

Analytic Discrete Self-Similar Solutions of Einstein-Klein-Gordon at Large D

Christian Ecker^{1,*}, Florian Ecker^{2,†} and Daniel Grumiller^{2,‡}

¹*Institute for Theoretical Physics, Goethe University, 60438 Frankfurt am Main, Germany*

²*Institute for Theoretical Physics, TU Wien, Wiedner Hauptstrasse 8–10/136, A-1040 Vienna, Austria*



(Received 29 January 2026; accepted 22 April 2026; published 12 May 2026)

Discretely self-similar solutions govern critical collapse and have been known only numerically since Choptuik's pioneering work. Using the large- D expansion, where D is the spacetime dimension, we construct an infinite family of analytic solutions of the Einstein-massless-Klein-Gordon equations. In this limit, the field equations simplify drastically, and the solutions are encoded in a single function of time. We characterize their structure and compare them with numerical critical solutions at finite D , identifying both universal features and effects specific to large D .

DOI: 10.1103/qgl5-513t

Introduction—Black hole formation is a dramatic physical process that is hard to describe using only paper and pencil. Even imposing spherical symmetry and restricting to Einstein gravity minimally coupled to a scalar field does not simplify the resulting PDEs sufficiently to permit analytic solutions in closed form [1–4].

Choptuik constructed numerically the first solutions that can be arbitrarily close to the threshold of black hole formation [5] and discovered that they exhibit discrete self-similarity (DSS). The symmetry enhancement of the critical solution resembles the emergent conformal symmetry observed in physical systems at a second-order phase transition, see, e.g., [6–8] for comparisons between critical collapse and other physical systems.

A related property is the existence of a self-similar horizon (SSH), i.e., a null hypersurface where the lattice vector ∂_τ associated with DSS, $\tau \rightarrow \tau + \Delta$ with echoing period Δ , becomes null. The SSH is an ingoing null hypersurface and terminates in the center in a naked singularity. Emanating from the latter is an outgoing Cauchy horizon. See Fig. 1 for a sketch of the past patch of a critical solution with DSS. This means one can interpret the critical solution as (conformal to) a spacetime crystal where the lattice vector ∂_τ is timelike in the region between center and the SSH (called past region), spacelike between the SSH and the Cauchy horizon (called outer region), and lightlike on the SSH.

This emergent discrete symmetry of the critical solution might, in principle, help to find it in closed form. Nevertheless, so far, the best one can do is to numerically construct the critical solution directly by imposing DSS as an input [9–11]. Since Einstein gravity has no dimensionless small parameter, there seems to be no obvious way to improve the situation.

However, if we allow deviations of the dimension D from four to arbitrarily large values, we can use $1/D$ as a small dimensionless parameter [12]. While it has been suggested in the past to exploit the large- D expansion to understand critical collapse [13–15], so far, the critical DSS solution seemed out of reach.

Numerically, going to large D is problematic because radial gradients get steeper with increasing dimension [16,17]. This numerical drawback has been turned into an advantage in the past to get analytical expressions for large- D observables [12,15,18–21]. Our main goal is to show that the same is true for DSS solutions.

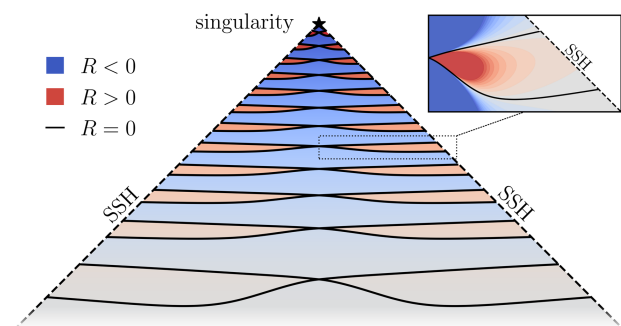


FIG. 1. Illustration of the past patch of the Choptuik spacetime with DSS. The closer one moves towards the top, the more rapidly the Ricci scalar oscillates until the singularity is reached. The inset zooms into one of the fundamental domains to highlight the Ricci contours, i.e., the magnitude of the Ricci scalar is bigger in the center and smaller at the SSH.

*Contact author: ecker@itp.uni-frankfurt.de

†Contact author: fecker@hep.itp.tuwien.ac.at

‡Contact author: grumil@hep.itp.tuwien.ac.at

In the present work, we explicitly construct an infinite family of DSS solutions of spherically symmetric Einstein gravity in D spacetime dimensions, minimally coupled to a massless scalar field ψ .

By construction, these solutions turn out to be regular between the center and the SSH and have a naked singularity in the center at the future end point of the SSH, just like for Choptuik's critical solution. We confront these solutions with numerical data to highlight the similarities and differences and show how to systematically improve features of the leading-order (LO) solutions by including next-to-leading-order (NLO) and next-to-next-to-leading-order (NNLO) corrections.

Field equations—Generalizing Choptuik's Eqs. (3)–(5) in [5] to arbitrary spacetime dimensions $D > 3$ yields

$$\frac{\partial_r a}{a} = \frac{(D-3)(1-a^2)}{2r} + \frac{r}{2(D-2)}(\hat{\Pi}^2 + \hat{\Phi}^2), \quad (1a)$$

$$\frac{\partial_r \alpha}{\alpha} = \frac{\partial_r a}{a} + \frac{(D-3)(a^2-1)}{r}, \quad (1b)$$

$$\partial_t \hat{\Phi} = \partial_r \left(\frac{\alpha}{a} \hat{\Pi} \right) \quad \partial_t \hat{\Pi} = \frac{1}{r^{D-2}} \partial_r \left(r^{D-2} \frac{\alpha}{a} \hat{\Phi} \right), \quad (1c)$$

where the D -dimensional metric $ds^2 = -\alpha^2(t, r)dt^2 + a^2(t, r)dr^2 + r^2 d^2\Omega_{S^{D-2}}$ is spherically symmetric, i.e., $d^2\Omega_{S^{D-2}}$ is the metric of the round unit S^{D-2} . The quantities $\hat{\Phi} = \partial_r \psi$, $\hat{\Pi} = (a/\alpha)\partial_t \psi$ are suitably chosen first derivatives of the scalar field ψ . (We use units $c = 8\pi G = 1$, which differs from Choptuik's choice $c = G = 1$.)

The coordinate system above is inconvenient for our purposes. Using the coordinate transformation [11] $\tau = -\ln(-t)$, $x = -r/t$ together with the redefinitions [22]

$$\Omega = \frac{(D-3)(a^2-1)}{x^2} \quad f = \frac{\alpha}{a} \quad \epsilon = \frac{1}{D-1} \quad (2)$$

the field equations above read

$$x\partial_x \Omega + x^2 \Omega^2 = \frac{1}{\epsilon}((1-2\epsilon)\Pi^2 - \Omega) + \Omega(\Pi^2 x^2 + x^4 \Psi^2) + \left(\frac{1}{\epsilon} - 2\right)x^2 \Psi^2, \quad (3a)$$

$$\frac{1}{x} \partial_x \ln f = \Omega, \quad (3b)$$

$$(\partial_\tau + x\partial_x)\Pi = fx\partial_x \Psi + f\left(\frac{1}{\epsilon} + x^2 \Omega\right)\Psi - \Pi, \quad (3c)$$

$$(\partial_\tau + x\partial_x)\Psi = f\frac{1}{x}\partial_x \Pi + f\Omega\Pi - 2\Psi, \quad (3d)$$

where we introduced the first-order matter variables

$$\Pi = \sqrt{\frac{\epsilon}{1-\epsilon}} \frac{1}{f} (\partial_\tau + x\partial_x)\psi \quad \Psi = \sqrt{\frac{\epsilon}{1-\epsilon}} \frac{1}{x} \partial_x \psi. \quad (4)$$

The DSS property means that the four free functions that characterize our solutions, Ω , f , Π , Ψ , are periodic in τ with some (echoing) period Δ . The D -dimensional metric in the gauge above is given by

$$ds^2 = e^{-2\tau} \left(1 + \frac{\epsilon x^2 \Omega}{1-2\epsilon} \right) ((x^2 - f^2)d\tau^2 - 2x d\tau dx + dx^2) + e^{-2\tau} x^2 d\Omega_{S^{D-2}}^2, \quad (5)$$

yielding a Ricci scalar $R = -[(1/\epsilon) + 1]e^{2\tau}(\Pi^2 - x^2 \Psi^2)a^{-2}$ that grows like $(1/\epsilon)$ for large dimensions.

Assuming the SSH conditions $f(\tau, x=1) = 1$, $f \geq x \geq 0$, any solution to (3) must obey the inequalities

$$0 \leq \Omega < \infty \quad 0 \leq x \leq f \leq 1. \quad (6)$$

Our goal is to solve the Eqs. (3) in an expansion of the dimension parameter $0 < \epsilon \ll 1$.

Leading order solution at large D —We expand all fields as Taylor series in ϵ , e.g., $\Omega = \Omega_{\text{LO}} + \epsilon \Omega_{\text{NLO}} + \epsilon^2 \Omega_{\text{NNLO}} + \mathcal{O}(\epsilon^3)$ and analogously for the SSH function f and the matter function Π . By contrast, as explained below, the other matter function Ψ starts at one order higher, $\Psi = \epsilon \Psi_{\text{LO}} + \epsilon^2 \Psi_{\text{NLO}} + \epsilon^3 \Psi_{\text{NNLO}} + \mathcal{O}(\epsilon^4)$.

All coefficients of Ω are determined algebraically from (3a) in terms of Π . This is so because, to LO, the terms proportional to $1/\epsilon$ in the first line on the right-hand side of (3a) must cancel, i.e.,

$$\Omega_{\text{LO}} = \Pi_{\text{LO}}^2. \quad (7)$$

This mechanism persists for all subleading orders.

Moreover, Ψ is determined without integration from Π and f using (3c). Similarly to before, the reason for this is the $1/\epsilon$ coefficient on the right-hand side, yielding

$$\Psi_{\text{LO}} = \frac{D_1 \Pi_{\text{LO}}}{f_{\text{LO}}} \quad D_n := \partial_\tau + x\partial_x + n. \quad (8)$$

Again, this mechanism persists for all subleading orders.

We only have two remaining PDEs that need to be solved. While in some general coordinate system these are still coupled, nonlinear PDEs and hard to solve, in the gauge (5) they decouple at each order.

Indeed, (3d) to LO simplifies to a nonlinear first-order PDE for the matter function Π_{LO} ,

$$\partial_x \Pi_{\text{LO}} = -x \Omega_{\text{LO}} \Pi_{\text{LO}} = -x \Pi_{\text{LO}}^3 \quad (9)$$

solved by

$$\Pi_{\text{LO}} = \frac{\beta(\tau)}{\sqrt{1 + \beta^2(\tau)x^2}} \quad (10)$$

with an arbitrary integration function $\beta(\tau)$ [23]. We have adapted this integration function such that Π_{LO} has no poles for any non-negative x . Also, we have chosen, without loss of generality, a branch of the square root function where Π_{LO} is positive for positive β .

The final equation we need to solve is (3b), which, to LO, reduces to

$$\partial_x \ln f_{\text{LO}} = x \Pi_{\text{LO}}^2 = \frac{x \beta^2(\tau)}{1 + \beta^2(\tau)x^2}. \quad (11)$$

We solve it with the boundary condition

$$\text{SSH: } f(\tau, x = 1) = 1 \quad (12)$$

that guarantees an SSH at $x = 1$, see, e.g., the discussion in [11]. Thus, we have no further free integration function and obtain

$$f_{\text{LO}} = \sqrt{\frac{1 + \beta^2(\tau)x^2}{1 + \beta^2(\tau)}}. \quad (13)$$

CSS and DSS solutions—Since $\beta(\tau)$ currently is unconstrained, there are infinitely many LO solutions at large D . If we choose $\beta = \text{const}$, we get continuously self-similar (CSS) solutions. They differ from the CSS solutions of Clark and Pimentel [25], see the Supplemental Material [24].

For periodic $\beta(\tau + \Delta) = \beta(\tau)$, we get DSS solutions. A prototypical choice is a function with zero average, a single zero in the interval $\tau \in [0, \Delta/2)$, and $\beta(\tau + \Delta/2) = -\beta(\tau)$. In this work we focus on such functions.

Thus, we have achieved our main goal, namely, to find explicit (and remarkably simple) expressions for DSS solutions of the Einstein-massless-Klein-Gordon model, to LO at large D , Eqs. (7)–(13). We address now some of the properties and shortcomings of these solutions.

Properties of leading order solution—The solutions (7)–(13) are regular in the whole past region of Fig. 1 and can also be extended into the outer region, until $x \rightarrow \infty$. To go beyond, one would need to change to a different coordinate system that allows one to reach the Cauchy horizon. This issue was resolved in [11], and the same resolution works here as well, so we focus in the remainder of our work on the past region where we can use the convenient coordinate system (5) for any $\tau \in \mathbb{R}$ and any $x \in [0, 1]$.

Another characteristic property of DSS solutions are the NEC lines [26], which are loci where the null energy condition saturates. The NEC lines coincide with lines of vanishing Ricci scalar and therefore also encode local geometry properties of the DSS solution. They are given by solutions to

$$\text{NEC-lines: } \Pi = \pm x \Psi \quad (14)$$

and for our solutions (7)–(13) they emanate from NEC vertices located at single zeros of the function $\beta(\tau)$.

A drawback of the $\mathcal{O}(1)$ approximation to (14) is that the NEC lines remain straight horizontal lines in the chosen coordinates until they reach the SSH, without any NEC angle. If we “cheat” and insert $\mathcal{O}(1)$ on the left-hand side and the LO $\mathcal{O}(\epsilon)$ on the right-hand side of (14), we get the correct NEC angle, but the NEC lines remain straight lines. By contrast, at any finite D the NEC lines curve downwards when emanating from the NEC vertices, see Fig. 3 in [26]. Another aspect quite different from finite D is that there is no unique DSS solution—any periodic function $\beta(\tau)$ with any echoing period Δ solves our equations of motion. Finally, another drawback of the LO approximation is that the maxima of the function f_{LO} , located at zeros of β , remain at $f = 1$ for all values of x . By contrast, at any finite D these maxima have a characteristic x profile [27].

Below, we show that adding NLO and NNLO corrections addresses all the issues above.

NLO solution—We obtain Ω_{NLO} from (3a) and Ψ_{NLO} from (3c),

$$\Omega_{\text{NLO}} = 2\Pi_{\text{LO}}\Pi_{\text{NLO}} - \omega_{\text{LO}}, \quad (15a)$$

$$\Psi_{\text{NLO}} = \frac{D_1 \Pi_{\text{NLO}}}{f_{\text{LO}}} - \left(x \partial_x + x^2 \Pi_{\text{LO}}^2 + \frac{f_{\text{NLO}}}{f_{\text{LO}}} \right) \Psi_{\text{LO}}, \quad (15b)$$

with $\omega_{\text{LO}} := 2\Pi_{\text{LO}}^2 - 2x^2 \Pi_{\text{LO}}^4$. The two remaining PDEs

$$\left(\frac{1}{x} \partial_x + 3\Pi_{\text{LO}}^2 \right) \Pi_{\text{NLO}} = \frac{D_2 \Psi_{\text{LO}}}{f_{\text{LO}}} + \Pi_{\text{LO}} \omega_{\text{LO}}, \quad (16a)$$

$$\left(\frac{1}{x} \partial_x - \Pi_{\text{LO}}^2 \right) f_{\text{NLO}} = 2f_{\text{LO}} \Pi_{\text{LO}} \Pi_{\text{NLO}} - f_{\text{LO}} \omega_{\text{LO}}, \quad (16b)$$

are solved by (prime denotes ∂_τ)

$$\Pi_{\text{NLO}} = \frac{4\beta^2 x^2 (1 + \beta^2)(\beta + \beta')^2 + p_1 \ln(1 + \beta^2 x^2)}{2\beta^3 (1 + \beta^2 x^2)^{5/2}}, \quad (17)$$

$$f_{\text{NLO}} = \frac{p_2 + p_3 \ln(1 + \beta^2) + p_4 \ln(1 + \beta^2 x^2)}{2\beta^4 (1 + \beta^2)^{3/2} (1 + \beta^2 x^2)^{3/2}}, \quad (18)$$

where p_n are polynomials in x^2 , $p_1 = (1 + \beta^2 x^2)[\beta^3(\beta'' - 4\beta') - \beta^2(3\beta'^2 + 2) + \beta(\beta'' - 5\beta') - 4\beta'^2]$, $p_2 = -(1 - x^2)\beta^2 \{p_1 + 4\beta^3 \beta' + 2\beta^2 \beta'^2 + 2\beta^2 x^2[\beta^4 + 4\beta^3 \beta' + \beta^2(1 + 2\beta'^2) + 2\beta\beta' + \beta'^2]\}$, $p_3 = p_1(1 + \beta^2 x^2)$, $p_4 = -p_1(1 + \beta^2)$.

Remarkably, both integration functions are fixed, the one for f_{NLO} by demanding the SSH condition (12) and the one for Π_{NLO} by redefining the LO integration function $\beta(\tau)$ such that $\Pi = \Pi_{\text{LO}} + \epsilon \Pi_{\text{NLO}} = \beta(\tau)$ at $x = 0$.

Echoing period at NLO—Since the exact solution and the LO solution obey the convexity conditions (6), we demand that also the NLO solution (15a)–(18) is consistent with them. Interestingly, this is not the case automatically. As we show now, resolving this issue can determine the echoing period Δ .

Consider an expansion of f near the SSH and additionally expand close to a zero of β ,

$$f_{\text{LO}} + \epsilon f_{\text{NLO}} = 1 - \epsilon(1-x)\beta(\beta'' + 3\beta') + \dots \quad (19)$$

If $\beta'' + 3\beta'$ does not vanish simultaneously with β , the subleading term in (19) is positive on one side of the zero of β and hence $f > 1$, thereby contradicting one of the convexity conditions (6).

Thus, at NLO, not all functions $\beta(\tau)$ are allowed anymore. However, all functions $\beta(\tau/\Delta)$ with period Δ can be converted into admissible functions by adjusting

$$\text{echoing period: } \Delta = \frac{|\beta''|}{3|\beta'|} \Big|_{\beta=0}. \quad (20)$$

The condition (20) guarantees $\partial_\tau^2 \beta(\tau/\Delta) + 3\partial_\tau \beta(\tau/\Delta)$ has a single zero at the zero of β so that $f \leq 1$ can be fulfilled. Moreover, it excludes simple functions like $\beta \propto \cos(2\pi\tau/\Delta)$ where the second (or first) derivative has zeros coincident with the zeros of β .

Example solution—In what follows, we analyze a specific DSS solution up to (N)NLO using

$$\beta(\tau) = \cos(2\pi\tau) + \frac{\sin(6\pi\tau)}{A} \quad A \approx 15.9476 \quad (21)$$

where the echoing period is fixed to $\Delta = 1$.

The choice (21) is motivated by simplicity: we normalized the amplitude of the leading Fourier mode to one, fixed its phase to a cosine mode, and allowed only one higher Fourier mode with orthogonal phase, a sine mode. Since we know from numerical solutions at finite D that the frequencies of higher modes are odd integer multiples of the fundamental frequency [27] we took thrice the fundamental frequency as argument of the sine mode. The amplitude $1/A$ of the second term is chosen to satisfy the condition (20) with $\Delta = 1$.

For each choice of β and each given order, there is a lower bound on the dimension D where the solution is (barely) viable; for the example (21) to NNLO this value is $D = 52$. For $D = 51$ there is a region near the SSH where $f > 1$, contradicting one of the convexity conditions (6), see Supplemental Material [24,28] and [29] for details.

Figure 2 displays all fields of the NNLO solution evaluated at $D = 300$ for the example (21). These plots can be compared and contrasted with the $D = 4$ plots in [11]: e.g., the shape of the function f (right upper graph) resembles the corresponding shape in $D = 4$, except that

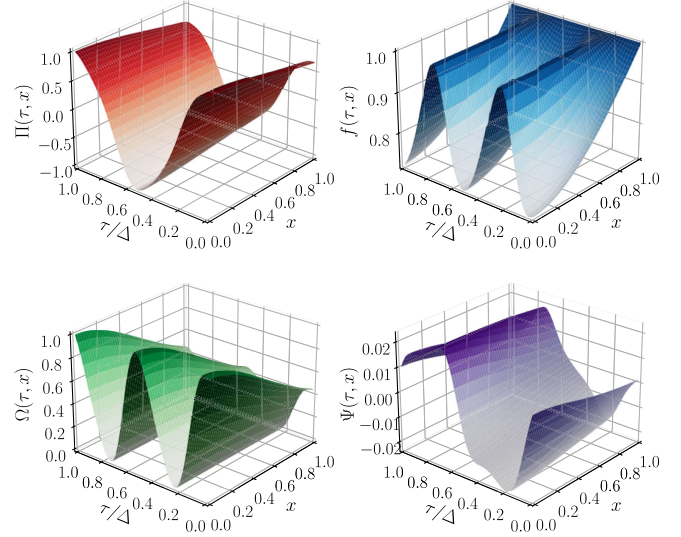


FIG. 2. NNLO solution for (21) with $D = 300$.

our value of the maximum decays only marginally from 1 near the SSH to slightly below 1 in the center, see (22) below, whereas in $D = 4$ it decays to about 0.456.

Figure 3 shows the NEC lines given by solutions to (14) for LO up to NNNLO for the same example. In the center, $x = 0$, all curves are close to each other but near the SSH there are differences between these curves. This suggests that the large- D expansion converges fast in the center and slowly at the SSH, see Fig. 1 in Supplemental Material [24]. The downward bending of the NEC lines observed in numerical simulations [26,27] happens not before NNLO.

Finally, to LO and NLO the maxima of f are at $f = 1$ for all values of x . To NNLO we get

$$\max f(\tau, x) = 1 - \frac{\epsilon^2}{4} (1 - x^4) (\beta'(\tau_0))^2 + \mathcal{O}(\epsilon^3), \quad (22)$$

where τ_0 is the NEC vertex time. The drop of the maximum of f from 1 to something proportional to $1/(D-1)^2$ is in qualitative agreement with numerical simulations at moderate values of D [27].

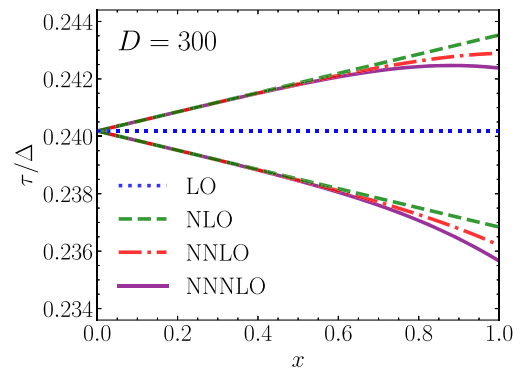


FIG. 3. NEC lines for (21) with $D = 300$.

Summary and outlook—We have provided a family of exact DSS solutions (7)–(13) of Einstein gravity in the large- D expansion and showed how to systematically improve the solution order by order in $1/D$. To all orders, the input is a single free function of time $\beta(\tau) \sim \beta(\tau + \Delta)$ with arbitrary echoing period Δ . We demonstrated that NLO corrections impose a consistency constraint (20) on Δ . For the example (21), we showed that several features of numerically obtained finite- D solutions are captured once NNLO corrections are included, in particular the decay of the maximum of the SSH function f from 1 at the SSH to a smaller central value (22) (see also Fig. 2), as well as the NEC lines in Fig. 3.

As in other applications of the large- D expansion in gravity [15,21], the resulting $1/D$ series is expected to be asymptotic rather than convergent. The main utility of the framework developed here is therefore to provide systematic analytic control over the structure of DSS critical solutions, with higher orders progressively improving the approximation at large but finite D .

Many rewarding questions remain open. Besides extending the perturbation theory past NNNLO, as outlined in the SM, it would be important to better understand the analytic structure of the $1/D$ expansion and whether it admits a finite radius of convergence or should instead be interpreted as an asymptotic series. The NEC lines in Fig. 3 up to NNNLO suggest fast convergence near the center but slow convergence near the SSH. It is also plausible that higher orders impose further consistency relations analogous to (20); if infinitely many exist, they would strongly constrain $\beta(\tau)$ and Δ , consistent with the numerical expectation that DSS solutions are unique at finite D [30,31]. A better quantitative match to numerical data at moderate D could be achieved by using more elaborate choices of $\beta(\tau)$ and by increasing the perturbative order.

Our discussion was restricted to the past patch (Fig. 1). As in [11], one may extend the solution through the outer patch to the future Cauchy horizon, and even study possible extensions beyond it.

Moreover, we focused on solutions with a regular SSH. If we drop that assumption we find another branch of solutions, to LO given by $f_{\text{LO}} = f(\tau)\sqrt{\beta^2(\tau)x^2 - 1}$, $\Omega_{\text{LO}} = \text{Eq. (7)}$, $\Pi_{\text{LO}} = \beta(\tau)/\sqrt{\beta^2(\tau)x^2 - 1}$, $\Psi_{\text{LO}} = \text{Eq. (8)}$, where we assume $\beta^2(\tau)x^2 > 1$, which excludes β having zeros. These solutions resemble the supercritical solutions above the threshold of black hole formation and contain a second arbitrary function, $f(\tau)$, besides $\beta(\tau)$. A detailed analysis of this branch would be interesting.

On the numerical side, it remains unclear how the echoing period Δ scales with D . Existing data [16,17,32] suggest $\Delta \rightarrow 0$ as $D \rightarrow \infty$, but the precise rate matters for the structure of the perturbative expansion [27]. Relatedly, semiclassical backreaction effects [33,34], problematic at finite D [35], may be consistently studied in the two-dimensional dilaton gravity formulation underlying the large- D limit [13].

Finally, we have not addressed the Choptuik exponent γ , which at finite D follows from linear perturbations around the DSS solution. Numerical and theoretical arguments suggest $\lim_{D \rightarrow \infty} \gamma = \frac{1}{2}$, hinting at a CSS limit [25]. These observations highlight that matching finite- D data requires insisting on DSS and expanding around the DSS solutions constructed here.

Acknowledgments—We thank Craig Clark for meeting with us and explaining in detail their work [25], Roberto Emparan for discussions on the large D limit of general relativity, Carsten Gundlach for sharing and explaining his vintage code allowing the numerical construction of the DSS solution in $D = 4$, and Maciej Maliborski for insightful feedback on our NEC angle discussion. D. G. is grateful to Peter Aichelburg and the other participants (Sascha Husa, Christiane Lechner, Michael Pürner, Jonathan Thornburg, and others) of the critical collapse seminar in 1999 for kindling his interest in this subject. C. E. acknowledges support by the DFG through the CRC-TR 211 “Strong-interaction matter under extreme conditions” Project No. 315477589–TRR 211. This work was supported by the Austrian Science Fund (FWF) [Grants DOI: 10.55776/P33789, 10.55776/P36619, 10.55776/PAT1871425].

Data availability—The data that support the findings of this article are openly available [29].

- [1] D. Christodoulou, *Commun. Math. Phys.* **105**, 337 (1986).
- [2] D. Christodoulou, *Commun. Math. Phys.* **106**, 587 (1986).
- [3] D. Christodoulou, *Commun. Math. Phys.* **109**, 591 (1987).
- [4] D. S. Goldwirth and T. Piran, *Phys. Rev. D* **36**, 3575 (1987).
- [5] M. W. Choptuik, *Phys. Rev. Lett.* **70**, 9 (1993).
- [6] T. Koike, T. Hara, and S. Adachi, *Phys. Rev. Lett.* **74**, 5170 (1995).
- [7] C. Gundlach, *Phys. Rep.* **376**, 339 (2003).
- [8] C. Gundlach and J. M. Martin-Garcia, *Living Rev. Relativity* **10**, 5 (2007).
- [9] C. Gundlach, *Phys. Rev. Lett.* **75**, 3214 (1995).
- [10] C. Gundlach, *Phys. Rev. D* **55**, 695 (1997).
- [11] J. M. Martin-Garcia and C. Gundlach, *Phys. Rev. D* **68**, 024011 (2003).
- [12] R. Emparan, R. Suzuki, and K. Tanabe, *J. High Energy Phys.* **06** (2013) 009.
- [13] R. Emparan, D. Grumiller, and K. Tanabe, *Phys. Rev. Lett.* **110**, 251102 (2013).
- [14] M. Rozali and B. Way, *J. High Energy Phys.* **11** (2018) 106.
- [15] R. Emparan and C. P. Herzog, *Rev. Mod. Phys.* **92**, 045005 (2020).
- [16] E. Sorkin and Y. Oren, *Phys. Rev. D* **71**, 124005 (2005).
- [17] J. Bland, B. Preston, M. Becker, G. Kunstatter, and V. Husain, *Classical Quantum Gravity* **22**, 5355 (2005).
- [18] R. Emparan and K. Tanabe, *Phys. Rev. D* **89**, 064028 (2014).
- [19] R. Emparan, R. Suzuki, and K. Tanabe, *Phys. Rev. Lett.* **115**, 091102 (2015).

- [20] R. Emparan, T. Shiromizu, R. Suzuki, K. Tanabe, and T. Tanaka, *J. High Energy Phys.* **06** (2015) 159.
- [21] R. Emparan, R. Suzuki, and K. Tanabe, *J. High Energy Phys.* **04** (2015) 085.
- [22] The idea to rescale a variable like $a^2 - 1$ by a linear factor in the dimension in a large- D context is due to Rozali and Way [14]. However, they also rescaled one of the coordinates, which makes the SSH inaccessible. Therefore, we do not follow their scaling regime but establish here a different one that allows a regular SSH in the large- D expansion.
- [23] The function $\beta(\tau)$ is equivalent to $\Pi(\tau, 0)$ and has a physical interpretation as Lorentz factor, $\beta = 1/\sqrt{1 - [\partial\psi_0(\tau)/\partial\tau]^2}$, where ψ_0 is the LO term in the scalar field, see Eq. (S2) in Supplemental Material [24].
- [24] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/qgl5-5l3t> for further details on CSS solutions, the reconstruction of the scalar field, NEC lines, extrema of the SSH function, and convexity conditions.
- [25] C. R. Clark and G. L. Pimentel, *J. High Energy Phys.* **11** (2025) 017.
- [26] C. Ecker, F. Ecker, and D. Grumiller, *Phys. Rev. D* **112**, L021505 (2025).
- [27] C. Ecker, F. Ecker, D. Grumiller, and T. Jechtl, [arXiv:2602.10185](https://arxiv.org/abs/2602.10185).
- [28] Achieving slightly lower dimensions is possible by generalizing the 2-mode ansatz (21). Allowing an arbitrary amplitude in the first term and a phase in the second term, we find that the phase needs to be close to zero and the optimal amplitude is close to 1.4. In that case, the lowest possible dimension compatible with the convexity conditions (6) turns out to be $D = 49$.
- [29] *Mathematica* notebook available at <https://github.com/EckrChristian/CritLargeD>.
- [30] C. Gundlach, *Adv. Theor. Math. Phys.* **2**, 1 (1998).
- [31] M. Reiterer and E. Trubowitz, *Commun. Math. Phys.* **368**, 143 (2019).
- [32] J. B. Bland, Dimension dependence of the critical phenomena in gravitational collapse of massless scalar field, Ph.D. thesis, University of Manitoba, 2006.
- [33] M. Tomašević and C.-H. Wu, [arXiv:2509.03584](https://arxiv.org/abs/2509.03584).
- [34] M. Tomašević and C.-H. Wu, [arXiv:2509.03587](https://arxiv.org/abs/2509.03587).
- [35] V. Frolov, P. Sutton, and A. Zelnikov, *Phys. Rev. D* **61**, 024021 (2000).