

Achieving the Quantum Fisher Information Bound in Pseudo-Hermitian Sensors

Ievgen I. Arkhipov^{1,*}, Franco Nori^{2,3} and Şahin K. Özdemir⁴

¹*Joint Laboratory of Optics of Palacký University and Institute of Physics of CAS, Faculty of Science, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic*

²*Quantum Information Physics Theory Research Team, Quantum Computing Center, RIKEN, Wakoshi, Saitama 351-0198, Japan*

³*Physics Department, The University of Michigan, Ann Arbor, Michigan 48109-1040, USA*

⁴*Department of Electrical and Computer Engineering, Saint Louis University, St. Louis, Missouri 63103, USA*



(Received 12 June 2025; revised 26 September 2025; accepted 27 January 2026; published 24 February 2026)

Non-Hermitian systems have attracted considerable interest over the last few decades due to their unique spectral and dynamical properties not encountered in Hermitian counterparts. An intensely debated question is whether non-Hermitian systems, described by pseudo-Hermitian Hamiltonians with real spectra, can offer enhanced sensitivity for parameter estimation when they are operated at or close to exceptional points. However, much of the current analysis and conclusions are based on mathematical formalism developed for Hermitian quantum systems, which is questionable when applied to pseudo-Hermitian Hamiltonians, whose Hilbert space is intrinsically nonflat. Here, we develop a covariant formulation of quantum Fisher information (QFI) defined on the deformed Hilbert space of pseudo-Hermitian Hamiltonians. This covariant framework ensures the preservation of the state norm and enables a consistent treatment of parameter sensitivity. We further show that the covariant QFI of pseudo-Hermitian systems is dual to the ordinary QFI of corresponding Hermitian systems. Importantly, this correspondence naturally imposes an upper bound on the covariant QFI and allows one to identify optimal projections which saturate the corresponding classical Fisher information to this ultimate limit. The developed framework also enables to set the criteria under which pseudo-Hermitian sensors can exhibit an advantage over their Hermitian counterparts of the same dimensionality.

DOI: [10.1103/physrevlett.136.080802](https://doi.org/10.1103/physrevlett.136.080802)

Introduction—Non-Hermitian systems described by pseudo-Hermitian Hamiltonians possessing real spectra, have recently attracted considerable interest in part due to the possibility to utilize them for enhanced quantum sensing. Specifically, these systems exhibit so-called exceptional points (EPs) in their spectrum, where both eigenvalues and eigenvectors of the Hamiltonian coalesce [1]. Near these EPs, the eigenvalue response to perturbations becomes nonanalytic, scaling as a fractional power of the perturbation, which has been suggested as a mechanism for enhanced parameter sensitivity in quantum metrology [2–4].

Several studies have demonstrated that quantum sensors operating near EPs can exhibit significant amplification of signal response, potentially outperforming their Hermitian counterparts in both classical and quantum regimes [5–18]. In parallel, other theoretical and experimental proposals suggested utilizing pseudo-Hermiticity away from EPs for

quantum sensing [19,20]. However, this apparent sensing supremacy has also been scrutinized. In particular, concerns have been raised about whether the increased sensitivity near EPs translates into a genuine advantage when quantum noise and other measurement imperfections are properly taken into account [21–28].

Recent studies have also explored the fundamental limits of quantum sensing in pseudo-Hermitian systems through the lens of quantum Fisher information (QFI) [29]. In particular, it has been argued that pseudo-Hermitian systems offer no intrinsic advantage in parameter estimation over their Hermitian counterparts when the optimal measurement strategy is employed [30,31]. These conclusions, however, rely on specific assumptions; most notably, the use of the dilation method, which introduces additional resources and encodes the system’s parameter dependence solely in the reduced non-Hermitian Hamiltonian. This also raises important questions about the generality of such no-go results. Furthermore, it remains unclear whether the conventional QFI formalism is directly applicable in pseudo-Hermitian settings, as naive application may lead to misleading or inconsistent results due to the nontrivial geometry of the underlying Hilbert space of pseudo-Hermitian Hamiltonians. Initial efforts to address this issue

*Contact author: ievgen.arkhipov@upol.cz

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI.

were undertaken in Refs. [16,32], where the QFI was reformulated using an appropriate metric on the deformed Hilbert space. Yet, that approach does not account for intricate system dynamics on such nonflat space.

In this Letter, we develop a general covariant quantum Fisher information (CQFI) framework for states governed by pseudo-Hermitian Hamiltonians. Introducing a covariant state derivative, our formulation faithfully captures the deformed Hilbert-space geometry while preserving state normalization. We show that the CQFI maps onto the standard QFI (SQFI) of an associated Hermitian system and, thus, reveal the duality between the two. This correspondence naturally sets an upper bound on the CQFI and enables one to determine the optimal measurement projections required to saturate the corresponding classical Fisher information to this ultimate limit. Crucially, our framework establishes criteria under which pseudo-Hermitian sensors can surpass Hermitian counterparts of the same Hilbert-space dimension. This dimension-preserving comparison contrasts with dilation-based approaches, where pseudo-Hermitian systems are benchmarked against higher-dimensional Hermitian embeddings [30].

Preliminaries on the geometry of pseudo-Hermitian Hilbert spaces—We begin by outlining key geometric concepts of the metric-modified Hilbert space induced by pseudo-Hermitian Hamiltonians [33]. As in Hermitian quantum mechanics, the dynamics of a state vector $|\psi^{\text{PH}}(t)\rangle$ in a pseudo-Hermitian system is governed by the pseudo-Hermitian Hamiltonian H^{PH} in the Schrödinger equation (by setting $\hbar = 1$): $i\partial_t|\psi^{\text{PH}}(t)\rangle = H^{\text{PH}}|\psi^{\text{PH}}(t)\rangle$.

To ensure that the state evolution is consistent with probability conservation, as in standard quantum mechanics, one must equip the Hilbert space with a positive-definite Hermitian metric operator η . This metric defines a modified inner product, $\langle\psi|\phi\rangle_\eta = \langle\psi|\eta|\phi\rangle$, which induces a Riemannian-like geometric structure on the η -deformed Hilbert space, such that the norm $\langle\psi^{\text{PH}}(t)|\eta|\psi^{\text{PH}}(t)\rangle$ is preserved in time. For pseudo-Hermitian Hamiltonians with real spectrum, the metric obeys the following condition [37] $(H^{\text{PH}})^\dagger\eta = \eta H^{\text{PH}}$. In this case, the Hamiltonian H^{PH} generates a unitarylike evolution in the metric-modified Hilbert space, even though it is not Hermitian.

An important consequence of the pseudo-Hermiticity is that any pseudo-Hermitian Hamiltonian H^{PH} can be mapped to a Hermitian Hamiltonian H^{H} through a similarity transformation [38]. Explicitly, since the metric operator η can always be written as $\eta = S^\dagger S$, for some invertible operator S , then the associated Hermitian Hamiltonian (up to an arbitrary unitary) can be defined as follows [39] $H^{\text{H}} = SH^{\text{PH}}S^{-1}$, $(H^{\text{H}})^\dagger = H^{\text{H}}$. This relation shows that pseudo-Hermitian quantum mechanics can be unitarily equivalent (in the η -inner product sense) to standard Hermitian quantum mechanics. Consequently, the corresponding state vectors in the flat and η -modified Hilbert spaces are related by

$$|\psi^{\text{H}}\rangle = S|\psi^{\text{PH}}\rangle. \quad (1)$$

Although, in certain cases, the operator S can be directly determined from the shape of the H^{PH} ; in general, however, it can be inferred from the metric [40].

Covariant derivative of states on non-flat Hilbert space—In order to consistently describe the vector transport on the curved space, one can borrow the concepts of connection and covariant derivative from differential geometry, which allows one to properly compare vectors at different points on a given curved manifold [42]. This is exactly the case for the η -modified Hilbert space. Indeed, when the Hamiltonian H^{PH} depends on a continuous parameter θ , the variation in θ can induce an additional geometric structure in its deformed Hilbert space. That is, the parameter space can be viewed as a base manifold over which the Hilbert space forms a vector bundle. Consequently, one can define a connection Γ_θ associated with the θ direction in order to enable a proper covariant differentiation with respect to this parameter. This newly induced connection ensures that a covariant state derivative respects the Hilbert space curvature induced by the metric $\eta(\theta)$. The explicit form of such a connection can read [40] $\Gamma_\theta = S^{-1}\partial_\theta S$. Moreover, when the eigenvectors of the metric η do not depend on the parameter θ , the connection can be alternatively written as $\Gamma_\theta = \eta^{-1}\partial_\theta\eta/2$ [40].

The associated covariant derivative along θ acting on the state vector $|\psi\rangle$ can then be defined as [40,42]

$$D_\theta|\psi\rangle = \partial_\theta|\psi\rangle + \Gamma_\theta|\psi\rangle. \quad (2)$$

Clearly, when the Hilbert space is flat, i.e., $\eta = I$, the covariant derivative is reduced to the ordinary derivative $D_\theta = \partial_\theta$, since $\Gamma_\theta = 0$ in this case [43].

Covariant quantum Fisher information—The accuracy with which one can estimate a parameter θ encoded in a quantum state is fundamentally constrained by the quantum Cramér-Rao inequality [44]. According to this bound, the standard deviation $\Delta\theta$ of any unbiased estimator is limited from below by $\Delta\theta \geq 1/\sqrt{mF_\theta}$, where m denotes the number of independent measurements and F_θ represents the QFI associated with the state.

The SQFI for a pure state $|\psi^{\text{H}}(\theta)\rangle$ of a Hermitian Hamiltonian H^{H} , parametrized by a real parameter θ , is related to the Fubini–Study metric on the projective Hilbert space (the manifold of rays) [45]:

$$F^{\text{H}}(\theta) = 4\|\partial_\theta\psi_\perp^{\text{H}}\|^2, \quad (3)$$

where we defined the vector's norm as $\|\cdot\|^2 = \langle\cdot|\cdot\rangle$, and $|\cdot_\perp\rangle = |\cdot\rangle - |\psi^{\text{H}}\rangle\langle\psi^{\text{H}}|\cdot\rangle$ is the projection of a vector $|\cdot\rangle$ onto the subspace orthogonal to the state $|\psi^{\text{H}}\rangle$.

In pseudo-Hermitian systems, one has to introduce the covariant derivative instead, which faithfully accounts for the change of a state $|\psi^{\text{PH}}(\theta)\rangle$ in a deformed Hilbert space.

In other words, if one naively applies the SQFI to pure states of pseudo-Hermitian systems, it would ignore the underlying geometric contribution to the QFI. To account for the geometry, one must generalize a SQFI to CQFI, defined as

$$F_c^{\text{pH}}(\theta) = 4 \left\| D_\theta \psi_\perp^{\text{pH}} \right\|_\eta^2, \quad (4)$$

where the η -norm is given by $\|\cdot\|_\eta^2 = \langle \cdot | \eta | \cdot \rangle$, and the perpendicular component is again projected as $|\cdot_\perp\rangle = (I - P_\psi)|\cdot\rangle$, with the properly redefined projector $P_\psi = |\psi^{\text{pH}}\rangle_\eta \langle \psi^{\text{pH}}|$.

Duality between CFQI and SQFI—With the covariant framework established above one finds a one-to-one correspondence between the SQFI of the Hermitian Hamiltonian and the CQFI of the pseudo-Hermitian system [40]:

$$F^{\text{H}}(\psi_\theta^{\text{H}}) = F_c^{\text{pH}}(\psi_\theta^{\text{pH}}). \quad (5)$$

This duality establishes CQFI as the natural extension of quantum Fisher information to pseudo-Hermitian systems and constitutes the central result of our Letter.

The equivalence in Eq. (5) entails an important consequence. Namely, the upper bound on the CQFI is identical to that on the SQFI. However, the upper bound of the SQFI is known, and it equals to [46] $F_{\text{max}}^{\text{H}} = (\lambda_{\text{max}} - \lambda_{\text{min}})^2$, where the $\lambda_{\text{max}} (\lambda_{\text{min}})$ is the maximal (minimal) eigenvalue of the Hermitian operator $h(\theta) = iV\partial_\theta V^\dagger$, where $V = \exp(-iH^{\text{H}}t)$, and therefore one has

$$F_c^{\text{pH}}(\psi_\theta^{\text{pH}}) \leq F_{\text{max}}^{\text{pH}}(\theta) = F_{\text{max}}^{\text{H}}(\theta). \quad (6)$$

For small values of $t \ll 1$, the Hermitian operator is substantially simplified and takes the form $h(\theta) \approx t\partial_\theta H^{\text{H}}$ [47]. In this case, the upper bound on the CQFI is straightforwardly found from the spectrum width of $\partial_\theta H^{\text{H}}$. For arbitrarily large $t > 1$, the explicit formula for $h(\theta)$ has been given in Ref. [48]. Importantly, the expression for the upper bound on the CQFI in Eq. (6) immediately gives the states $|\psi^{\text{pH}}\rangle$ which maximize the CQFI, and which explicitly read $|\psi_{\text{max}}^{\text{pH}}\rangle \equiv S^{-1}[\lambda_{\text{max}}\rangle + e^{i\phi}|\lambda_{\text{min}}\rangle]$, where $|\lambda\rangle$ is an eigenstate of the operator $h(\theta)$, and ϕ is an arbitrary phase.

At first glance, the revealed duality in Eq. (5) suggests that pseudo-Hermitian systems offer no fundamental sensing advantage over their Hermitian counterparts: under optimal measurements, sensitivities coincide. These preliminary conclusions agree with Refs. [30,31], which, however, rely solely on the quantum dilation framework, thus necessitating additional resources and assumptions.

Nevertheless, and most crucially, in the linear sensing regime pseudo-Hermitian systems can still exhibit enhanced performance. This is because linear perturbations in the pseudo-Hermitian frame may map to nonlinear

perturbations in the Hermitian frame via the similarity transformation S , thus effectively amplifying sensitivity. It is well established that, when the parameter under interest couples to the Hermitian system nonlinearly, it results in enhanced sensing precision [49]. We further substantiate these conclusions below by analyzing two minimal paradigmatic models.

Optimal projections saturating the CQFI bound—In a real experiment, however, any measurement on a quantum system yields a classical Fisher information (CsFI), determined by the measured probabilities $p(x_\theta|\theta)$, which reaches the QFI only when optimal state projections are applied [46,50]. These optimal projections can be determined from the symmetric logarithmic derivative (SLD) operator L , defined in flat Hilbert space for a state ρ_θ as $\dot{\rho}_\theta = (L\rho + \rho L)/2$ [51,52]. The significance of L lies in the fact that its variance equals the QFI, $F_Q = \langle L^2 \rangle$, since $\langle L \rangle = 0$, so the eigenvectors of the SLD are precisely the optimal states that saturate the CsFI to the QFI [45,53]. For pure states the SLD operator attains the form $L = 2(|\partial_\theta \psi\rangle\langle\psi| + |\psi\rangle\langle\partial_\theta \psi|)$.

In this regard, we can promote the standard SLD operator, defined on the flat Hilbert space, to a covariant SLD operator on the η -modified Hilbert space of pseudo-Hermitian systems. Indeed, the correspondence in Eq. (5) implies that the variances of the ordinary L and covariant L_c SLD operators, averaged over the corresponding states in Eq. (1), must be dual to each other, i.e., $\langle L^2 \rangle = \langle L_c^2 \rangle_\eta$, where the pseudo-Hermitian L_c operator reads,

$$L_c = 2(|\partial_\theta \psi\rangle\langle\psi|_\eta + |\psi\rangle\langle\partial_\theta \psi|_\eta), \quad L_c^\dagger = \eta L_c \eta^{-1}. \quad (7)$$

The correspondence between these two SLD operators immediately provides a prescription for finding the optimal projection in pseudo-Hermitian sensors that enables the ultimate precision in parameter estimation. Indeed, knowing the state $|\psi_{\text{max}}^{\text{pH}}\rangle$, which maximizes the CQFI, allows one to find the corresponding optimal projectors from the eigenvectors $|\nu_i\rangle$ of L_c in Eq. (7).

Therefore, realizing measurements via the corresponding projectors $\Pi_i = |\nu_i\rangle\langle\nu_i|_\eta$ enables one to saturate the covariant CsFI,

$$F_{\text{cl}}^{\text{pH}}(\theta) = \sum (\partial_\theta p_i)^2 / p_i, \quad p_i(\theta) = \left\langle \psi_{\text{max}}^{\text{pH}} | \Pi_i | \psi_{\text{max}}^{\text{pH}} \right\rangle_\eta, \quad (8)$$

to the upper bound of the CQFI, i.e., $F_{\text{cl}}^{\text{pH}}(\theta) = F_{\text{max}}^{\text{pH}}(\theta)$.

Needless to say, performing projections with respect to the η -modified inner product on the associated Hilbert space can be experimentally challenging. However, recent experiments have demonstrated the feasibility of metric-dependent projections in various non-Hermitian quantum platforms [54–56], thus indicating that achieving

‘covariant’ ultimate precision in pseudo-Hermitian quantum sensors is now within experimental reach.

Nonreciprocal systems—

Example 1a: Now we examine conditions when quantum sensing can or cannot exhibit enhancement in the pseudo-Hermitian systems in comparison to their Hermitian counterparts. We first analyze a specific setup analogous to that in Refs. [19,20,31], where the Hamiltonian is parametrized by a multiplicative parameter θ , namely,

$$H^{\text{pH}}(\theta) = \theta \begin{pmatrix} 0 & \delta^{-1} \\ \delta & 0 \end{pmatrix}, \quad \delta \neq 0. \quad (9)$$

The form of the metric operator is readily found $\eta = \text{diag}[1, \delta^{-2}]$ [40], which does not depend on θ . The associated H^{H} gets the form [40]: $H^{\text{H}}(\theta) = \theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Comparing both Hamiltonians, one comes to the conclusion that for this type of parameterization, the H^{pH} does not provide any advantage in enhancing the QFI, since the multiplicative nature of the parametrization remains in the associated Hermitian Hamiltonian H^{H} . Specifically, for $t \ll 1$, the upper bound on the CQFI for this type of pseudo-Hermitian Hamiltonian reads,

$$F_c^{\text{pH}}(\theta) \leq 4t^2, \quad (10)$$

which does not depend on θ .

Example 1b: On the other hand, by choosing a parameterization as follows:

$$H^{\text{pH}}(\theta) = \begin{pmatrix} 0 & \delta^{-1} + \theta \\ \delta + \theta & 0 \end{pmatrix}, \quad \delta \neq 0, \quad (11)$$

one can achieve enhancement in quantum sensing. This kind of nonreciprocal sensor can be implemented in quantum photonic settings [57,58]. The associated H^{H} has the perturbed off-diagonal terms $\sqrt{(\delta + \theta)(\delta^{-1} + \theta)}$ [40], which are nonlinear in θ , in contrast to linear perturbations in Eq. (11). This means that for the same fixed measurement protocol based on a linear disturbance, one can achieve better performance in the QFI in the pseudo-Hermitian system governed by H^{pH} in Eq. (11).

Again, assuming for simplicity $t \ll 1$, the CQFI for this type of parameterization is bounded as follows:

$$F_c^{\text{pH}}(\theta) \leq F_{\text{max}}^{\text{pH}}(\theta) = \frac{(\delta^2 + 2\theta\delta + 1)^2}{\delta(\delta + \theta)(\delta\theta + 1)} t^2. \quad (12)$$

Clearly, the CQFI bound $F_{\text{max}}^{\text{pH}}(\theta)$ diverges as the system approaches the EP, which corresponds to the zeros in the denominator of Eq. (12) and is associated with the Jordan form of the pseudo-Hermitian Hamiltonian in Eq. (11) (see also Fig. 1). Consequently, initializing the system near the

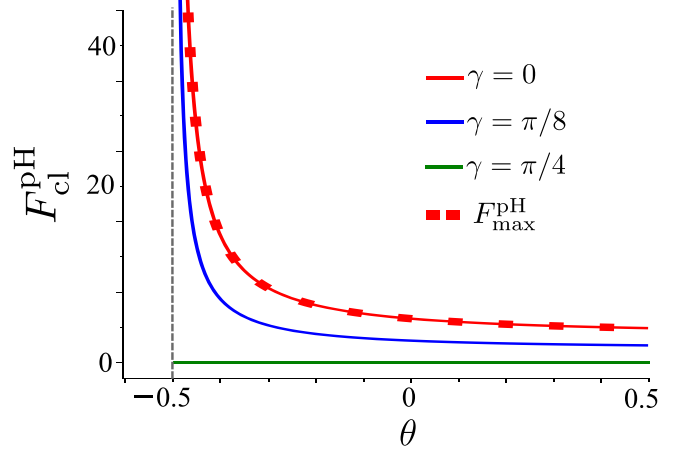


FIG. 1. Covariant classical Fisher information obtained from different measurement projectors in the nonreciprocal system of Example 1b. The parameter γ controls the deviation of the measurement basis $|\chi(\gamma)\rangle$ from the optimal axis defined by the eigenbasis of the covariant SLD operator (see the main text for details). The upper bound of the covariant quantum Fisher information for each case is shown by a red dotted curve. The vertical dash gray line corresponds to the value of $\theta = \theta_{\text{EP}} = -0.5$ [arb. units], when the pseudo-Hermitian sensor is at the exceptional point.

EP can enhance sensitivity in parameter estimation. Importantly, the CQFI faithfully captures the phase transition occurring at the EP, signaling spontaneous pseudo-Hermiticity symmetry breaking. This observation is in line with Hermitian systems, where the SQFI is utilized as a witness for quantum phase transitions [59–62]. It is also instructive to compare these results with those obtained within the flat Hilbert space formalism which leads to erroneous conclusions [40].

Next we find the optimal projections, i.e., eigenstates of the covariant SLD operator in Eq. (7), which saturate the corresponding CsFI to the CQFI bound given in Eq. (12). As the initial state we choose the vacuum $|0\rangle$. The time-evolving state is then $|\psi_t\rangle = e^{-iH^{\text{pH}}t}|0\rangle$. In the basis $\{|0\rangle, |1\rangle\}$, the η -normalized state takes the form

$$|\psi_t\rangle = \cos(\sqrt{k_1 k_2} t) |0\rangle - i \sqrt{\frac{k_2}{k_1}} \sin(\sqrt{k_1 k_2} t) |1\rangle, \quad (13)$$

which satisfies $\langle \psi_t | \eta | \psi_t \rangle = 1$, with η given in [40]. Here we denote $k_1 = \delta^{-1} + \theta$ and $k_2 = \delta + \theta$. A straightforward calculation shows that the CQFI of this state attains its upper bound $F_{\text{max}}^{\text{pH}}(\theta)$ in Eq. (12), diverging when approaching the EP.

After some algebra, one finds that the eigenstates $|\nu_{\pm}\rangle$ of the L_c operator take the form [40]

$$|\nu_{\pm}\rangle = (|\psi_t\rangle \pm |u_t\rangle) / \sqrt{2}, \quad (14)$$

where $|u_t\rangle = \sin(\sqrt{k_1 k_2} t)|0\rangle + i\sqrt{k_2/k_1} \cos(\sqrt{k_1 k_2} t)|1\rangle$, satisfying $\langle\psi_t|u_t\rangle_\eta = 0$, and $\langle\nu_i|\nu_j\rangle_\eta = \delta_{ij}$. The optimal projectors then read $\Pi_\pm(\theta) = |\nu_\pm\rangle\langle\nu_\pm|_\eta$.

Evidently, the optimal projectors generally depend on the parameter θ , which means that suitable measurement strategies are required to saturate the CsFI [63]. Such measurement approaches involve measuring the projections at the fixed basis at some reference points $x_0 = (\delta_0, \theta_0, t_0)$, so that evaluating the corresponding probabilities $p_\pm(\theta) = \langle\psi_t(\theta)|\Pi_\pm(x_0)|\psi_t(\theta)\rangle_\eta$ allows one to achieve the CQFI bound [63]. Remarkably, the projectors $\Pi_\pm(x_0)$ saturate the CsFI for any fixed system parameters x_0 , implying that these optimal projectors are universal and the corresponding nonreciprocal pseudo-Hermitian sensor is basis robust [64].

In other words, even though the measurement outcomes $p_\pm(\theta, x_0)$ depend on the given choice of the basis $\Pi_\pm(x_0)$, this dependence vanishes in the expression for CsFI in Eq. (8), thus leading to $F_{\text{cl}}^{\text{PH}}(\theta) = F_{\text{max}}^{\text{PH}}(\theta)$ (see Supplemental Material [40] for details). For example, one can verify that the set $\{|0\rangle, |1\rangle\}$ is a particular choice of the eigenstates $|\nu_\pm\rangle$ in Eq. (14) for certain x_0 . Hence, by implementing sensing measurements in this frame already allows one to saturate the CsFI.

On the other hand, to assess the impact of *nonoptimal* measurements on the CsFI, we consider projections onto

$$\Pi(\gamma, \phi) = |\chi(\gamma, \phi)\rangle\langle\chi(\gamma, \phi)|_\eta, \quad \text{and} \quad \mathbb{1} - \Pi(\gamma, \phi), \quad (15)$$

where $|\chi(\gamma, \phi)\rangle = \cos \gamma |\nu_+\rangle + e^{i\phi} \sin \gamma |\nu_-\rangle$ denotes a generic state rotated away from the optimal basis in Eq. (14). Substituting this form into Eq. (8) with $\phi = \pi/2$ and taking, without loss of generality, the projectors' reference point x_0 to coincide with the parameter point of the probe state $|\psi_t\rangle$, the CsFI reduces to the “suppressed” form

$$F_{\text{cl}}^{\text{PH}}(\theta, \gamma) = F_{\text{max}}^{\text{PH}}(\theta) \cos^2 2\gamma, \quad (16)$$

as shown in Fig. 1. The general case is discussed in [40].

In addition to nonreciprocal systems, another class of pseudo-Hermitian sensors, namely, $\mathcal{PT}$ -symmetric systems, is often considered in the context of enhanced sensing. Accordingly, in the Appendix, we analyze these systems, deriving the upper bound of their CQFI and presenting the explicit form of the optimal projectors that saturate this limit.

Conclusions—In summary, we developed a covariant formulation of QFI for pure states evolving under pseudo-Hermitian Hamiltonians, properly accounting for the metric-modified geometry of the underlying Hilbert space. We also uncovered a duality between the standard and covariant QFI, which provides a natural upper bound for the latter; and explicit criteria for when pseudo-Hermitian systems can outperform Hermitian counterparts in quantum

sensing. Furthermore, this duality allows to define a covariant symmetric logarithmic derivative operator, whose eigenbasis determines the optimal projectors that saturate the covariant QFI bound. Recent advances in state measurements on metric-dependent Hilbert spaces [54–56] make our results directly relevant for experiments, thus opening a path toward practically attaining ultimate precision in pseudo-Hermitian quantum sensors operating at or in close vicinity of EPs.

Acknowledgments—I. I. A. acknowledges support from the Air Force Office of Scientific Research (AFOSR) Award No. FA8655-24-1-7376, and from the Ministry of Education, Youth and Sports of the Czech Republic Grant OP JAC No. CZ.02.01.01/00/22_008/0004596. F. N. is supported in part by the Japan Science and Technology Agency (JST) via the CREST Quantum Frontiers program Grant No. JPMJCR24I2, the Quantum Leap Flagship Program (Q-LEAP), the Moonshot R&D Grant No. JPMJMS2061], and the Office of Naval Research (ONR) Global (via Grant No. N62909-23-1-2074). Ş. K. Ö. acknowledges support of the Air Force Office of Scientific Research (AFOSR) Multidisciplinary University Research Initiative (MURI) Award on Programmable Systems with Non-Hermitian Quantum Dynamics (Grant No. FA9550-21-1-0202).

Data availability—No data were created or analyzed in this study.

-
- [1] T. Kato, *Perturbation Theory for Linear Operators*, Classics in Mathematics (Springer, Berlin, 1995).
 - [2] J. Wiersig, Enhancing the sensitivity of frequency and energy splitting detection by using exceptional points: Application to microcavity sensors for single-particle detection, *Phys. Rev. Lett.* **112**, 203901 (2014).
 - [3] J. Wiersig, Sensors operating at exceptional points: General theory, *Phys. Rev. A* **93**, 033809 (2016).
 - [4] Ş. K. Özdemir, S. Rotter, F. Nori, and L. Yang, Parity-time symmetry and exceptional points in photonics, *Nat. Mater.* **18**, 783 (2019).
 - [5] Z.-P. Liu, J. Zhang, Ş. K. Özdemir, B. Peng, H. Jing, X.-Y. Lü, C.-W. Li, L. Yang, F. Nori, and Y.-X. Liu, Metrology with $\mathcal{PT}$ -symmetric cavities: Enhanced sensitivity near the $\mathcal{PT}$ -phase transition, *Phys. Rev. Lett.* **117**, 110802 (2016).
 - [6] W. Chen, Ş. K. Özdemir, G. Zhao, J. Wiersig, and L. Yang, Exceptional points enhance sensing in an optical microcavity, *Nature (London)* **548**, 192 (2017).
 - [7] H. Hodaie, A. U. Hassan, S. Wittek, H. Garcia-Gracia, R. El-Ganainy, D. N. Christodoulides, and M. Khajavikhan, Enhanced sensitivity at higher-order exceptional points, *Nature (London)* **548**, 187 (2017).
 - [8] H.-K. Lau and A. A. Clerk, Fundamental limits and non-reciprocal approaches in non-Hermitian quantum sensing, *Nat. Commun.* **9**, 4320 (2018).

- [9] Y.-H. Lai, Y.-K. Lu, M.-G. Suh, Z. Yuan, and K. Vahala, Observation of the exceptional-point-enhanced Sagnac effect, *Nature (London)* **576**, 65 (2019).
- [10] L. Bao, B. Qi, D. Dong, and F. Nori, Fundamental limits for reciprocal and nonreciprocal non-Hermitian quantum sensing, *Phys. Rev. A* **103**, 042418 (2021).
- [11] L. Bao, B. Qi, F. Nori, and D. Dong, Exponential sensitivity revival of noisy non-Hermitian quantum sensing with two-photon drives, *Phys. Rev. Res.* **6**, 023216 (2024).
- [12] R. Kononchuk, J. Cai, F. Ellis, R. Thevamaran, and T. Kottos, Exceptional-point-based accelerometers with enhanced signal-to-noise ratio, *Nature (London)* **607**, 697 (2022).
- [13] S. Yu, Y. Meng, J.-S. Tang, X.-Y. Xu, Y.-T. Wang, P. Yin, Z.-J. Ke, W. Liu, Z.-P. Li, Y.-Z. Yang, G. Chen, Y.-J. Han, C.-F. Li, and G.-C. Guo, Experimental investigation of quantum $\mathcal{PT}$ -enhanced sensor, *Phys. Rev. Lett.* **125**, 240506 (2020).
- [14] M. Zhang, W. Sweeney, C. W. Hsu, L. Yang, A. D. Stone, and L. Jiang, Quantum noise theory of exceptional point amplifying sensors, *Phys. Rev. Lett.* **123**, 180501 (2019).
- [15] K. J. H. Peters and S. R. K. Rodriguez, Exceptional precision of a nonlinear optical sensor at a square-root singularity, *Phys. Rev. Lett.* **129**, 013901 (2022).
- [16] X. Yu, X. Zhao, L. Li, X.-M. Hu, X. Duan, H. Yuan, and C. Zhang, Toward Heisenberg scaling in non-Hermitian metrology at the quantum regime, *Sci. Adv.* **10**, eadk7616 (2024).
- [17] Q.-C. Wu, Y.-H. Zhou, T. Liu, Y.-H. Kang, Q.-P. Su, C.-P. Yang, and Z.-W. Zhou, Enhanced quantum sensing in time-modulated non-Hermitian systems, *Chin. J. Phys.* **98**, 1116 (2025).
- [18] J. Naikoo, R. W. Chhajlany, and A. Miranowicz, Enhanced quantum sensing with hybrid exceptional-diabolic singularities, *New J. Phys.* **27**, 064505 (2025).
- [19] Y. Chu, Y. Liu, H. Liu, and J. Cai, Quantum sensing with a single-qubit pseudo-Hermitian system, *Phys. Rev. Lett.* **124**, 020501 (2020).
- [20] L. Xiao, Y. Chu, Q. Lin, H. Lin, W. Yi, J. Cai, and P. Xue, Non-Hermitian sensing in the absence of exceptional points, *Phys. Rev. Lett.* **133**, 180801 (2024).
- [21] J. Zhang, B. Peng, Ş. K. Özdemir, K. Pichler, D. O. Krimer, G. Zhao, F. Nori, Y. X. Liu, S. Rotter, and L. Yang, A phonon laser operating at an exceptional point, *Nat. Photonics* **12**, 479 (2018).
- [22] W. Langbein, No exceptional precision of exceptional-point sensors, *Phys. Rev. A* **98**, 023805 (2018).
- [23] H. Wang, Y.-H. Lai, Z. Yuan, M.-G. Suh, and K. Vahala, Petermann-factor sensitivity limit near an exceptional point in a Brillouin ring laser gyroscope, *Nat. Commun.* **11**, 1610 (2020).
- [24] C. Chen, L. Jin, and R.-B. Liu, Sensitivity of parameter estimation near the exceptional point of a non-Hermitian system, *New J. Phys.* **21**, 083002 (2019).
- [25] H. Loughlin and V. Sudhir, Exceptional-point sensors offer no fundamental signal-to-noise ratio enhancement, *Phys. Rev. Lett.* **132**, 243601 (2024).
- [26] J. Naikoo, R. W. Chhajlany, and J. Kołodźński, Multiparameter estimation perspective on non-Hermitian singularity-enhanced sensing, *Phys. Rev. Lett.* **131**, 220801 (2023).
- [27] J.-X. Peng, B. Zhu, W. Zhang, and K. Zhang, Dissipative quantum Fisher information for a general Liouvillian parametrized process, *Phys. Rev. A* **109**, 012432 (2024).
- [28] X. Zheng and Y. D. Chong, Noise constraints for nonlinear exceptional point sensing, *Phys. Rev. Lett.* **134**, 133801 (2025).
- [29] J. Ma, X. Wang, C. Sun, and F. Nori, Quantum spin squeezing, *Phys. Rep.* **509**, 89 (2011).
- [30] W. Ding, X. Wang, and S. Chen, Fundamental sensitivity limits for non-Hermitian quantum sensors, *Phys. Rev. Lett.* **131**, 160801 (2023).
- [31] N. Zeng, T. Liu, K. Xia, Y.-R. Zhang, and F. Nori, Non-Hermitian sensing from the perspective of post-selected measurements, *Phys. Rev. Res.* **7**, 043219 (2025).
- [32] J. Naikoo, R. W. Chhajlany, J. Kołodźński, and A. Miranowicz, Metric formalism for precision assessment in non-Hermitian systems, [arXiv:2506.22571](https://arxiv.org/abs/2506.22571).
- [33] Pseudo-Hermitian Hamiltonians can be physically realized through several approaches. For instance, they can be implemented via the Naimark dilation method, where a pseudo-Hermitian system can be embedded into a higher-dimensional Hermitian system [34]. Alternatively, one can realize them through the postselection procedure, where by properly monitoring the environment of an open quantum system, one can effectively induce the continuous nonunitary dynamics [35,36].
- [34] U. Günther and B. F. Samsonov, Naimark-dilated $\mathcal{PT}$ -symmetric brachistochrone, *Phys. Rev. Lett.* **101**, 230404 (2008).
- [35] F. Minganti, A. Miranowicz, R. W. Chhajlany, I. I. Arkhipov, and F. Nori, Hybrid-Liouvillian formalism connecting exceptional points of non-Hermitian Hamiltonians and Liouvillians via postselection of quantum trajectories, *Phys. Rev. A* **101**, 062112 (2020).
- [36] I. I. Arkhipov, P. Lewalle, F. Nori, Ş. K. Özdemir, and K. B. Whaley, Topological state permutations in time-modulated non-Hermitian multiqubit systems with suppressed nonadiabatic transitions, *Phys. Rev. Res.* **7**, 033242 (2025).
- [37] A. Mostafazadeh, Pseudo-Hermitian representation of quantum mechanics, *Int. J. Geom. Methods Mod. Phys.* **07**, 1191 (2010).
- [38] C.-Y. Ju, A. Miranowicz, F. Minganti, C.-T. Chan, G.-Y. Chen, and F. Nori, Einstein's quantum elevator: Hermitization of non-Hermitian Hamiltonians via a generalized vielbein formalism, *Phys. Rev. Res.* **4**, 023070 (2022).
- [39] Note that the associated Hermitian Hamiltonian H^H is not unique given a metric η , since by redefining $S \rightarrow TS$, where T is some unitary, the metric remains invariant, though the Hermitian Hamiltonian is transformed as $H^H \rightarrow TH^HT^\dagger$. Since this unitary freedom corresponds to a gauge choice and therefore does not modify the underlying physics, one may, without loss of generality, choose T to be parameter independent.
- [40] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/qk5r-h851>, which also include Ref. [41] for detailed derivations of the expressions presented in the main text.
- [41] I. I. Arkhipov and F. Minganti, Emergent non-Hermitian localization phenomena in the synthetic space of zero-dimensional bosonic systems, *Phys. Rev. A* **107**, 012202 (2023).
- [42] M. Nakahara, *Geometry, Topology and Physics*, 2nd ed. (IOP Publishing, Bristol, 2003).

- [43] The covariant derivative with the metric-compatible connection in Eq. (2) satisfies the following relation [42]: $\partial_\theta \langle \psi | \psi \rangle_\eta = \langle D_\theta \psi | \psi \rangle_\eta + \langle \psi | D_\theta \psi \rangle_\eta$. In particular, when $\eta = I$, implying $\Gamma = 0$, the covariant derivative is reduced to the standard derivative $D_\theta = \partial_\theta$. As a result, the derivative of the scalar $\partial_\theta \langle \psi | \psi \rangle$ simplifies to the ordinary derivative of the inner product of the two state-vectors on the standard flat Hilbert space.
- [44] C. Helstrom, Minimum mean-squared error of estimates in quantum statistics, *Phys. Lett.* **25A**, 101 (1967).
- [45] S. L. Braunstein and C. M. Caves, Statistical distance and the geometry of quantum states, *Phys. Rev. Lett.* **72**, 3439 (1994).
- [46] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum metrology, *Phys. Rev. Lett.* **96**, 010401 (2006).
- [47] R. M. Wilcox, Exponential operators and parameter differentiation in quantum physics, *J. Math. Phys. (N.Y.)* **8**, 962 (1967).
- [48] S. Pang and T. A. Brun, Quantum metrology for a general Hamiltonian parameter, *Phys. Rev. A* **90**, 022117 (2014).
- [49] S. Boixo, A. Datta, M. J. Davis, S. T. Flammia, A. Shaji, and C. M. Caves, Quantum metrology: Dynamics versus entanglement, *Phys. Rev. Lett.* **101**, 040403 (2008).
- [50] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, *Nat. Photonics* **5**, 222 (2011).
- [51] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
- [52] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (Springer, Basel, 2011).
- [53] M. G. A. Paris, Quantum estimation for quantum technology, *Int. J. Quantum. Inform.* **07**, 125 (2009).
- [54] H. Zhang, K. Wang, L. Xiao, and P. Xue, Self-normal and biorthogonal dynamical quantum phase transitions in non-Hermitian quantum walks, *Light Sci. Appl.* **14**, 253 (2025).
- [55] X. Wang and G. Ma, Experimental measurement of non-Hermitian left eigenvectors, *Front. Phys.* **20**, 54202 (2025).
- [56] Y.-T. Wang, Z.-A. Wang, Z.-P. Li *et al.*, Experimental investigation of direct non-Hermitian measurement and uncertainty relation towards high-dimensional quantum domain, [arXiv:2407.05332](https://arxiv.org/abs/2407.05332).
- [57] S. Soleymani, Q. Zhong, M. Mokim, S. Rotter, R. El-Ganainy, and Ş. K. Özdemir, Chiral and degenerate perfect absorption on exceptional surfaces, *Nat. Commun.* **13**, 599 (2022).
- [58] Z. Zhang, Z. Xu, R. Huang *et al.*, Chirality-induced quantum non-reciprocity, *Nat. Photonics* **19**, 840 (2025).
- [59] T.-L. Wang, L.-N. Wu, W. Yang, G.-R. Jin, N. Lambert, and F. Nori, Quantum Fisher information as a signature of the superradiant quantum phase transition, *New J. Phys.* **16**, 063039 (2014).
- [60] S. Yin, J. Song, Y. Zhang, and S. Liu, Quantum Fisher information in quantum critical systems with topological characterization, *Phys. Rev. B* **100**, 184417 (2019).
- [61] Q. Guan and R. J. Lewis-Swan, Identifying and harnessing dynamical phase transitions for quantum-enhanced sensing, *Phys. Rev. Res.* **3**, 033199 (2021).
- [62] V. Montenegro, C. Mukhopadhyay, R. Yousefjani, S. Sarkar, U. Mishra, M. G. Paris, and A. Bayat, Review: Quantum metrology and sensing with many-body systems, *Phys. Rep.* **1134**, 1 (2025).
- [63] Q. Liu, Z. Hu, H. Yuan, and Y. Yang, Fully-optimized quantum metrology: Framework, tools, and applications, *Adv. Quantum Technol.* **7**, 2400094 (2024).
- [64] We note that in evaluating the probabilities $p_\pm(\theta) = \langle \psi_t(\theta) | \Pi_\pm(x_0) | \psi_t(\theta) \rangle_\eta$, the inner product involves states defined, in general, at different points in parameter space. Consequently, the states $|\nu(x_0)\rangle$ and $|\psi_t\rangle$ may correspond to distinct local metrics. This ambiguity can be resolved either by parallel transporting one state to the parameter location of the other, or by flattening the η -deformed Hilbert space. Here we adopt the latter approach and discuss it in Supplemental Material [40].
- [65] C.-Y. Ju, A. Miranowicz, G.-Y. Chen, and F. Nori, Non-Hermitian Hamiltonians and no-go theorems in quantum information, *Phys. Rev. A* **100**, 062118 (2019).

End Matter

Appendix: Example 2. $\mathcal{PT}$ -symmetric systems—Here we analyze another well-known type of pseudo-Hermitian systems, namely, $\mathcal{PT}$ -symmetric systems.

Specifically, we consider a $\mathcal{PT}$ -symmetric Hamiltonian of the form

$$H^{\mathcal{PT}} = \begin{pmatrix} re^{i\tau} & s \\ s & re^{-i\tau} \end{pmatrix}, \quad r, \tau, s \in \mathbb{R}. \quad (\text{A1})$$

Such Hamiltonian can describe, e.g., a coupled optical dimer with loss and gain [4]. The $\mathcal{PT}$ -symmetry condition is satisfied by identifying the parity operator with the Pauli matrix, i.e., $\mathcal{P} = \sigma_x$, and $\mathcal{T}$ with complex conjugation, such that $[H^{\mathcal{PT}}, \mathcal{PT}] = 0$. This Hamiltonian undergoes a spontaneous $\mathcal{PT}$ -symmetry breaking at the EP defined with the relation $s_{\text{EP}} = r \sin \tau$. Such that when $s > r \sin \tau$ the $H^{\mathcal{PT}}$

is in the exact $\mathcal{PT}$ -phase, meaning that its spectrum is real. For $s < r \sin \tau$, on the other hand, the spectrum becomes complex, implying that the Hamiltonian is in the broken $\mathcal{PT}$ -phase.

The metric operator η for this $\mathcal{PT}$ -symmetric Hamiltonian has been already calculated in Ref. [65], which reads,

$$\eta = \frac{1}{\cos \alpha} \begin{pmatrix} 1 & -i \sin \alpha \\ i \sin \alpha & 1 \end{pmatrix}, \quad \alpha = \arcsin \left(\frac{r}{s} \sin \tau \right). \quad (\text{A2})$$

By diagonalizing the metric $\eta = U \Lambda U^\dagger$, we find

$$\Lambda = \text{diag} \left[\sec \alpha + \sqrt{\sec^2 \alpha - 1}, \sec \alpha - \sqrt{\sec^2 \alpha - 1} \right], \quad (\text{A3})$$

and the corresponding unitary

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}. \quad (\text{A4})$$

Knowing the metric eigenspace, one can straightforwardly find the associated Hermitian Hamiltonian H^H :

$$H^H = \begin{pmatrix} r \cos \tau & i\sqrt{s^2 - r^2 \sin^2 \tau} \\ -i\sqrt{s^2 - r^2 \sin^2 \tau} & r \cos \tau \end{pmatrix}. \quad (\text{A5})$$

Evidently, the Hamiltonian H^H in Eq. (A5) is Hermitian as long as the H^{PT} is in the unbroken PT -phase.

Having established the H^H , one can now determine the upper bound on the CQFI for a linearly perturbed pseudo-Hermitian system described by the perturbed Hamiltonian $H^{PT} \rightarrow H^{PT} + \theta \sigma_x$, where the symmetric coupling is modified by θ as $s \rightarrow s + \theta$. For sufficiently short evolution times $t \ll 1$, the upper bound attains the form

$$F_c^{PT}(\theta) \leq F_{\max}^{PT}(\theta) = \frac{4(s + \theta)^2}{(s + \theta)^2 - r^2 \sin^2 \tau} t^2. \quad (\text{A6})$$

Again, the CQFI bound diverges when the system approaches the EP (the denominator becomes zero at the EP), implying the system undergoes a spectral phase transition corresponding to spontaneous PT -symmetry breaking. Importantly, the linear perturbations in the PT -symmetric systems are equivalent to the nonlinear perturbations in their Hermitian counterparts. Hence, the PT -symmetric sensors can exhibit enhanced sensing supremacy over the Hermitian systems when the estimated parameter is encoded via the linear perturbations.

We now turn to the evaluation of the optimal projectors for this PT -symmetric sensor, which saturates the CsFI to the CQFI bound given in Eq. (A6). As in Example 1b in the main text, we focus on the time-evolving state starting from the vacuum, i.e., $|\xi_t\rangle = \exp(-iH^{PT}t)|0\rangle$, or more specifically,

$$|\xi_t\rangle = S^{-1}(\cos \Omega t |0\rangle - \sin \Omega t |1\rangle), \quad (\text{A7})$$

where $\Omega = \sqrt{(s + \theta)^2 - r^2 \sin^2 \tau}$, and the matrix S , which is the similarity transformation between Hamiltonians in Eqs. (A1) and (A5), explicitly reads as

$$S = \sqrt{\Lambda} U^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} i\kappa_+ & \kappa_+ \\ -i\kappa_- & \kappa_- \end{pmatrix}, \quad (\text{A8})$$

with $\kappa_\pm = \sqrt{\sec \alpha \pm \tan \alpha}$, and $\alpha = \arcsin[(r/s) \sin \tau]$.

One can readily check that for the state $|\xi_t\rangle$ the CQFI attains its upper bound, i.e.,

$$F_c^{PT}(\xi_t) = F_{\max}^{PT}(\theta). \quad (\text{A9})$$

Accordingly, the optimal projectors follow directly from the eigenstates of the covariant SLD operator L_c defined by the state $|\xi_t\rangle$ in Eq. (7), and which read,

$$|\mu_\pm\rangle = S^{-1}(w_\pm |0\rangle \pm w_\mp |1\rangle)/\sqrt{2}, \quad \langle \mu_i | \mu_j \rangle_\eta = \delta_{ij}, \quad (\text{A10})$$

where $w_\pm = \cos \Omega t \pm \sin \Omega t$.

Clearly, the eigenbasis of L_c depends on the parameter θ . Nonetheless, analogous to Example 1b, one finds that by fixing the eigenbasis $|\mu_\pm(x_0)\rangle$ at an arbitrary reference point $x_0 = (r_0, \tau_0, s_0, \theta_0, t_0)$ in the exact- PT phase, and by measuring the outcomes

$$p_\pm(\theta) = \langle \xi_t(\theta) | \Pi_\pm(x_0) | \xi_t(\theta) \rangle_\eta, \quad (\text{A11})$$

with projectors

$$\Pi_\pm(x_0) = |\mu_\pm(x_0)\rangle \langle \mu_\pm(x_0)|_\eta, \quad (\text{A12})$$

one can always saturate the corresponding CsFI to the CQFI bound, i.e., $F_{\text{cl}}^{PT}(\theta) = F_{\max}^{PT}(\theta)$. Indeed, by combining Eqs. (A18) and (A12) one obtains

$$p_\pm(\theta) = \frac{1 \pm \sin 2(\Omega_0 t_0 - \Omega t)}{2}. \quad (\text{A13})$$

Inserting now Eq. (A13) into Eq. (8), one arrives at

$$F_{\text{cl}}^{PT}(\theta) = \frac{4(s + \theta)^2}{(s + \theta)^2 - r^2 \sin^2 \tau} t^2 = F_{\max}^{PT}(\theta), \quad (\text{A14})$$

thus confirming that the CsFI saturates the CQFI bound, independent of the chosen reference point x_0 . Note that in the calculations above it is important to properly handle the metric-induced inner product for two states at different system parameter points, where the local metric may differ. We provide further details on this matter in Sec. VII of Supplemental Material [40].

To examine the effect of nonoptimal measurements in the PT -symmetric system, we again use projectors rotated relative to the optimal ones, i.e.,

$$\Pi^0(\gamma, \phi) = |\chi(\gamma, \phi)\rangle \langle \chi(\gamma, \phi)|_\eta, \quad (\text{A15})$$

[cf. Eq. (15)], where the state $|\chi\rangle$ is now realized in the SLD basis in Eq. (A10) again at some arbitrary reference point x_0 , namely,

$$|\chi(\gamma, \phi)\rangle = \cos \gamma |\mu_+(x_0)\rangle + e^{i\phi} \sin \gamma |\mu_-(x_0)\rangle. \quad (\text{A16})$$

Similar to Example 1b, for simplicity, hereafter we set $\phi = \pi/2$.

Now, to obtain the CsFI, we need to calculate the probability of the projection measurements

$$p(\theta) = \langle \xi_t(\theta) | \Pi^0(\gamma) | \xi_t(\theta) \rangle_\eta. \quad (\text{A17})$$

By combining Eqs. (8) and (A15)–(A17), one obtains

$$p(\theta) = \frac{1 + \cos(2\gamma) \sin 2(\Omega_0 t_0 - \Omega t)}{2}. \quad (\text{A18})$$

As a result the CsFI attains the form

$$F_{\text{cl}}^{\mathcal{PT}}(\theta, \gamma) = F_{\text{max}}^{\mathcal{PT}}(\theta) \frac{\cos^2 2\gamma \cos^2 2(\Omega_0 t_0 - \Omega t)}{1 - \cos^2 2\gamma \sin^2 2(\Omega_0 t_0 - \Omega t)}. \quad (\text{A19})$$

Evidently, when the reference point x_0 is chosen to coincide with the parameter space point of the probe state $|\xi_t\rangle$, the expression in Eq. (A19) simplifies to

$$F_{\text{cl}}^{\mathcal{PT}}(\theta, \gamma) = F_{\text{max}}^{\mathcal{PT}}(\theta) \cos^2 2\gamma. \quad (\text{A20})$$

We illustrate the impact of reference-point choice x_0 for different (nonoptimal) projectors on the corresponding CsFI in Fig. 2.

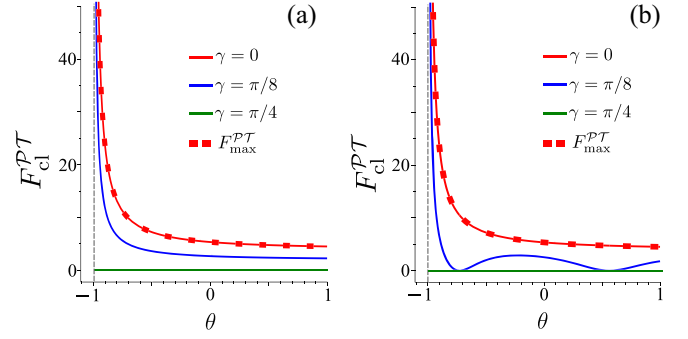


FIG. 2. Covariant classical Fisher information in the $\mathcal{PT}$ -symmetric system, obtained from different measurement projectors: (a) when the reference point x_0 coincides with the parameter point of the probe state $|\xi_t\rangle$, and (b) when x_0 differs from the probe-state parameter point. The parameter γ controls the deviation of the measurement basis $|\chi(\gamma)\rangle$ from the optimal axis defined by the eigenbasis of the covariant SLD operator. The upper bound of the covariant quantum Fisher information for each case is shown by a red dotted curve. In both panels, the vertical dash gray line corresponds to the value of $\theta = \theta_{\text{EP}}$, when the corresponding sensor is at the exceptional point.