\text{EW}}(\mu_\chi)]^2 - \Delta S_{\text{EW}}^e. \quad (23)$$

Besides providing a convenient normalization channel, this convention also avoids mass corrections $\simeq m_\tau^2/M_W^2 \simeq 5 \times 10^{-4}$ [120,121], as would arise when using G_F from muon decay instead. The chiral LECs are contained in $G_{\text{EM}}(s)$, depending linearly on $X_\ell(\mu_\chi)$ [83]:

$$G_{\text{EM}}(s)|_{X_\ell(\mu_\chi)=0} = G_{\text{EM}}(s)|_{X_\ell(\mu_\chi)=\bar{X}_\ell(\mu_\chi)} + e^2 \bar{X}_\ell(\mu_\chi). \quad (24)$$

The analysis of $G_{\text{EM}}(s)$ in Refs. [82,83] is performed for the reference value $\bar{X}_\ell(M_\rho) = 14 \times 10^{-3}$ [65], which can be easily adjusted to an arbitrary value by means of Eq. (24). It is evident that only the product $S_{\text{EW}}^{\pi\pi} G_{\text{EM}}(s)$ is scale and scheme independent, and in the following we will therefore only consider the combined effect. Since both $S_{\text{EW}}^{\pi\pi}$ and the relevant terms in $G_{\text{EM}}(s)$ do not depend on the $\pi\pi$ invariant mass s , the respective IB corrections for the HVP contribution to a_μ can simply be obtained by scaling $\Delta \bar{a}_\mu[S_{\text{EW}}^{\pi\pi}] = -12.166(56)(8) \times 10^{-10}$ accordingly (quantities taken from Refs. [82,83] are indicated by a bar), so that the analysis of Refs. [82,83] can be updated via

$$\begin{aligned} \Delta a_\mu[\pi\pi, \tau] - \Delta \bar{a}_\mu[\pi\pi, \tau] &= \left[\frac{\Delta S_{\text{EW}}^{\pi\pi} - \Delta \bar{G}_{\text{EM}}}{\Delta S_{\text{EW}}^{\pi\pi}} - 1 \right] \Delta \bar{a}_\mu[S_{\text{EW}}^{\pi\pi}] \\ &= -0.07(4) \times 10^{-10}, \end{aligned} \quad (25)$$

where $\Delta S_{\text{EW}}^{\pi\pi} = S_{\text{EW}}^{\pi\pi} - 1$ from Eq. (23), $\Delta \bar{S}_{\text{EW}}^{\pi\pi} = 0.0233$, and

$$\Delta \bar{G}_{\text{EM}} = -e^2 \left(\bar{X}_\ell(M_\rho) - \frac{1}{4\pi^2} \log \frac{m_\tau^2}{M_\rho^2} \right). \quad (26)$$

Including phase-space corrections in addition to $S_{\text{EW}}^{\pi\pi}$ and $G_{\text{EM}}(s)$, we obtain

$$\Delta a_\mu[\pi\pi, \tau] = -24.9(1)_{\text{exp}}(5)_{\text{th}}(1)_{\text{SD}} \times 10^{-10}, \quad (27)$$

with central value almost identical to Refs. [82,83], but the uncertainty from the short-distance matching, previously

estimated as 1.3×10^{-10} [77], has become negligible. Accordingly, the uncertainty of isospin breaking in the matrix elements, $|F_\pi^V/f_+|$, estimated as 4.7×10^{-10} in Ref. [77], now constitutes by far the dominant limitation for the use of hadronic τ data in the evaluation of the HVP contribution to a_μ .

Conclusions—In this Letter, we presented the matching of short-distance contributions and hadronic matrix elements for pion β decay and $\tau \rightarrow \pi\pi\nu_\tau$ at NLL accuracy. To this end, we first derived the required matching relations between ChPT and LEFT, both with the spurion method, which yields explicit expressions for the relevant LECs, and based on general properties of the amplitude related to Ward identities, current algebra, and the IR structure, which confirms the spurion analysis and reveals a remarkable universality feature of the matching relation.

For the phenomenological analysis, we combined our matching relations with recent results for the hadronic matrix elements in lattice QCD and NLL corrections in the strong coupling. As key results, we substantially improved the radiative corrections to pion β decay and the short-distance contributions to $\tau \rightarrow \pi\pi\nu_\tau$. The remaining theory uncertainty in V_{ud} extracted from pion β decay now lies a factor of five below the precision goal of the PIONEER experiment, rendering the phase-space factor and thus the pion mass difference the limiting factor besides the measurement of the branching fraction. For $\tau \rightarrow \pi\pi\nu_\tau$, the main application concerns IB corrections for the HVP contribution to a_μ , in which case the remaining uncertainties from the short-distance matching are now negligible.

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End Matter

Derivation of the matching conditions and universality—We arrived at the matching conditions (5) and (6) between LEFT and ChPT in two ways: first, following Refs. [26,60] we performed the matching at the level of Green's functions obtained by differentiation of the generating functional with respect to charge and weak spurions. The results could not be immediately read off of Ref. [60], because for the LEFT a Pauli-Villars regulator with on-shell subtraction scheme was used, while we work in dimensional regularization and $\overline{\text{MS}}$ subtraction scheme (to resum logarithms to NLL_s

accuracy). Following the logic of Ref. [60], we arrived at an expression for $X_\ell(\mu_\chi, \mu, a)$ in terms of the correlation function $\int dx e^{ik \cdot x} \langle 0 | T \{ V_\mu^a(x) V_\nu^b(0) \} | \pi^c \rangle$ [88], where a, b, c are SU(3) flavor indices. Finally, using a soft-pion theorem we related this matrix element to the object appearing in the γW box, namely, $\int dx e^{ik \cdot x} \langle \pi^d | T \{ V_\mu^a(x) A_\nu^b(0) \} | \pi^c \rangle$, up to higher-order chiral corrections. One of the advantages of the spurion method is that it allows one to derive representations for any individual LEC, and hence the combination X_ℓ .

Second, we matched physical amplitudes for $\pi^+ \rightarrow \pi^0 e^+ \nu_e$ in LEFT and ChPT [88]. The key point is that the $\mathcal{O}(\alpha)$ term in the matrix element $\langle f | O_\beta | i \rangle$ in LEFT can be written in terms of convolutions of known kernels with nonperturbative matrix elements of weak and electromagnetic quark currents between hadronic states. Current algebra relates three-point and two-point correlation functions [7,9]. Inspection of the IR behavior of these convolution integrals [5,7] allows one to identify the parts of the LEFT amplitude that are contained in the ChPT loops, leaving the rest as contributions to the LECs. While this analysis gives us access only to the linear combination of LECs contributing to the decay rate, it reveals universal features of the matching: Eq. (6) formally applies to the Fermi component of any β decay, provided one computes the T_3 form factor appearing in $\bar{\square}^V(\mu_0)$, as defined in Eq. (7), in the appropriate initial and final state. Up to effects suppressed by E_e/M_{had} , m_e/M_{had} , where M_{had} denotes the mass of the decaying hadronic state and m_e , E_e mass and energy of the electron, the long-distance corrections are also universal (independent of the mass and spin of the hadrons involved in the decay) and provided by the so-called Sirlin function [5], up to a constant that depends on the choice of ultraviolet (UV) regulator and is fixed by the matching procedure. Consistently with the explicit calculation of the neutron decay [26,143] we find for the pion β decay

$$\begin{aligned} \frac{d\Gamma}{d\Gamma(0)} &= [g_V^\pi(\mu_\chi)]^2 \left[1 + \frac{\alpha}{2\pi} \left(\frac{5}{4} + 3 \log \frac{\mu_\chi}{m_e} + \hat{g}(E_e, E_0) \right) \right] \\ &= 1 + 2\bar{\square}_\pi^V(\mu_0) + \frac{\alpha}{2\pi} \left(4 \left(1 - \frac{\alpha_s(\mu_0)}{4\pi} \right) \log \frac{M_Z}{\mu_0} \right. \\ &\quad \left. + 3 \log \frac{\mu_0}{m_e} + \hat{g}(E_e, E_0) \right) + \mathcal{O}(\alpha^2, \alpha\alpha_s), \end{aligned} \quad (\text{A1})$$

where in the second line we have provided the fixed-order expression (up to $\mathcal{O}(\alpha\alpha_s)$ terms that are not logarithmically enhanced) and $\hat{g}(E_e, E_0)$ is the “subtracted” Sirlin function [5], from which the large logarithm of M_{had}/m_e has been subtracted:

$$\begin{aligned} \hat{g}(E_e, E_0) &= -\frac{3}{4} + \left[\frac{1+\beta^2}{\beta} + \frac{1}{12\beta} \left(\frac{\bar{E}}{E_e} \right)^2 \right] \log \frac{1+\beta}{1-\beta} \\ &\quad + 4 \left[\frac{1}{2\beta} \log \frac{1+\beta}{1-\beta} - 1 \right] \left[\log \frac{2\bar{E}}{m_e} - \frac{3}{2} + \frac{\bar{E}}{3E_e} \right] \\ &\quad + \frac{1}{\beta} \left[-4\text{Li}_2 \left(\frac{2\beta}{1+\beta} \right) - \log^2 \left(\frac{1+\beta}{1-\beta} \right) \right]. \end{aligned} \quad (\text{A2})$$

Here $\beta = |\mathbf{p}_e|/E_e$ and $\bar{E} = E_0 - E_e$, with the electron endpoint energy E_0 , which, to this order, is the only source through which the hadron mass enters the radiative corrections. The ChPT results [62] capture additional M_π -dependent nonuniversal corrections, but the integral over the Sirlin function,

$$\begin{aligned} \Delta I_{\pi\ell} &\simeq \frac{\alpha}{2\pi} \left[\frac{5}{4} + 3 \log \frac{2E_0}{m_e} + \frac{64}{M_\pi^5 I_{\pi\ell}} \int_{m_e}^{E_0} dE_e E_e^2 \bar{E}^2 \beta \hat{g}(E_e, E_0) \right] \\ &\simeq \frac{\alpha}{\pi} \left(\frac{187}{40} - \frac{2\pi^2}{3} \right) \simeq -0.0044, \end{aligned} \quad (\text{A3})$$

explains the main features of the phase-space correction $\Delta I_{\pi\ell}$. In particular, the last line, applicable in the limit $m_e \rightarrow 0$ and neglecting recoil corrections, demonstrates that with the choice $\log(\mu_\chi/2E_0)$ in Eq. (15) indeed all logarithms disappear in $\Delta I_{\pi\ell}$ [129].

Low-energy RG—The low-energy RG for $X_\ell(\mu_\chi, \mu, a)$ needs to be formulated at the level of $g_V^\pi(\mu_\chi)$ to account for the evolution of $\alpha_\chi(\mu_\chi)$, leading to

$$\begin{aligned} \mu_\chi \frac{dg_V^\pi(\mu_\chi)}{d\mu_\chi} &= \gamma(\alpha_\chi) g_V^\pi(\mu_\chi), \\ \gamma(\alpha_\chi) &= \tilde{\gamma}_0 \frac{\alpha_\chi}{4\pi} + \tilde{\gamma}_1 \frac{\alpha_\chi^2}{(4\pi)^2} + \mathcal{O}(\alpha_\chi^3), \\ \tilde{\gamma}_0 &= -3, \quad \tilde{\gamma}_1 = \frac{10\tilde{n}}{3} + \frac{5}{2} - \frac{8\pi^2}{3}, \end{aligned} \quad (\text{B1})$$

where

$$\tilde{n}(\mu_\chi) = \sum_{\ell=e,\mu} Q_\ell^2 n_\ell \theta(\mu_\chi - m_\ell) + \frac{1}{4} \sum_\pi Q_\pi^2 n_\pi \theta(\mu_\chi - M_\pi) \quad (\text{B2})$$

counts the number of active spin-1/2 (ℓ) and spin-0 (π) degrees of freedom weighted by their charges $Q_{\ell,\pi}$, and $\alpha_\chi(\mu_\chi)$ is evaluated as in Ref. [26]. To obtain the $\mathcal{O}(\alpha_\chi^2)$ term in Eq. (B1) we used universality of the corresponding anomalous dimension [29,144], which allows us to also resum NLL in the low-energy theory. At one-loop order the dependence of α_χ on μ_χ can be ignored, and $\tilde{\gamma}_0$ reproduces the RG relation obtained using ChPT beta functions [113,114]

$$\mu_\chi \frac{\partial}{\partial \mu_\chi} X_\ell(\mu_\chi, \mu, a) = \frac{3}{8\pi^2}. \quad (\text{B3})$$