

Topological Response in Open Quantum Systems with Weak Symmetries

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In open quantum systems, the interaction of the system with its environment gives rise to two types of symmetry: a strong one, where the system's symmetry charge is conserved exactly, and a weak one, where the system can exchange symmetry charge with the environment but still preserve symmetry at the ensemble level. While generic open quantum systems feature weak symmetries only, the symmetry-protected topological response for bosonic (spin) systems has only been considered in the stricter setup with additional strong symmetries. Here, we address the generic case and demonstrate that weak symmetries alone can protect topological responses that distinguish different phases of matter. For bosonic systems, focusing on one-dimensional mixed states described by locally purifiable density operators, we propose a quantized response characterizing qualitatively distinct phases. It is detectable via the decay behavior of different string order parameters. We illustrate our general results through a noisy Affleck-Kennedy-Lieb-Tasaki model. In particular, we show that the coupling to the environment can induce a phase transition to a state protected by weak symmetries, without a pure-state or strong-symmetry analog.

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Introduction—Inevitable coupling to the environment transforms pure states into mixed-state ensembles, enriching the interplay between symmetry and topology. In particular, the notion of symmetry is refined into two classes, *weak* and *strong* symmetries [1], distinguished by whether the symmetry charge can be exchanged with the environment, or not. Considerable efforts have been made to pursue these new aspects, broadly divided into two categories. For fermionic topological phases [2–9] with weak symmetry [6], topological order parameters have been constructed in all symmetry classes [7,9]. Meanwhile, for bosonic (spin) systems, nontrivial symmetry-protected topological (SPT) phases have been established only under strict conditions that require the presence of at least one *strong symmetry* [10–16]. However, generic open quantum systems exhibit only *weak symmetries*, making topological signatures without fine-tuning in these bosonic systems challenging at first sight. A robust topological signature protected solely by weak symmetries is thus highly desirable, and could be detected on emerging experimental platforms, e.g., trapped ions [17–19], Rydberg atoms [20],

and superconducting circuits [21], which offer capabilities beyond traditional solid-state setups [17–26]. Specifically, these platforms enable the measurement of nonlocal observables, e.g., full counting statistics in optical lattices [22] and string order parameters in programmable quantum simulators [20,24].

In this Letter, we consider the more physical setup for bosonic systems with only weak symmetries. Focusing on one dimension, we rigorously demonstrate robust topological properties, characterized by quantized symmetry charge responses to twisted boundary conditions (or symmetry flux insertion). In particular, this quantized response defines topological invariants that extend beyond current frameworks, and are experimentally detectable via qualitatively distinct behaviors of string order parameters across different phases. This exact result is based on two ingredients: (i) a finite on-site weak symmetry group, where the quantized response is defined given a pair of commuting elements; and (ii) the tensor network ansatz [27–29], where mixed states are efficiently described by locally purified density operators (LPDOs) and the purifications are assumed to be short-range correlated. We illustrate these results in an exemplary mixed state resulting from the Affleck-Kennedy-Lieb-Tasaki (AKLT) state exposed to on-site weak-symmetry preserving noise channels. Remarkably, we observe a noise-driven transition to a mixed-state phase with quantized responses that cannot occur in previously studied systems with strong symmetries.

Quantized response in pure states—To set the stage for mixed states, we first recap relevant concepts of quantized

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response in pure SPT states [30–33] in the language of matrix product states (MPSs) [34–37]. Consider a topological state $|\psi\rangle$ of a one-dimensional spin chain protected by an on-site [38] finite-group symmetry G , i.e., $U_g|\psi\rangle \sim |\psi\rangle$ up to a phase factor. Here, $g \in G$ labels a group element, and $U_g = u_g^{\otimes N}$ with N being the system size. Recall the following MPS representation with finite bond dimension:

$$|\psi(X)\rangle = \sum_{\{i_j\}} \text{tr}(XA^{i_1}A^{i_2}\cdots A^{i_N})|i_1, i_2, \dots, i_N\rangle, \quad (1)$$

where i_j labels the local basis states at site j , A is a rank-3 tensor with A^{i_j} a matrix in the virtual space, and “tr” (“Tr”) denotes tracing over the virtual (physical) indices. We choose $X = \mathbb{1}$ (periodic boundary condition) to represent the state $|\psi\rangle \equiv |\psi(\mathbb{1})\rangle$, while other boundary conditions will be discussed later to probe the SPT order.

The characteristic feature of SPT states in the MPS representation is the symmetry transformation rule of the tensor A . In particular, A can be chosen to satisfy [37,41]

$$\sum_j u_g^{ij} A^j = e^{i\theta_g} V_g A^i V_g^\dagger. \quad (2)$$

Here, V_g forms a projective representation of the symmetry group, i.e., $V_{g_1}V_{g_2} = V_{g_1g_2}\omega(g_1, g_2)$, with ω being a phase factor [42]. The choice of the V_g and ω is not unique, due to the gauge transformations: $V_g \rightarrow V_g e^{i\phi(g)}$ and $\omega(g_1, g_2) \rightarrow \omega(g_1, g_2) e^{i\phi(g_1g_2) - i\phi(g_1) - i\phi(g_2)}$. This redundancy defines equivalence classes, classified by the group cohomology [31,32,42].

The transformation rule [Eq. (2)] ensures symmetry under U_g in the bulk for $|\psi(X)\rangle$,

$$U_g|\psi(X)\rangle = e^{iN\theta_g}|\psi(V_g^\dagger X V_g)\rangle, \quad (3)$$

while the boundary lacks this symmetry unless $V_g^\dagger X V_g \propto X$ (up to a phase factor) such as when $X = \mathbb{1}$. This property can be utilized to detect SPT phases through the g_2 -symmetry charge induced from a g_1 -twisted boundary condition, where g_1 and g_2 denote two commuting group elements hereafter. Specifically, taking $X = V_{g_1}$, the boundary is g_2 -symmetric since $V_{g_2}^\dagger V_{g_1} V_{g_2} = [\omega(g_1, g_2)/\omega(g_2, g_1)]V_{g_1}$, yielding

$$U_{g_2}|\psi(V_{g_1})\rangle = e^{iN\theta_{g_2}} \frac{\omega(g_1, g_2)}{\omega(g_2, g_1)} |\psi(V_{g_1})\rangle. \quad (4)$$

Physically, $|\psi(V_{g_1})\rangle$ is the g_1 -flux inserted MPS state [34,35]. The extensive phase factor $e^{iN\theta_{g_2}}$ represents the symmetry charge without flux insertion [Eq. (3)], while the ratio $\omega(g_1, g_2)/\omega(g_2, g_1)$ captures the additional quantized topological g_2 charge induced by the g_1 -flux insertion, i.e.,

$$\frac{\omega(g_1, g_2)}{\omega(g_2, g_1)} = \frac{\langle\psi(V_{g_1})|U_{g_2}|\psi(V_{g_1})\rangle}{\langle\psi|U_{g_2}|\psi\rangle}. \quad (5)$$

The response is quantized because the phase accumulates multiplicatively with each additional g_1 flux, and adding $|g_1|$ (the order of g_1 , i.e., $g_1^{|g_1|} = \mathbb{1}$) fluxes results in a trivial response [35].

Quantized response in mixed states—We now show that the above discussion extends to mixed states. We will focus on those described by LPDOs, with purifications admitting short-range correlated MPS representations [27–29],

$$|\Psi(X)\rangle = \sum_{\{i_j, a_j\}} \text{tr}(XA^{i_1 a_1} \cdots A^{i_N a_N})|i_1 a_1, \dots, i_N a_N\rangle. \quad (6)$$

Here, i_j and the additional index a_j label the physical and ancilla states at site j , respectively. Similar to the pure-state case, we introduce a boundary matrix X acting on the virtual space. For later convenience, we define the following graphical notation for $(A^{ia})_{\alpha\beta}$:

α —— β , where the red leg represents ancillas, and connecting legs of tensors represent the contraction of corresponding indices in later formulas. The system density matrix is obtained by tracing over the ancillas, i.e., $\rho(X) = \text{Tr}_a |\Psi(X)\rangle\langle\Psi(X)|$, with $\rho(\mathbb{1}) \equiv \rho$.

Symmetries in mixed states, unlike the pure-state case, bifurcate into two classes [1]: a symmetry g is weak for ρ if $[U_g, \rho] = 0$, and strong if $U_g \rho \sim \rho$ up to a phase factor. We consider symmetries that admit a local realization in the extended Hilbert space including ancillas, i.e., $U_g \otimes u_g^a = (u_g \otimes u_g^a)^{\otimes N}$ [12,14], representing the symmetry of the purified MPS. Similarly to Eq. (2), with a proper gauge choice, $u_g \otimes u_g^a$ acts on A^{ia} as

$$\begin{array}{c} u_g \quad u_g^a \\ \circ \quad \circ \\ | \quad | \\ \text{---} A \text{---} \end{array} = e^{i\theta_g} \begin{array}{c} \text{---} V_g \text{---} A \text{---} V_g^\dagger \text{---} \end{array}. \quad (7)$$

The key distinction between these two classes of symmetries lies in u_g^a : The strong one requires the ancilla to be g -charge neutral ($u_g^a = \mathbb{1}$), whereas the weak one imposes no such restriction. We focus on the weak symmetry case hereafter, where the weak g_1 -flux inserted state [43], obtained by taking $X = V_{g_1}$ and denoted as $\rho(V_{g_1})$, preserves the g_2 weak symmetry.

Now we define the g_2 -charge response from the g_1 -flux insertion, and demonstrate its quantization, as it remains a multiplicative phase factor upon adding g_1 fluxes. Here unlike the pure-state (or strong-symmetry) case, the g_2 symmetry holds only on average: Each eigenstate of the density matrix is g_2 -symmetric but can have distinct symmetry charges. Correspondingly, in the $N \gg 1$ limit, the ensemble average of U_{g_2} decays with system size due to statistical cancellation,

$$\text{Tr}[\rho(V_{g_1})U_{g_2}] \sim e^{-\Theta_{g_2}N + iQ(g_1, g_2)}. \quad (8)$$

As demonstrated below, the extensively decaying amplitude arises from nonuniversal bulk contributions, independent of the boundary condition $X = V_{g_1}$, whereas the phase term $e^{iQ(g_1, g_2)}$ captures topological information. These results follow from the tensor diagram,

$$\begin{aligned} \text{Tr}[\rho(V_{g_1})U_{g_2}] &= \text{tr}[(V_{g_1}^* \otimes V_{g_1})T(g_2)^N], \end{aligned} \quad (9)$$

which is contracted horizontally. The g_2 -twisted transfer matrix $T(g_2)$ can be written as

$$\begin{aligned} T(g_2) &\equiv \text{Diagram} = \lambda_{g_2}^{(0)} \left(\text{Diagram 1} + \text{Diagram 2} \right) + \delta T(g_2) \\ &= \lambda_{g_2}^{(0)} |R_{g_2}^0\rangle \langle L_{g_2}^0| + \delta T(g_2). \end{aligned} \quad (10)$$

Here, $\lambda_{g_2}^{(0)}$ is the largest eigenvalue of $T(g_2)$ in magnitude, assumed to be nondegenerate. The corresponding right and left eigenvectors $|R_{g_2}^0\rangle$ and $\langle L_{g_2}^0|$ are normalized [i.e., $\langle L_{g_2}^0|R_{g_2}^0\rangle = 1$]. The remainder $\delta T(g_2)$ has spectrum radius $|\lambda_{g_2}^{(1)}| < |\lambda_{g_2}^{(0)}|$, with $\lambda_{g_2}^{(1)}$ the subleading eigenvalue in magnitude, indicating a finite spectral gap in $T(g_2)$,

$$\Delta(g_2) \equiv |\lambda_{g_2}^{(0)}| - |\lambda_{g_2}^{(1)}| > 0, \quad (11)$$

referred to as the g_2 -symmetry gap hereafter.

The g_2 -symmetry gap protects the quantized response, given by

$$e^{iQ(g_1, g_2)} = \frac{\text{Tr}[\rho(V_{g_1})U_{g_2}]}{\text{Tr}(\rho U_{g_2})} = \frac{\text{tr}[(V_{g_1}^* \otimes V_{g_1})T(g_2)^N]}{\text{tr}[T(g_2)^N]}. \quad (12)$$

The denominator $\text{Tr}(\rho U_{g_2})$ removes the extensive factor $e^{-\Theta_{g_2}N} = [\lambda_{g_2}^{(0)}]^N$ that is independent of g_1 . Crucially, $\Delta(g_2)$ ensures that, in the $N \gg 1$ limit, $T(g_2)^N / \text{tr}[T(g_2)^N]$ approaches $|R_{g_2}^0\rangle \langle L_{g_2}^0|$, a projector onto the leading eigenvector of $T(g_2)$. This yields

$$e^{iQ(g_1, g_2)} = \langle L_{g_2}^0 | (V_{g_1}^* \otimes V_{g_1}) | R_{g_2}^0 \rangle, \quad (13)$$

expressing $e^{iQ(g_1, g_2)}$ as a single quantum amplitude, allowing the standard quantization argument used in

pure-state cases: $(V_{g_1}^* \otimes V_{g_1})$ is a symmetry of $T(g_2)$ (see Appendix A for a diagrammatic proof), so $e^{iQ(g_1, g_2)}$ is an eigenvalue of $V_{g_1}^* \otimes V_{g_1}$, associated with the eigenvector $|R_{g_2}^0\rangle$. We thus confirm that the g_2 -charge response is multiplicative upon inserting multiple g_1 fluxes, and, moreover, remains quantized for any finite group G .

Changing the quantized response $e^{iQ(g_1, g_2)}$ requires closing the g_2 -symmetry gap, whereas notably, long-range correlations in local operators are generally not required—a point we will explore later through a concrete model. Furthermore, unlike pure states where $e^{iQ(g_1, g_2)}$ is given by the group cohomology invariant $\omega(g_1, g_2)/\omega(g_2, g_1)$ [see Eq. (5)], we will show that weakly symmetric open systems can exhibit different values, suggesting the need for a more general mathematical framework. Finally, while the LPDO discussion does not require strong injectivity, we note that our discussion naturally applies to states described by strong injective matrix product density operators [14,15] (MPDOs) as well. For these states, weak symmetries can be pushed to virtual space similarly to Eq. (7), and the above discussion remains applicable replacing $V_{g_1}^* \otimes V_{g_1}$ with the corresponding transformations in the virtual space.

String order parameters—One might be concerned that the quantized phase above is a ratio of two exponentially small numbers in system size, making it challenging to measure. To address this practical issue, we propose using string order parameters [41,44–46] on finite segments as probes of $Q(g_1, g_2)$ in a large system:

$$\mathcal{S}(g_2, \chi_{g_1}^L, \chi_{g_1}^R) = \text{Tr} \left[\rho \chi_{g_1}^L \otimes \left(\bigotimes_{i=j}^{j+l-1} u_{g_2} \right) \otimes \chi_{g_1}^R \right]. \quad (14)$$

Here, $\left(\bigotimes_{i=j}^{j+l-1} u_{g_2} \right)$ creates a symmetry-twisted domain of length l . The operators $\chi_{g_1}^{L/R}$, separating twisted and untwisted domains, carry a g_1 charge,

$$u_{g_1} \chi_{g_1}^{L/R} u_{g_1}^\dagger = e^{\pm i\phi(g_1)} \chi_{g_1}^{L/R}. \quad (15)$$

In the pure-state limit (i.e., for ordinary SPT states), $Q(g_1, g_2)$ is associated with the group cohomology invariant $\omega(g_1, g_2)/\omega(g_2, g_1)$, known to be captured by the dependence of the string order parameter on $\phi(g_1)$, as reflected in the corresponding selection rule [46].

For general cases, the topological charge $Q(g_1, g_2)$ reveals itself analogously. Tuning the parameter $\phi(g_1)$ by varying $\chi_{g_1}^{L/R}$ results in distinct decay behaviors of the string order parameter with string length l : Scaling behavior $\mathcal{S}(g_2, \chi_{g_1}^L, \chi_{g_1}^R) \sim [\lambda_{g_2}^{(0)}]^l$ occurs only if $e^{iQ(g_1, g_2)} = e^{i\phi(g_1)}$, while other choices yield subleading scaling, e.g., $[\lambda_{g_2}^{(1)}]^l$, as demonstrated later with a concrete example, and proven in [43]. To illustrate this behavior, we introduce a normalized string order parameter by dividing out the extensive part that leads to exponential decay,

$$\mathcal{S}^{(n)}(g_2, \chi_{g_1}^L, \chi_{g_1}^R) = \frac{\mathcal{S}(g_2, \chi_{g_1}^L, \chi_{g_1}^R)}{|\text{Tr}(\rho U_{g_2})|^{1/N}}, \quad (16)$$

satisfying the following selection rule,

$$\mathcal{S}^{(n)}(g_2, \chi_{g_1}^L, \chi_{g_1}^R)|_{l \rightarrow \infty} = \begin{cases} \mathcal{O}(1), & \text{if } e^{i\mathcal{Q}(g_1, g_2)} = e^{i\phi(g_1)} \\ 0, & \text{if } e^{i\mathcal{Q}(g_1, g_2)} \neq e^{i\phi(g_1)}. \end{cases} \quad (17)$$

Besides open quantum systems, similar ideas dealing with exponentially decaying string order parameters, associated with emergent higher-form symmetries, have been used to define the Fredenhagen-Marcu string order parameters [47–49] for detecting topological order. Additionally, this construction parallels earlier probes of bulk topology and boundary anomaly using defect operators [5, 7, 9, 50–53], and aligns with results derived from field theory in open quantum systems [9].

Example: AKLT state—The key concept that allows us to define phases and phase transitions is the g_2 -symmetry gap [Eq. (11)], i.e., the spectral gap of the twisted transfer matrix $T(g_2)$. Meanwhile, the correlation length for local observables is set by the gap of $T(1)$, whose closing signals long-range correlations. In the strong symmetry case, $T(g_2)$ and $e^{i\theta_{g_2}} T(1)$ are related by similarity transformations which preserve the spectrum. Consequently, all g_2 -symmetry gaps open and close at the same time, and thus the phase transition point must develop long-range correlations in the state. By contrast, for weak symmetries, this condition no longer holds. A phase transition can occur by closing one of the g_2 -symmetry gaps while $T(1)$ remains gapped, indicating finite correlation length for local observables as anticipated above. In the following, we will demonstrate this physics by constructing a prototype example.

To generate an LPDO with a short-range correlated purification, as considered within our framework, one representative way is to apply local noise channels to short-range correlated MPSS. We consider the pure spin-1 AKLT state described by the following MPS tensor [37, 39], i.e.,

$$A^{-1} = \sqrt{\frac{2}{3}}\sigma^+, \quad A^0 = -\sqrt{\frac{1}{3}}\sigma^z, \quad A^{+1} = -\sqrt{\frac{2}{3}}\sigma^-, \quad (18)$$

where we use standard notation for the Pauli matrices, and the physical indices label local spin- z basis. This is the ground state of the AKLT model, and has mutually commuting $\mathbb{Z}_2$ symmetries generated by global π rotations of spin along different axes, represented as $R_a^{\otimes N}$. Here, $R_a \equiv e^{i\pi S_a}$, and $S_a \in \{S_0, S_x, S_y, S_z\}$ with S_0 being the identity matrix and $S_{x,y,z}$ the usual spin-1 operators. Applying on-site noisy channels to the AKLT state ρ_0 ,

we obtain

$$\rho = \mathcal{N}_1 \circ \mathcal{N}_2 \circ \cdots \mathcal{N}_N[\rho_0], \quad \text{and} \quad \mathcal{N}_i[\cdot] = \sum_{\alpha} K_{\alpha,i}(\cdot) K_{\alpha,i}^\dagger, \quad (19)$$

where the Kraus operators $\{K_{\alpha,i}\}$ of $\mathcal{N}_i$ act locally on the spin-1 Hilbert space at site i as $\{\sqrt{1-p}S_0, \sqrt{p}S_xS_y, \sqrt{p}S_yS_z, \sqrt{p}S_zS_x\}$, with $p \in [0, 1]$ representing the noise rate. The noise breaks the AKLT state's strong $\mathbb{Z}_2$ symmetries down to weak ones. The resulting state ρ can be locally purified to a short-range correlated state by considering Stinespring's dilation [54] for each onsite channel.

Below we summarize key findings, while a detailed analysis is provided in Appendix B. This model exhibits two distinct phases that can be characterized by the quantized responses $(e^{i\mathcal{Q}(R_x, R_z)}, e^{i\mathcal{Q}(R_y, R_z)})$. The conventional AKLT phase appears in the low noise regime ($p < 1/2$), with $(e^{i\mathcal{Q}(R_x, R_z)}, e^{i\mathcal{Q}(R_y, R_z)}) = (-1, -1)$. As the noise rate increases ($p > 1/2$), a new phase emerges, characterized by $(-1, +1)$. This phase intrinsically relates to weak symmetry: The sign change in $e^{i\mathcal{Q}(R_y, R_z)}$ signals a switch of the leading eigenvector in $T(R_z)$, accompanied by the closing of the R_z -symmetry gap while preserving the $T(1)$ gap—a phenomenon impossible in pure-state or strong-symmetry cases, which in turn gives rise to a quantized response beyond the group cohomology description. Moreover, these quantized charges result in qualitatively distinct behavior in the string order parameters [cf. Eq. (14)], expressed as

$$\mathcal{S}_\alpha \equiv \mathcal{S}(R_z, S_\alpha, S_\alpha) = \text{Tr} \left[\rho S_\alpha \otimes \left(\bigotimes_{i=j}^{j+l-1} R_z \right) \otimes S_\alpha \right], \quad (20)$$

and their normalized counterparts $\mathcal{S}_\alpha^{(n)}$. Figure 1(a) shows that $\mathcal{S}_y^{(n)}$ distinguishes two phases: It vanishes in the $(-1, -1)$ phase due to the selection rule [Eq. (17)], but reaches an order-one value in the weak-symmetry protected phase. Conversely, $\mathcal{S}_x^{(n)}$ vanishes in both phases.

In practice, we can probe this transition by contrasting qualitatively different decay behaviors of string order parameters $\mathcal{S}_\alpha$ as functions of string length. These behaviors are characterized by the relevant decay exponents, defined as

$$\xi_\alpha = -\lim_{l \rightarrow \infty} \frac{1}{l} \log |\mathcal{S}_\alpha|. \quad (21)$$

Here, the α dependence of ξ_α reflects the selection rule, as ξ_α asymptotically converges to the $T(R_z)$ eigenvalue with dominant contribution to $\mathcal{S}_\alpha$. This is confirmed by the analytical results plotted in Fig. 1(b).

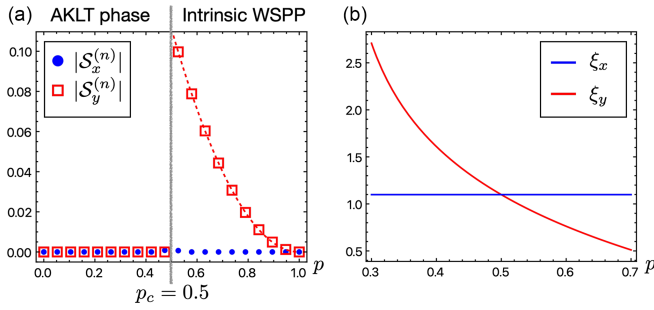


FIG. 1. Phase detection for the decohered AKLT state at different noise rate p via string order parameters. Panel (a) shows the normalized string order parameters in a 200-site periodic chain with string length 50. The dashed line is analytically computed in the limit of $N \gg l \gg 1$. The quantized responses ($e^{iQ(R_x, R_z)}$, $e^{iQ(R_y, R_z)}$) identify two phases: $(-1, -1)$, representing the AKLT phase, and $(-1, +1)$, an intrinsic weak-symmetry protected phase (WSPP). These phases manifest as distinct patterns in the normalized string order parameters, with $S_y^{(n)} = 0$ for the former, and $\mathcal{O}(1)$ for the latter. Panel (b) presents the analytically computed decay exponents ξ_x and ξ_y [Eq. (21)]; their crossing point marks the transition [55].

Discussion and outlook—Our work demonstrates that weak symmetries can protect topological responses in mixed-state bosonic systems, extending beyond the strong-symmetry requirement discussed in previous studies. Our approach differs from previous ones extending the circuit approach for pure states to define mixed-state phases via local quantum channels [56,57] or fast local Lindbladian evolution [11], where existing nontrivial generalizations [10,12] of bosonic SPT phases rely on strong symmetry. This distinction arises from different perspectives in generalizing from the pure-state cases, where SPT phases can be described by their topological responses as order parameters [35,46,58,59] or, more formally with the circuit approach, as equivalence classes under symmetric local unitary transformations [42]. In mixed states, these approaches lead to different notions of phases. The order-parameter approach reveals a weak-symmetry protected phase, characterized by quantized responses and the selection rule for string order parameters (see Fig. 1). This notion of phase has a clear operational meaning and is detectable in current experimental platforms [20,24,60,61].

Central to our findings is the symmetry gap of the symmetry-twisted transfer matrix, which ensures the robustness of topological responses. Consequently, phase transitions occur via symmetry gap closures without standard thermodynamic (local) signatures, a situation well described within the tensor network framework and amenable to efficient numerical simulations. Similar patterns are observed in fermionic systems and the decohered toric code [9,62,63], where topological phase transitions have been described as the loss of topological modes [9,63]. Future directions include extending the tensor-network

approach to higher-dimensional and fermionic systems, and developing a systematic framework for classifying mixed-state phases protected by weak symmetry and its symmetry gap. These may ultimately yield a unified understanding of mixed-state topological phases, with implications for quantum error correction [62–65].

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Data availability—The data that support the findings of this article are openly available [66].

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End Matter

Appendix A: Proof of $[V_{g_1}^* \otimes V_{g_1}, T(g_2)] = 0$ —We present a diagrammatic proof of the identity,

$$[V_{g_1}^* \otimes V_{g_1}, T(g_2)] = 0, \quad \text{for commuting } g_1, g_2, \quad (\text{A1})$$

or, equivalently,

$$T(g_2) = (V_{g_1}^* \otimes V_{g_1}) T(g_2) (V_{g_1}^T \otimes V_{g_1}^\dagger). \quad (\text{A2})$$

The proof directly follows from the tensor network diagram below,

The first equality rewrites u_{g_2} as $u_{g_1}^\dagger u_{g_2} u_{g_1}$ for commuting g_1, g_2 , and then introduces $u_{g_1}^a$ on the ancilla line using the identity $u_{g_1}^\dagger u_{g_1}^a = \mathbb{1}$. The second equality follows from applications of the weak symmetry transformation rule of A [see Eq. (7)].

Appendix B: Analytical results of the decohered AKLT state—We present analytical results for the N -site AKLT state under decoherence in the $N \gg 1$ limit, including the spectrum of the (symmetry-twisted) transfer matrix, the quantized response, and the string order parameters.

Building on the MPS representation of the AKLT state [Eq. (18)], we first calculate the spectra of transfer matrices with and without a symmetry twist, namely, $T(\mathbb{1})$ and $T(R_z)$, for the decohered state obtained after applying on-site noisy channels [Eq. (19)] at different noise rate p . Specifically for $T(\mathbb{1})$, the spectrum is unchanged under decoherence due to the trace-preserving nature of the on-site channels:

TABLE I. Summary of the eigenvalues and eigenvectors of $T(\mathbb{1})$, together with the corresponding $V_{R_\mu}^* \otimes V_{R_\mu}$ eigenvalues e^{iq_μ} ($\mu = x, y$).

Eigenvalue	Eigenvector	e^{iq_x}	e^{iq_y}
1	$(1/\sqrt{2})(1, 0, 0, 1)^T$	1	1
$-\frac{1}{3}$	$(1/\sqrt{2})(-1, 0, 0, 1)^T$	-1	-1
$-\frac{1}{3}$	$(1/\sqrt{2})(0, 1, 1, 0)^T$	1	-1
$-\frac{1}{3}$	$(1/\sqrt{2})(0, -1, 1, 0)^T$	-1	1

$$T(\mathbb{1}) = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{B1})$$

whose eigenvalues and eigenvectors are shown in Table I. By contrast, the spectrum of $T(R_z)$ depends on p :

$$T(R_z) = \begin{pmatrix} \frac{1}{3}(1-2p) & 0 & 0 & -\frac{2}{3}(1-p) \\ 0 & \frac{1}{3}(-1+2p) & -\frac{2p}{3} & 0 \\ 0 & -\frac{2}{3}p & \frac{1}{3}(-1+2p) & 0 \\ -\frac{2}{3}(1-p) & 0 & 0 & \frac{1}{3}(1-2p) \end{pmatrix}, \quad (\text{B2})$$

with eigenvalues and eigenvectors shown in Table II.

In the thermodynamic limit, the quantized response [namely, $e^{iQ(R_\mu, R_z)}$ with $\mu = x, y$] is determined by the leading eigenvector of $T(R_z)$, with its value given by the corresponding $V_{R_\mu}^* \otimes V_{R_\mu}$ eigenvalue. This can be directly read from Table II: (i) For $p < \frac{1}{2}$, the leading eigenvector is $(1/\sqrt{2})(-1, 0, 0, 1)^T$, with quantized responses $(e^{iQ(R_x, R_z)}, e^{iQ(R_y, R_z)}) = (-1, -1)$, corresponding to the same phase as the AKLT state. (ii) For $p > \frac{1}{2}$, the leading eigenvector becomes $(1/\sqrt{2})(0, -1, 1, 0)^T$, and the quantized responses shift to $(e^{iQ(R_x, R_z)}, e^{iQ(R_y, R_z)}) = (-1, +1)$. The transition occurs through an exchange of leading eigenvectors, with the critical point at $p = \frac{1}{2}$, where the R_z -symmetry gap closes.

We now consider the string order parameters $\mathcal{S}_\alpha$ [Eq. (20)] and their normalized counterparts $\mathcal{S}_\alpha^{(n)}$ in the limit of $N \gg l \gg 1$, which can be computed knowing the eigenvalues and eigenvectors of $T(\mathbb{1})$ and $T(R_z)$. In this limit, the leading contributions to the normalized string order parameters $\mathcal{S}_x^{(n)}$ and $\mathcal{S}_y^{(n)}$, as discussed in the main text (cf. Fig. 1), are given by

TABLE II. Summary of the eigenvalues and eigenvectors of $T(R_z)$, together with the corresponding $V_{R_\mu}^* \otimes V_{R_\mu}$ eigenvalues e^{iq_μ} ($\mu = x, y$).

Eigenvalue	Eigenvector	e^{iq_x}	e^{iq_y}
$[(3-4p)/3]$	$(1/\sqrt{2})(-1, 0, 0, 1)^T$	-1	-1
$[(-1+4p)/3]$	$(1/\sqrt{2})(0, -1, 1, 0)^T$	-1	1
$-\frac{1}{3}$	$(1/\sqrt{2})(1, 0, 0, 1)^T$	1	1
$-\frac{1}{3}$	$(1/\sqrt{2})(0, 1, 1, 0)^T$	1	-1

$$|\mathcal{S}_x^{(n)}| = 0, \quad |\mathcal{S}_y^{(n)}| = \begin{cases} 0 & p < \frac{1}{2} \\ \left[\frac{2}{3}(1-p)\right]^2 & p > \frac{1}{2} \end{cases}. \quad (\text{B3})$$

This highlights the distinct behavior between the $(-1, -1)$ AKLT phase and the $(-1, +1)$ intrinsic weak-symmetry protected phase. Specifically, the analytical results for $\mathcal{S}_x$ and $\mathcal{S}_y$ are

$$\begin{aligned} \mathcal{S}_x &= \left[\frac{2}{3}(1-p)\right]^2 \left[\left(-\frac{1}{3}\right)^l + \left(-\frac{1}{3}\right)^{N-l} \right] \\ &\rightarrow \left[\frac{2}{3}(1-p)\right]^2 \left(-\frac{1}{3}\right)^l \end{aligned} \quad (\text{B4})$$

and

$$\begin{aligned} \mathcal{S}_y &= \left[\frac{2}{3}(1-p)\right]^2 \left[\left(\frac{-1+4p}{3}\right)^l + \left(-\frac{1}{3}\right)^N \right] \\ &\rightarrow \left[\frac{2}{3}(1-p)\right]^2 \left(\frac{-1+4p}{3}\right)^l, \end{aligned} \quad (\text{B5})$$

while the leading term of the normalization factor $|\text{Tr}(\rho R_z^{\otimes N})|^{1/N}$ [see Eq. (16)] is given by

$$|\text{Tr}(\rho R_z^{\otimes N})|^{1/N} \rightarrow \begin{cases} \left(\frac{3-4p}{3}\right)^l & p < \frac{1}{2} \\ \left(\frac{-1+4p}{3}\right)^l & p > \frac{1}{2} \end{cases}. \quad (\text{B6})$$