

Dynamic Focusing to Suppress Emittance Transfer in Crab-Crossing Flat Beam Collisions

Derong Xu^{1,*}, J. Scott Berg¹, Michael M. Blaskiewicz¹, Yue Hao², Yun Luo¹, Christoph Montag¹, Sergei Nagaitsev¹, Boris Podobedov¹, Vadim Ptitsyn¹, Ferdinand Willeke¹, and Binping Xiao¹

¹Brookhaven National Laboratory, Upton, New York, USA

²Michigan State University, East Lansing, Michigan, USA



(Received 27 June 2025; revised 4 December 2025; accepted 11 February 2026; published 1 April 2026)

Flat hadron beam collisions, though expected to enhance peak luminosity by about an order of magnitude, have not yet been demonstrated. We identify a new mechanism in which the hourglass-induced synchrotron resonance, when coupled to unavoidable machine fluctuations, drives emittance transfer and degrades the flat-beam condition. This effect becomes a critical limitation for maintaining flat hadron beams on operational timescales. Using Electron-Ion Collider (EIC) parameters, we show that it imposes stringent tolerances on machine noise. We further demonstrate that a dynamic focusing scheme, implemented by combining sextupoles with crab cavities, suppresses the resonance and relaxes these constraints, providing a practical path toward robust flat-beam collisions in future lepton-hadron colliders.

DOI: [10.1103/PhysRevLett.136.135001](https://doi.org/10.1103/PhysRevLett.136.135001)

Electron-Ion Colliders (EICs) are powerful tools for probing nuclear structure and testing quantum chromodynamics (QCD) [1]. After the success of HERA [2], which pioneered high-energy electron-proton collisions and deep inelastic scattering [3], the next-generation EIC at Brookhaven National Laboratory aims to deliver high-luminosity, polarized lepton-hadron collisions with broad energy tunability [4]. Similar proposals, including the LHeC at CERN [5] and the EicC in China [6], reflect a global drive to push the QCD frontier. To reach high luminosity, the EIC adopts a flat hadron beam and a crab crossing scheme. These features introduce new beam dynamics challenges and motivate this Letter.

A large crossing angle in the interaction region (IR) is essential in colliders to avoid parasitic collisions [7], reduce background near the interaction point (IP), and accommodate the detector and IR magnets. Although the crossing angle reduces beam overlap, the luminosity loss can be mostly recovered by the crab crossing scheme. In this scheme, radio-frequency (RF) crab cavities apply time-dependent transverse kicks, rotating the bunches to enable effective head-on collisions [8]. Crabbing of proton bunches has been demonstrated at CERN's SPS [9], and crab crossing of electron beams at KEKB enabled record luminosities in lepton colliders [10]. As a result, crab crossing has become a key feature in modern collider

designs, including HL-LHC [11], EIC, and EicC. In addition to restoring geometric luminosity, crab cavities also suppress low-order synchrotron resonances from the crossing angle. The EIC baseline design includes both first- and second-order harmonic crab cavities in the hadron ring [12].

Luminosity is the primary performance metric of a collider, and beam-beam interactions can limit the maximum achievable luminosity in both lepton and hadron machines. The beam-beam parameter, which characterizes the maximum tune shift induced by collisions, quantifies the strength of the electromagnetic interaction between the opposing beams. Defining beam flatness as the aspect ratio at the IP, $\kappa \equiv \sigma_y^*/\sigma_x^*$, the luminosity and beam-beam parameters for Gaussian-distributed colliding beams can be expressed as [13]

$$L \propto \frac{1}{\kappa}, \quad \xi_x \propto \frac{\beta_x^*}{1 + \kappa}, \quad \xi_y \propto \frac{\beta_y^*}{\kappa(1 + \kappa)}, \quad (1)$$

where $\beta_{x,y}^*$ denote the β functions at the IP, L is luminosity, and $\xi_{x,y}$ are beam-beam parameters. By scaling $\beta_x^* \propto 1 + \kappa$ and $\beta_y^* \propto \kappa(1 + \kappa)$, one can increase the luminosity with a smaller κ while keeping the beam-beam parameters constant.

The successful demonstration of a flat hadron beam at relativistic heavy ion collider (RHIC) [14] supports the adoption of a flat beam configuration in the EIC baseline, which aims to achieve a peak luminosity of $10^{34} \text{ cm}^{-2} \text{ s}^{-1}$. The EIC design chooses the flatness $\kappa = \beta_y^*/\beta_x^* = \epsilon_y/\epsilon_x < 0.1$, where $\epsilon_{x,y}$ are the transverse beam emittances. The resulting small vertical β_y^* , in both hadron and electron rings, becomes comparable

*Contact author: dxu@bnl.gov

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International license](https://creativecommons.org/licenses/by/4.0/). Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

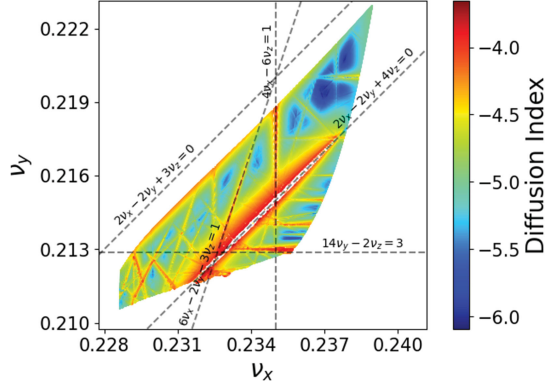


FIG. 1. Frequency map showing the resonance $2\nu_x - 2\nu_y + p\nu_z = 0$, induced by the hourglass effect. Particles span transverse planes $(0, 6\sigma_x^*) \times (0, 6\sigma_y^*)$, with a longitudinal offset $z = 3\sigma_z$.

to the hadron bunch length σ_z , giving rise to the hourglass effect [15]. Although this effect does not significantly reduce luminosity, it introduces nonlinear longitudinal-transverse coupling that excites higher-order synchrotron resonances [16].

As shown in Fig. 1, frequency map analysis reveals a clear resonance line at

$$2\nu_x - 2\nu_y + p\nu_z = 0, \quad (2)$$

where p is an integer and $\nu_{x,y,z}$ are the horizontal, vertical, and longitudinal tunes, respectively.

When the beam dynamics is governed by this resonance, the motion possesses two invariants, $J_x + J_y = \text{const}$ and $2J_z + pJ_y = \text{const}$ [17]. As shown in the Supplemental Material [18], these invariants, together with the Hamiltonian, render the system integrable, and the oscillations of J_y remain confined. For a real beam, however, the actions vary across particles, and both the resonance detuning and the resonance driving terms depend on amplitude. As a result, the intrinsic oscillation frequencies of J_y differ from particle to particle: those away from the resonant condition exhibit relatively fast oscillations, whereas those closer to the separatrix oscillate more slowly. This spread of intrinsic frequencies causes the ensemble to evolve toward a stable transverse equilibrium.

However, in real-world accelerator operations, physical noise is unavoidable. When the frequency spectrum of these fluctuations coincides with the intrinsic oscillation frequencies of J_y , they can efficiently pump energy into the system and drive vertical emittance growth. The formation of resonance islands broadens the distribution of intrinsic frequencies, including very low frequencies near the separatrix, which makes the system more sensitive to external noise.

The physical noise sources vary widely in amplitude and frequency spectrum. For simplicity, this Letter

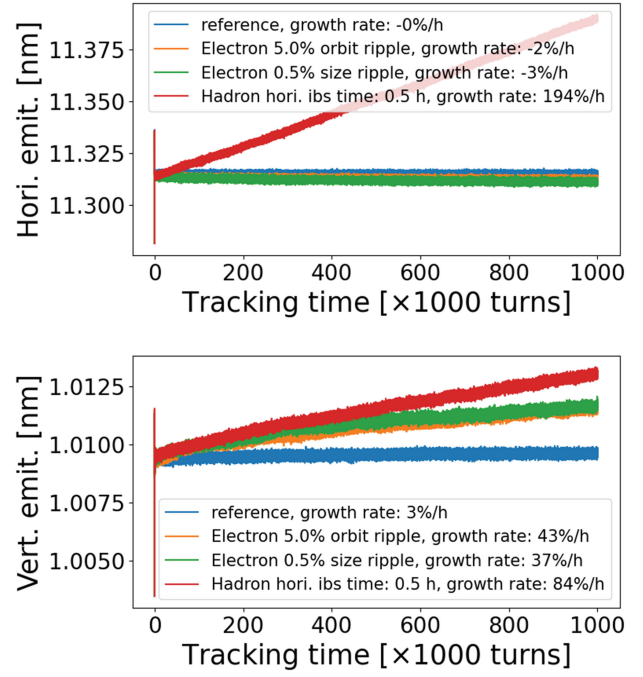


FIG. 2. Emittance evolution under three types of fluctuations: electron beam orbit ripple, electron beam size ripple, and hadron intrabeam scattering. Orbit and size ripples are modeled as narrow-band signals centered at 60 Hz with a 10 Hz bandwidth, representative of power supply noise. Their RMS amplitudes are normalized to the electron size at the IP. IBS is modeled as a wide-band stochastic process, with strength defined by the $1/e$ horizontal emittance growth time. More simulation results are available in the Supplemental Material [18].

examines robustness against three representative types: low-frequency electron beam orbit ripple, electron beam size ripple, and hadron intrabeam scattering (IBS). The orbit and size ripples are primarily driven by magnet power supply noise and typically exhibit narrow-band spectra well separated from the hadron betatron tunes [19]. In contrast, IBS acts as a broadband, white-noise source that overlaps with the hadron tune lines, though its diffusion amplitude is much smaller.

Figure 2 shows significant vertical emittance growth under all three types of fluctuations. For the electron orbit and size ripples, the horizontal emittance decreases over time, indicating transverse emittance transfer. In the IBS case, the horizontal emittance growth rate is reduced from the nominal 200%/h to a fitted 194%/h, again reflecting redistribution to the vertical plane. Compared with the orbit ripple, the electron beam size ripple causes greater vertical growth at the same magnitude, as it induces both direct diffusion and tune modulation, enhancing resonance crossing and amplifying emittance exchange.

Since practical cooling for high-energy hadron beams is still under development [20–22], vertical emittance growth presents a distinct challenge. Emittance tends to transfer from the plane with larger emittance to the one with

smaller emittance. Unlike dispersion-based mechanisms that redistribute diffusion or cooling rates [23,24], synchrotron resonance induces a unidirectional emittance flow without enabling cross-plane cooling. This necessitates dedicated vertical cooling: vertical growth driven by horizontal IBS cannot be suppressed by horizontal cooling alone.

Unequal transverse emittances, synchrotron resonance, and physical fluctuations together form the essential ingredients for driving emittance transfer. One mitigation strategy is to reduce the fluctuation amplitudes from known sources [25]. While effective in principle, this approach increases the cost of accelerator subsystems or even requires technologies beyond the current state of the art. Dedicated feedback systems could alleviate such constraints, but unknown or unpredictable noise sources during machine operation pose an ongoing challenge. These considerations highlight the need for an active strategy to suppress emittance transfer.

Another possible mitigation strategy is to avoid resonance lines within the beam-beam tune footprint. However, for the EIC, the pursuit of high luminosity necessitates a large beam-beam footprint. In addition, the short hadron bunch length and constraints on the RF system together limit the available range of the longitudinal tune. As a result, the resonance condition discussed earlier becomes unavoidable. Since this resonance originates from the hourglass effect, it can be weakened by leveraging the concept of dynamic focusing.

Originally proposed by Brinkmann for THERA [26], the dynamic focusing scheme employs RF quadrupoles (RFQs). Particles receive time-dependent focusing based on their longitudinal positions, aligning their beta functions at the collision point (CP). While the concept has been considered to mitigate head-tail-like instabilities from strong hourglass effects [27] and mentioned as a potential option for future colliders [28], designing RFQs to provide such dynamic focusing remains technically challenging. No implementation has yet been demonstrated in an operational machine.

Sextupoles can be used to suppress specific beam-beam resonances, as is well demonstrated in the crab-waist scheme [29]. A side effect of this method is the redistribution of the vertical beta function within the overlap region, yielding a modest geometric luminosity gain [30].

The hourglass-induced resonance originates from the modulation of transverse beam-beam kick by longitudinal position. An alternative dynamic focusing scheme can be implemented by combining crab cavities with sextupoles, allowing this modulation to be canceled. A similar scheme, referred to as the “travel-focus” scheme, was explored for SuperKEKB [31].

Our simulation adopts Hirata’s framework [32]. The CP is defined where a test particle encounters a slice of the opposite beam. The particle undergoes a virtual drift from

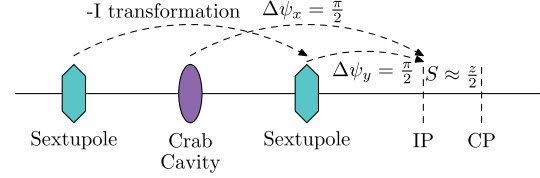


FIG. 3. Locations of sextupoles and crab cavities to generate dynamic focusing. The other side of the IP is omitted for clarity, assuming symmetry and a locally closed crab crossing scheme. The values of $\Delta\psi_{x,y}$ represent the remainders of the corresponding phase advances modulo π .

the IP to the CP, receives the beam-beam kick there, and then, it is transported back to the IP. The hourglass effect, manifested as vertical betatron modulation, arises from this IP-to-CP drift.

Since the hadron bunch is much longer than the electron bunch (for the EIC, the RMS bunch lengths are 6 cm for hadrons and 0.7 cm for electrons), the CP is effectively located at a distance $S \approx z/2$ from the IP [33], where z is the longitudinal coordinate of the hadron particle.

The crab cavity imparts a longitudinally varying transverse kick, characterized by the crab dispersion [34]. Downstream of the cavity, this kick translates into a horizontal displacement. A sextupole placed in this region provides time-dependent focusing that cancels the change in the vertical beta function associated with the IP-to-CP drift. This mechanism resembles the chromaticity compensation scheme in the final focusing section [35], though it relies on nonzero crab dispersion rather than momentum dispersion.

Figure 3 illustrates the relative position in a crab crossing scheme to generate dynamic focusing. A horizontal local crab crossing scheme is assumed, in which the crab dispersion is confined between the upstream and downstream crab cavities around the IP. The horizontal phase advance between crab cavities and the IP is $\pi/2$.

To compensate for the vertical IP-to-CP shift, an inner sextupole is inserted between the crab cavity and the IP with the vertical phase advance of $\Delta\psi_y \bmod \pi = \pi/2$. The required integrated sextupole strength is given by

$$K_2 L = \frac{\sqrt{\beta_x^* / \beta_{s,x}}}{4\theta_c \beta_{s,y} \beta_y^* \cos \Delta\psi_x}. \quad (3)$$

Here, θ_c is the half crossing angle, $\Delta\psi_x$ is the horizontal phase advance from the sextupole to the IP, and $\beta_{s,x}$ and $\beta_{s,y}$ are the beta functions at the inner sextupole. It is assumed that the crossing angle is fully compensated by the crab cavities.

To cancel the second-order geometric terms introduced by the inner sextupole, another outer sextupole is placed outside the crab cavities, such that the two sextupoles form a $-I$ transformation. This cancellation is imperfect because

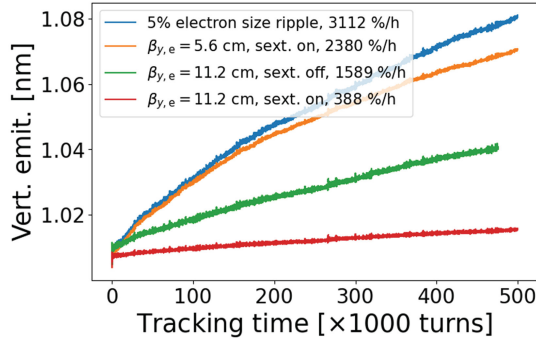


FIG. 4. Hadron vertical emittance evolution under a 5% electron beam size fluctuation with and without dynamic focusing and increased electron vertical beta function.

the crab dispersion vanishes at the outer sextupole. As a result, the residual kick at the CP is

$$\Delta p_x = -\frac{\beta_{s,x}}{\beta_{s,x}^* \beta_{s,y}^*} \left(\frac{\theta_c z_0^2}{4} + \frac{x_0 z_0}{2} \right), \quad (4)$$

where x_0 and z_0 are the horizontal and longitudinal phase space coordinates at the IP. This residual term can be minimized by reducing the ratio $\beta_{s,x}/\beta_{s,y}$, thereby ensuring that the perturbation to horizontal dynamics is negligible.

Figure 4 demonstrates the effectiveness of the sextupole correction scheme under a 5% electron beam size fluctuation, which otherwise induces substantial vertical emittance growth. In addition to the hadron beam's vertical beta function, the electron vertical beta function $\beta_{y,e}^*$ also contributes to the hourglass effect. Doubling $\beta_{y,e}^*$ or enabling the sextupoles alone provides only partial reduction, whereas their combination yields a pronounced suppression of emittance growth.

Figure 5 shows the corresponding frequency map with the dynamic focusing sextupoles enabled and increased $\beta_{y,e}^*$. The resonance in Eq. (2), which is clearly visible in Fig. 1, is completely suppressed. These results demonstrate that the vertical emittance growth originates from

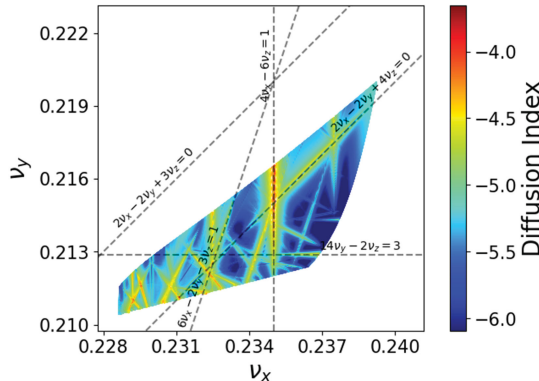


FIG. 5. Frequency map with dynamic sextupoles on. Other conditions are the same as in Fig. 1.

the hourglass effect and that the proposed correction strategy effectively eliminates the underlying resonance mechanism.

The $\beta_{y,e}^*$, however, is also constrained by beam-beam interactions. In a linac-ring collider, the condition $\beta_{y,e}^* > \sigma_z$ is more readily satisfied. In contrast, in a ring-ring collider, $\beta_{y,e}^*$ is limited by the maximum sustainable beam-beam parameter; exceeding this threshold leads to electron vertical emittance blowup. Simulations indicate an upper limit around 0.15 [36], while typical values achieved in circular lepton collider operations are slightly above 0.1 [37,38]. Beam-beam studies for the EIC are consistent with these limits. Strong-strong simulations show that the electron emittance remains stable for $5.6 \text{ cm} < \beta_{y,e}^* < 8.4 \text{ cm}$, corresponding to a vertical beam-beam parameter of 0.10–0.15. Our Letter favors operating at a larger $\beta_{y,e}^*$, within this feasible range, to suppress the impact of the hourglass effect on the hadron beam.

Implementing the dynamic focusing scheme with an increased $\beta_{y,e}^* = 7.2 \text{ cm}$ improves tolerance to electron beam orbit ripple, size fluctuation, and hadron IBS diffusion by factors of 2 to 3, accompanied by a slight gain in luminosity. This enhanced robustness against real-world noise underscores the scheme's practical advantage.

Beyond the EIC, this method offers broader applicability to future lepton-hadron colliders. The EIC design constrains vertical divergence to match the horizontal at the IP [39], limiting hadron vertical beta function $\beta_{y,h}^*$ reduction. Dynamic focusing decouples $\beta_{y,h}^*$ from the hadron bunch length σ_z . By reducing $\beta_{y,h}^*$ while holding σ_y^* constant, the emittance ratio ϵ_y/ϵ_x improves, the vertical beam-beam parameter decreases, and resonance driving term strengths diminish. Simulation results in Fig. 6 show that, at a vertical beta of 2.4 cm, even a 5% electron beam size fluctuation induces no vertical emittance growth. This strategy opens

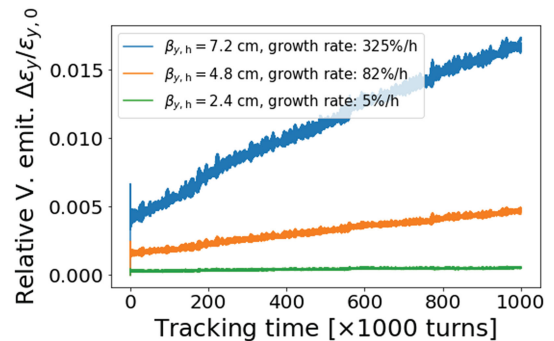


FIG. 6. Hadron vertical emittance evolution under a 5% electron beam size ripple with dynamic focusing. Emittance is normalized to its initial value. Reducing hadron $\beta_{y,h}^*$ to 2.4 cm eliminates vertical emittance growth. In contrast, the same fluctuation causes a growth rate exceeding 3000%/h without dynamic focusing (see Fig. 4).

new opportunities for robust flat hadron beam collisions in future high-luminosity colliders.

However, achieving the required beam optics poses notable challenges. The crab cavities demand a specific horizontal phase advance and a large horizontal beta function to compensate for the crossing angle, whereas the sextupole requires precise vertical phase control relative to the IP. Sextupole placement further benefits from a large vertical beta function and a small horizontal beta function to reduce both its required strength and residual nonlinear effects. In the EIC-like local crabbing scheme, the proximity of crab cavities to the final focusing quadrupoles limits available space for additional sextupoles due to layout constraints. Nonetheless, these design challenges appear manageable within the scope of collider optics.

One possible solution is to place the crab cavities further away from the IP, allowing better separation of the horizontal and vertical beta peaks and providing more degrees of freedom to control phase advances. An alternative is the global crabbing scheme, successfully operated at KEKB and considered for the HL-LHC [40]. It has also been evaluated for the EIC, with no adverse beam-beam effects identified [41]. Its potential to facilitate dynamic focusing merits further exploration.

In summary, the emittance transfer in flat hadron beam collisions arises from the interplay of hourglass-induced synchrotron resonance and unavoidable external fluctuations. The dynamic focusing scheme suppresses this resonance and relaxes the associated noise constraints. This insight informs the design and operation of future flat hadron beam collision facilities.

Acknowledgments—The authors thank S. Y. Lee for helpful discussions about synchrotron resonance. This work was supported by Brookhaven Science Associates, LLC under Contract No. DE-SC0012704 with the U.S. Department of Energy, and by a U.S. Department of Energy Early Career Award. This research used resources of the National Energy Research Scientific Computing Center (NERSC), a U.S. Department of Energy User Facility (Project No. m2503).

Data availability—The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

- [1] A. Accardi, J. Albacete, M. Anselmino, N. Armesto, E. Aschenauer, A. Bacchetta, D. Boer, W. Brooks, T. Burton, N. B. Chang *et al.*, Electron-ion collider: The next QCD frontier: Understanding the glue that binds us all, *Eur. Phys. J. A* **52**, 268 (2016).
- [2] F. Willeke, The HERA lepton–proton collider, in *Challenges and Goals for Accelerators in the XXI Century* (World Scientific Publishing Company, Singapore, 2016) Chap. 15, pp. 225–242.
- [3] ZEUS Collaboration, Measurement of inelastic production in deep inelastic scattering at HERA, *Eur. Phys. J. C* **44**, 13 (2005).
- [4] F. Willeke and J. Beebe-Wang, Electron Ion Collider Conceptual Design Report 2021, Technical Report No. BNL-221006–2021-FORE, Brookhaven National Lab. (BNL), Upton, NY (USA); Thomas Jefferson National Accelerator Facility (TJNAF), Newport News, VA (USA), 2021.
- [5] J. Fernandez, C. Adolphsen, P. Adzic, A. Akay, H. Aksakal, J. Albacete, B. Allanach, S. Alekhin, P. Allport, V. Andreev *et al.*, On the relation of the LHeC and the LHC, *arXiv:1211.5102*.
- [6] D. P. Anderle, V. Bertone, X. Cao, L. Chang, N. Chang, G. Chen, X. Chen, Z. Chen, Z. Cui, L. Dai *et al.*, Electron-ion collider in China, *Front. Phys.* **16**, 64701 (2021).
- [7] K. Hirata, Don't be afraid of beam-beam interactions with a large crossing angle, Technical Report No. SCAN-9411248, 1994.
- [8] R. Palmer, Energy scaling, crab crossing and the pair problem, Technical Report No. SLAC-PUB-4707, Stanford Linear Accelerator Center, 1988.
- [9] R. Calaga, A. Alekou, F. Antoniou, R. Appleby, L. Arnaudon, K. Artoos, G. Arduini, V. Baglin, S. Barriere, H. Bartosik *et al.*, First demonstration of the use of crab cavities on hadron beams, *Phys. Rev. Accel. Beams* **24**, 062001 (2021).
- [10] T. Abe, K. Akai, M. Akemoto, A. Akiyama, M. Arinaga, K. Ebihara, K. Egawa, A. Enomoto, J. Flanagan, S. Fukuda *et al.*, Compensation of the crossing angle with crab cavities at KEKB, in *2007 IEEE Particle Accelerator Conference (PAC)* (IEEE, Albuquerque, New Mexico, 2007), pp. 27–31.
- [11] G. Apollinari, O. Brüning, T. Nakamoto, and L. Rossi, High luminosity large hadron collider HL-LHC, *arXiv:1705.08830*.
- [12] D. Xu, Y. Hao, Y. Luo, C. Montag, and J. Qiang, Study of harmonic crab cavity in EIC beam-beam simulations, in *Proceedings of the IPAC'21*, International Particle Accelerator Conference No. 12 (JACoW Publishing, Geneva, Switzerland, 2021), pp. 2595–2597, [10.18429/JACoW-IPAC2021-WEPAB009](https://doi.org/10.18429/JACoW-IPAC2021-WEPAB009).
- [13] A. W. Chao, M. Tigner, H. Weise, and F. Zimmermann, *Handbook of Accelerator Physics and Engineering* (World Scientific, Singapore, 2023).
- [14] Y. Luo, D. Xu, M. Blaskiewicz, and C. Montag, Experimental demonstration of a large transverse emittance ratio 11:1 in the Relativistic Heavy Ion Collider for the electron-ion collider, *Phys. Rev. Lett.* **132**, 205001 (2024).
- [15] M. A. Furman, Hourglass effects for asymmetric colliders, in *Proceedings of the 1991 IEEE Particle Accelerator Conference* (IEEE, San Francisco, CA, 1991), Vol. 1, pp. 422–424.
- [16] D. Xu, Y. Hao, Y. Luo, and J. Qiang, Synchrotron resonance of crab crossing scheme with large crossing angle and finite bunch length, *Phys. Rev. Accel. Beams* **24**, 041002 (2021).
- [17] S. Y. Lee, Single particle dynamics at synchro-betatron coupling resonances, *Phys. Rev. E* **49**, 5706 (1994).

- [18] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/rf7b-96z1> for a simplified theoretical model and more simulation results.
- [19] B. Podobedov and M. Blaskiewicz, Eddy current shielding of the magnetic field ripple in the EIC Electron Storage Ring vacuum chambers, Technical Report No. BNL-224904-2023-TECH, [Brookhaven National Laboratory (BNL), Upton, NY (USA), 2023].
- [20] M. Blaskiewicz, Cooling of high-energy hadron beams, *Annu. Rev. Nucl. Part. Sci.* **64**, 299 (2014).
- [21] V.N. Litvinenko and Y.S. Derbenev, Coherent electron cooling, *Phys. Rev. Lett.* **102**, 114801 (2009).
- [22] J. Jarvis, V. Lebedev, A. Romanov, D. Broemmelsiek, K. Carlson, S. Chattopadhyay, A. Dick, D. Edstrom, I. Lobach, S. Nagaitsev, H. Piekarczyk, P. Piot, J. Ruan, J. Santucci, G. Stancari, and A. Valishev, Experimental demonstration of optical stochastic cooling, *Nature (London)* **608**, 287 (2022).
- [23] J. Wei and G. Parzen, Intra-beam scattering scaling for very large hadron colliders, in *PACS2001. Proceedings of the 2001 Particle Accelerator Conference (Cat. No. 01CH37268)* (IEEE, Chicago, 2001), Vol. 1, pp. 42–44.
- [24] P. Baxevanis and G. Stupakov, Transverse dynamics considerations for microbunched electron cooling, *Phys. Rev. Accel. Beams* **22**, 081003 (2019).
- [25] B. Podobedov *et al.*, Physics-driven specifications for the EIC ESR magnet power supply ripple, in *Proceedings of the 16th International Particle Accelerator Conference, IPAC'25–16th International Particle Accelerator Conference No. 16* (JACoW Publishing, Geneva, Switzerland, 2025), pp. 746–749.
- [26] R. Brinkmann and M. Dohlus, A method to overcome the bunch length limitation on β_p^* for electron-proton colliders, Technical Report No. DESY M-95-11, 1995.
- [27] L. Wang, Beam-beam dynamics with strong hourglass effect, *ICFA mini workshop: Beam-Beam Effects in Circular Colliders BB24*, EPFL, Lausanne, (Sep. 2-5-2024), 2024.
- [28] Y. Acar, A. Akay, S. Beser, A. Canbay, H. Karadeniz, U. Kaya, B. Oner, and S. Sultansoy, Future circular collider based lepton–hadron and photon–hadron colliders: Luminosity and physics, *Nucl. Instrum. Methods Phys. Res., Sect. A* **871**, 47 (2017).
- [29] D. Shatilov, E. Levichev, E. Simonov, and M. Zobov, Application of frequency map analysis to beam-beam effects study in crab waist collision scheme, *Phys. Rev. ST Accel. Beams* **14**, 014001 (2011).
- [30] M. Zobov *et al.*, Test of “crab-waist” collisions at the DAΦNE Φ factory, *Phys. Rev. Lett.* **104**, 174801 (2010).
- [31] Y. Ohnishi, Recent report on design of SuperKEKB, in *ICFA Beam Dynamics Newsletter No. 48* [International Committee for Future Accelerators (ICFA). 2009], pp. 252–259.
- [32] K. Hirata, H. W. Moshhammer, and F. Ruggiero, A symplectic beam-beam interaction with energy change, *Part. Accel.* **40**, 205 (1992), <https://inspirehep.net/literature/338873>.
- [33] D. Xu, V.S. Morozov, D. Sagan, Y. Hao, and Y. Luo, Enhanced beam-beam modeling to include longitudinal variation during weak-strong simulation, *Phys. Rev. Accel. Beams* **27**, 061002 (2024).
- [34] D. Xu, Y. Luo, and Y. Hao, Combined effects of crab dispersion and momentum dispersion in colliders with local crab crossing scheme, *Phys. Rev. Accel. Beams* **25**, 071002 (2022).
- [35] P. Raimondi and A. Seryi, Novel final focus design for future linear colliders, *Phys. Rev. Lett.* **86**, 3779 (2001).
- [36] K. Ohmi, M. Tawada, Y. Cai, S. Kamada, K. Oide, and J. Qiang, Luminosity limit due to the beam-beam interactions with or without crossing angle, *Phys. Rev. ST Accel. Beams* **7**, 104401 (2004).
- [37] P. Y. Shatunov, D. E. Berkaev, Y. M. Zharinov, I. M. Zemlyansky, A. S. Kasaev, A. N. Kyrpotin, I. A. Koop, A. P. Lysenko, A. V. Otboev, E. A. Perevedentsev, V. P. Prosvetov, Y. A. Rogovsky, A. L. Romanov, A. I. Senchenko, A. N. Skrinsky, Y. M. Shatunov, and D. B. Shwartz, Status and perspectives of the VEPP-2000, *Phys. Part. Nucl. Lett.* **13**, 995 (2016).
- [38] D. Zhou, K. Ohmi, Y. Funakoshi, and Y. Ohnishi, Luminosity performance of SuperKEKB, *J. Instrum.* **19**, T02002 (2024).
- [39] V. Ptitsyn, Accelerator physics challenges for EIC, in *Proceedings of the 14th International Particle Accelerator Conference, IPAC'23–14th International Particle Accelerator Conference No. 14* (JACoW Publishing, Geneva, Switzerland, 2023), pp. 2621–2626.
- [40] Y.-P. Sun, R. Assmann, R. Tomás, and F. Zimmermann, Crab dispersion and its impact on the CERN large hadron collider collimation, *Phys. Rev. ST Accel. Beams* **13**, 031001 (2010).
- [41] D. Xu *et al.*, Assessing global crabbing scheme feasibility for Electron-Ion Collider, in *Proceedings of the 15th International Particle Accelerator Conference, IPAC'24–15th International Particle Accelerator Conference No. 15* (JACoW Publishing, Geneva, Switzerland, 2024), pp. 226–229.