

Gravitational Wave Scattering via the Born Series: Scalar Tidal Matching to $\mathcal{O}(G^7)$ and Beyond

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We introduce a novel method to compute gravitational wave amplitudes within the framework of effective field theory. By reinterpreting the Feynman diagram expansion as a Born series, our method offers several key advantages. It directly yields partial wave amplitudes, streamlining the matching with black hole perturbation theory. Long-distance gravitational interactions are unambiguously factorized from short-distance tidal effects, including dissipation, which are systematically incorporated via an in-in worldline effective action. Crucially, at every order in perturbation theory, integrals are expressed in terms of harmonic polylogarithms, enabling an end-to-end computation scalable to arbitrary orders. We illustrate the method with new predictions for scalar black hole Love numbers and their renormalization group equations to $\mathcal{O}(G^7)$.

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Introduction—The advent of gravitational wave astronomy has led to the observation of black hole and neutron star binaries [1,2], driving a surge in the application of field-theoretic methods to describe these systems [3–15]. The finite-size nature of such compact objects is characterized by their Love numbers, which quantify the tidal response to an external gravitational field [16,17]. Crucially, Love numbers leave a distinctive imprint on the gravitational wave signal [18–20] and have already been constrained for neutron stars [21–23]. Within the framework of effective field theory (EFT), Love numbers are rigorously defined as the Wilson coefficients of operators in the worldline effective action of the compact object [5–8]. Interestingly, the static Love numbers of black holes vanish identically [15–17,24–29].

In recent work [29–33], Love numbers were determined by comparing scattering amplitudes in the EFT to black hole perturbation theory (BHPT) [34–44]. Crucially, scattering amplitudes are gauge invariant, and linking them to Love numbers provides a robust framework that avoids ambiguities inherent in coordinate-dependent approaches. Nonetheless, it remains a challenge to systematically compute scattering amplitudes to high orders in perturbation theory; for instance, matching the leading dynamical Love number in gravity requires a five-loop calculation at $\mathcal{O}(G^6)$

in Newton’s constant G , which is far beyond the current state-of-the-art at two loops [32].

In this Letter we introduce a new method that enables the EFT computation of gravitational wave amplitudes and consequent matching of tidal Love numbers at the required high orders in perturbation theory. We study scalar wave scattering and present new predictions for scalar black hole Love numbers at $\mathcal{O}(G^7)$, reproducing the $\mathcal{O}(G^3)$ results of [32].

To study the tidal responses of a compact object of mass M , we consider the scattering of a wave with frequency ω in the regime

$$GM\omega \ll 1 \ll GM^2, \quad (\hbar = c = 1). \quad (1)$$

The left inequality means that the probe’s wavelength is much larger than the Schwarzschild radius, while the right inequality is the classical limit, i.e., the limit of large gravitational charge $M \gg M_{\text{Planck}}$. As noted in [45], the Feynman diagram expansion in this regime takes the form of a Born series

$$M \omega \text{ --- } V + \text{---} V \text{---} V + \text{---} V \text{---} V \text{---} V + \dots$$

where the *potential* V collects two-particle irreducible diagrams in a perturbative series in G . In our case, V includes the expected gravitational long-distance interactions but crucially differs from it in the region near the compact object, which is replaced by effective contact interactions that model tidal effects. These are precisely the object’s Love numbers.

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Setup: The effective wave equation—Because of the connection with the Born series, we can interpret the scattering amplitude calculated in the EFT as arising from an underlying wave equation, which we refer to as the *effective wave equation*. This equation can be derived in the standard manner from an effective action, where the parameters are determined by matching the physical scattering phases to those of the complete underlying theory. In our scalar model, given that $\omega \ll M$ as indicated by (1), the scalar field ϕ does not produce its own gravitational field. As a result, its long-distance dynamics are governed solely by the action

$$S_{\text{Grav}} = -\frac{1}{2} \int d^d x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \quad (2)$$

where $g^{\mu\nu}$ is the Schwarzschild metric in d -spacetime dimensions,

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_{d-2}^2 \quad (3)$$

with

$$f(r) = 1 - \frac{2GMn_d \mu^{4-d}}{r^{d-3}}, \quad n_d \equiv \frac{4\pi^{\frac{3-d}{2}} \Gamma(\frac{d-1}{2})}{d-2}. \quad (4)$$

The metric $g^{\mu\nu}$ is understood to be perturbatively expanded in Newton's constant G , whose units are kept as in four dimensions via the introduction of the dimensionful scale μ . This allows us to accurately replicate the Feynman diagram computation in dimensional regularization where as usual $d = 4 - 2\epsilon$. As we will see, the perturbative diagrams are dominated by the single scale $r \sim \omega^{-1} \gg GM$ and sensitivity to the near-horizon region will appear in the form of ultraviolet (UV) divergences regulated by ϵ .

We supplement (2) with a worldline effective action modeling the tidal response of the compact object,

$$S_{\text{Love}} = \sum_{\ell, n} \frac{\mu^{4-d}}{\ell!} \int d\tau [C_{\ell, n} (\nabla_\ell \phi_+) \partial_\tau^n (\nabla_\ell \phi_-)], \quad (5)$$

where the spatial derivative is defined as $\nabla^\mu = (g^{\mu\nu} + u^\mu u^\nu) \nabla_\nu$, and $u^\mu = (1, 0, 0, 0)$ is the 4-velocity of the compact object. The symbol ∇_ℓ denotes the symmetrized and trace-subtracted product of ℓ spatial derivatives. The coefficients $C_{\ell, n}$ are the Love numbers, and we again inserted a power of the scale μ to ensure that their units are as in four-dimensions: $[C_{\ell, n}] = -2\ell - n - 1$. Being supported on $\mathbf{r} = 0$, where $x^\mu = (t, \mathbf{r})$, the worldline action (5) adds a potential proportional to $\delta^{d-1}(\mathbf{r})$ to the wave equation.

An essential aspect of tidal physics is dissipation. As for other open systems, an action can be written using the in-in field doubling. We use the Keldysh basis [46,47] in which $\phi_+ = \frac{1}{2}(\phi_1 + \phi_2)$ represents the classical field and

$\phi_- = \phi_1 - \phi_2$ the source. For conservative effects, the in-in action is simply the difference $S[\phi_1] - S[\phi_2]$ of two conventional actions [as in (2)]

$$S[\phi] \supset \int dt \frac{\phi \partial_t^n \phi}{2} \Rightarrow S_{\text{in-in}} \supset \frac{1 + (-1)^n}{2} \int dt \phi_+ \partial_t^n \phi_- \quad (6)$$

while the nonconservative Love numbers, $C_{\ell, n}$ with n odd, can only be written in the form (5). We will focus on the retarded two-point amplitude $\langle \phi_- \phi_+ \rangle$, which satisfies the same boundary conditions as the classical problem and reduces to it in the classical limit [48,49] (see Supplemental Material [50] for further details, which includes Refs. [51–59]).

The effective wave equation is the equation of motion derived from varying ϕ_- in the action $S = S_{\text{Grav, SK}} + S_{\text{Love}}$. After projecting the scalar field onto spherical harmonics $\phi_+(\mathbf{r}) = [\psi(r) Y_\ell^m(\hat{\mathbf{r}}) / r^{(d-2)/2}]$ we find

$$\left[\frac{d^2}{dr^2} - \frac{(\ell - \epsilon)(\ell - \epsilon + 1)}{r^2} + \omega^2 \right] \psi(r) = V(r) \psi(r), \quad (7)$$

where the potential is given by $V(r) = V_{\text{Grav}}(r) + V_{\text{Love}}(r)$, and ℓ is the total angular momentum eigenvalue. The first term is the long-distance gravitational potential derived from S_{Grav} in (2),

$$V_{\text{Grav}}(r) = \sum_{n=1}^{\infty} \left(\frac{2GMn_d \mu^{2\epsilon}}{r^{1-2\epsilon}} \right)^n \left[\frac{2\epsilon - 1}{r} \frac{d}{dr} + \frac{\ell^2 + \ell + 1 - \epsilon(3 + 2\ell) + 2\epsilon^2}{r^2} - (n+1)\omega^2 \right], \quad (8)$$

which is known to reproduce the family of “fan” Feynman diagrams [60–63]



The second term is the short-distance potential due to the worldline effective action S_{Love} in (5),

$$V_{\text{Love}}(r) = -\mu^{2\epsilon} \sum_{\ell} F_{\ell}(\omega) \frac{(-1)^{\ell}}{\ell!} \nabla_{\ell} \delta^{d-1}(\mathbf{r}) \nabla_{\ell}, \quad (9)$$

with

$$F_{\ell}(\omega) = \sum_{n=0}^{\infty} (i\omega)^n C_{\ell, n}, \quad C_{\ell, n} = \frac{\text{diagram}}{\ell, n}. \quad (10)$$

Born series, S matrix, and boundary conditions—The radial wave equation (7) can be solved perturbatively via

the Born series as in quantum mechanics,

$$\begin{aligned} \psi(r) = & \Psi_0(r) + \int_0^\infty dr_1 G(r, r_1) V(r_1) \Psi_0(r_1) \\ & + \int_0^\infty dr_1 G(r, r_1) V(r_1) \int_0^\infty dr_2 G(r_1, r_2) V(r_2) \Psi_0(r_2) \dots, \end{aligned} \quad (11)$$

where the Green's function $G(r, r')$ and the free solution $\Psi_0(r)$ satisfy the free radial wave equation given by (7) when $V(r) = 0$. The free solution $\Psi_0(r)$ is a linear combination of Riccati-Hankel functions [64],

$$\Psi_0(r) = A_{\text{in}} h_{\ell-\epsilon}^-(\omega r) + A_{\text{out}} h_{\ell-\epsilon}^+(\omega r), \quad (12)$$

which behave as free radial waves at infinity,

$$h_{\ell-\epsilon}^\pm(r \rightarrow \infty) = e^{\pm i(\omega r - \frac{\pi}{2}(\ell-\epsilon))}. \quad (13)$$

As is customary in scattering theory [64,65], we assume that interactions decay sufficiently rapidly at infinity, $V(r \rightarrow \infty) \lesssim 1/r$, ensuring that $\psi(r \rightarrow \infty) \approx \Psi_0(r)$. The S matrix, or more precisely, the partial wave amplitude, is defined as the ratio of the coefficients of the outgoing and ingoing free waves,

$$S_\ell(\omega) \equiv -\frac{A_{\text{out}}}{A_{\text{in}}}. \quad (14)$$

Moreover, it is also assumed that the centrifugal potential $[(\ell - \epsilon)(\ell - \epsilon + 1)/r^2]$ dominates at small distances over the interaction potential $V(r \rightarrow 0)$, so that $\psi(r \rightarrow 0) \approx \Psi_0(r)$. The Riccati-Hankel functions in this limit behave as linear combinations of “regular” and “irregular” terms

$$\begin{aligned} h_{\ell-\epsilon}^+(r \rightarrow 0) - h_{\ell-\epsilon}^-(r \rightarrow 0) & \propto r^{1+\ell-\epsilon}, \\ h_{\ell-\epsilon}^+(r \rightarrow 0) + h_{\ell-\epsilon}^-(r \rightarrow 0) & \propto r^{\ell-\epsilon}. \end{aligned} \quad (15)$$

At $r \rightarrow 0$ the solution then assumes the form

$$\psi(r \rightarrow 0) = B_{\text{reg}} r^{1+\ell-\epsilon} + \frac{B_{\text{irr}} \mu^{2\epsilon}}{2\ell + 1 - 2\epsilon} r^{\ell-\epsilon}. \quad (16)$$

When $V = 0$ the ϕ_+ equation of motion at the origin imposes the boundary condition $B_{\text{irr}} = 0$. This can be seen from the fact that the irregular solution produces a δ -function term (recall that $\phi_+ \propto \Psi/r^{(d-2/2)}$):

$$\vec{\nabla}^2 \frac{r^\epsilon}{r^{\frac{d-2}{2}}} = -\frac{4\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-3}{2})} \delta^{d-1}(\mathbf{r}). \quad (17)$$

Using this identity and its derivatives, it is easy to see that canceling the δ function in the equation of motion including the Love number potential V_{Love} now imposes a modified boundary condition relating the two coefficients, where using (9) we find specifically

$$\frac{B_{\text{irr}}}{B_{\text{reg}}} = c_\ell F_\ell(\omega), \quad c_\ell = \frac{2^\ell}{2\pi^{\frac{3}{2}-\epsilon}} \Gamma\left(\frac{3}{2} + \ell - \epsilon\right). \quad (18)$$

This closely resembles the definition of Love numbers found in the GR literature [16,17,66]. The advantage here is that $F_\ell(\omega)$ has a gauge-invariant definition based on the worldline action (5). Equation (18) thus connects the GR and EFT definitions of Love numbers, which is one of our main conceptual results. Because it allows to trade $V_{\text{Love}}(r)$ for a redefinition of the boundary condition at $r \rightarrow 0$, it unambiguously factorizes short-distance and long-distance effects.

It may seem surprising that the gravitational potential V_{grav} does not modify this boundary condition. An intuitive explanation is that in dimensional regularization, the $r \rightarrow 0$ behavior is in effect regulated by treating $r^{-1+2\epsilon}$ as small when $r \rightarrow 0$.

In the free case ($V(r) = 0$), the boundary condition $B_{\text{irr}} = 0$ imposes that $A_{\text{in}} = -A_{\text{out}}$ via the relation (15). Consequently, this implies $S_\ell(\omega) = 1$, as expected. In the general case, however, the relationship between A_{out} and A_{in} is more intricate and can be determined perturbatively in the potential $V(r)$ using the Born series (11).

The assumptions that the interacting solution $\psi(r)$ reduces to the free solution $\Psi_0(r)$ both at infinity ($r \rightarrow \infty$) and at the origin ($r \rightarrow 0$), while satisfied within dimensional regularization, are in fact *not* satisfied when $\epsilon = 0$. This is the fundamental reason why the Born series and the partial wave amplitude $S_\ell(\omega)$ exhibit IR and UV divergences which we need to regularize by keeping $\epsilon \neq 0$, in close parallel with the Feynman diagram computation [32]. The IR divergences are entirely due to the Coulomb term $n = 1$ in $V_{\text{Grav}}(r)$ and obey a very simple exponentiation [64]. The UV divergences, on the other hand, which must cancel between V_{Love} and V_{Grav} contributions, originate from structure at the Schwarzschild radius which is not resolved by the perturbative expansion in the region $r \sim \omega^{-1} \gg GM$.

Concretely, the relationship between partial waves $S_\ell(\omega)$ and momentum-space Feynman diagrams involves an integral over Legendre polynomials [32]. Notably, the EFT matching with BHPT is performed at the level of the partial waves [32] and, crucially, our method yields directly the partial wave $S_\ell(\omega)$ within the EFT framework, eliminating the need for this intermediate step.

General strategy and near-far factorization—With the boundary conditions and the connection to the amplitude in (14) established, we turn to the computation of the iterated integrals in the Born series (11), where $V(r) = V_{\text{Grav}}(r) + V_{\text{Love}}(r)$ are given by (8) and (9). As mentioned, some

integrals diverge when $\epsilon \rightarrow 0$. Unfortunately, the Riccati-Hankel functions $h_{\ell-\epsilon}^{\pm}(\omega r)$, from which the Green's function $G(r, r')$ can be built of, do not have a closed form in terms of elementary functions for $\epsilon \neq 0$, making these integrals hard to solve in full generality.

Our strategy is to separate the problem in two different regions which we refer to as *far-zone* and *near-zone* where we can make distinct approximations. The idea is similar to the method of “matched asymptotic expansions” [67] and the “method of regions” in the context of Feynman integrals [68].

In the far-zone region we choose the Green's function $G^{\text{far}}(r, r')$ to only have support for ordered radii $0 < r < r'$. This eliminates any possible UV divergence from $r' = 0$ and allows for a regular expansion in ϵ where the starting point are Riccati-Hankel functions $h_{\ell}^{\pm}(\omega r)$ that reduce to elementary functions for integer ℓ . Moreover, in the far-zone region, the Love numbers drop out of the calculation due to their compact support at $r' = 0$. The iterated integrals over $V_{\text{Grav}}(r)$ which contribute to $S_{\ell}(\omega)$ can then be conveniently expressed in terms of harmonic polylogarithms (HPLs), as we detail in the next section.

In the near-zone region, on the other hand, we choose a Green's function $G^{\text{near}}(r, r')$ that only has support for $r' < r$. Here we have to keep ϵ finite in order to regulate UV divergences and simplifications come from the fact that the Riccati-Hankel functions $h_{\ell-\epsilon}^{\pm}(\omega r)$ at small r (or ω) admit simple power series starting with the regular and irregular terms in (15). The near-zone Born series then proceeds by starting with the boundary condition (16) incorporating the Love numbers via (18), and iteratively integrating over V_{grav} and G^{near} .

When the dust settles we have two solutions of the same differential equation in two different regions. They are made to match by bringing the far-zone solution to $r \rightarrow 0$ and expanding the near-zone solution in ϵ . This leads to the linear relation

$$\begin{bmatrix} A_{\text{in}} \\ A_{\text{out}} \end{bmatrix} \propto W_{\text{far-near}} \cdot \begin{bmatrix} 1 \\ c_{\ell} F_{\ell}(\omega) \end{bmatrix}, \quad (19)$$

where $W_{\text{far-near}}$ is the Wronskian matrix [67] that connects the two independent far-zone and near-zone solutions at $r \rightarrow 0$ and whose calculation depends only on $V_{\text{Grav}}(r)$. Equation (14) then determines the partial wave amplitude $S_{\ell}(\omega)$ in terms of the Love numbers in $F_{\ell}(\omega)$, enabling the latter to be fixed by matching $S_{\ell}(\omega)$ to the BHPT answer [34–43].

Far-zone Born series and harmonic polylogarithms—In the far zone we perturbatively expand the Born series (11) in a double series in G and ϵ , treating $[\epsilon(1 + 2\ell - \epsilon)/r^2] + V_{\text{Grav}}(r)$ as the perturbing potential where $V_{\text{Grav}}(r)$ given in (8) is expanded in ϵ . The solution $\Psi_0(r)$ for $\epsilon = 0$ is a linear combination of Riccati-Hankel functions $h_{\ell}^{\pm}(\omega r)$ with integer argument ℓ ,

$$h_{\ell}^{\pm}(z) = e^{\pm iz} \sum_{n=0}^{\ell} \frac{a_{n,\ell}^{\pm}}{z^n}, \quad a_{n,\ell}^{\pm} = \frac{(\pm i)^{n-\ell} (\ell + n)!}{2^n n! (\ell - n)!}, \quad (20)$$

and the Green's function is given by (with $z = \omega r$)

$$G^{\text{far}}(r, r') = \frac{h_{\ell}^{+}(z) h_{\ell}^{-}(z') - (z \leftrightarrow z')}{2i\omega} \Theta(r' - r). \quad (21)$$

It is crucial for our method that (20) consists of simple exponentials times polynomials in z^{-1} . With the use of integration-by-parts identities the far-zone Born integrals (11) are expressible in terms of iterated integrals defined recursively by

$$\mathcal{I}_{\dots j_n}(z_{n+1}) \equiv - \int_{z_{n+1}}^{\infty} \frac{dz_n}{z_n} e^{\sigma_n z_n} \frac{\log\left(\frac{z_{n+1}}{z_n}\right)^{j_n-1}}{(j_n - 1)!} \mathcal{I}_{\dots}(z_n), \quad (22)$$

with seed $\mathcal{I}(z_1) = 1$ and $\sigma_n = 2i(-1)^{n-1}$. Because of the exponential factor, these integrals are *not* simply related to harmonic polylogarithms [69]. Nonetheless, to compute the partial wave $S_{\ell}(\omega)$ via (19) we only need their limit as $r \rightarrow 0$ which remarkably *can* be calculated using harmonic polylogarithms [70]. To see this, we make the recursive change of variables

$$\frac{z_{k+1}}{z_k} = \frac{1 - u_k}{1 - u_{k-1}} \left(\frac{u_{k-1}}{u_k} \right)^{\delta_{k,\text{even}}}, \quad (23)$$

starting at $k = 1$ and setting $u_0 \equiv 0$, and where the Kronecker delta is nonzero when k is even. This change is designed to collect all exponentials into the outermost integral over z_1 , which can then be readily performed as $z_{n+1} \rightarrow 0$, where it becomes a complete integral. This limit introduces divergences that we regulate by replacing the innermost logarithmic power in (22) by $[(z_n)^{\eta}/\eta^{j_n-1}]$. The complete regulated integral is then expressed as an integral over the region $\Gamma = \{0 < u_1 < \dots < u_{n-1} < 1\}$,

$$\begin{aligned} \mathcal{I}_{j_1 \dots j_n}^{(\eta)} = & \mathcal{N} \int_{\Gamma} d \log(1 - u_1) d \log\left(\frac{1 - u_2}{u_2}\right) d \log(1 - u_3) \dots \\ & \times \prod_{k=1}^{n-1} \frac{\log\left(\frac{z_{k+1}}{z_k}\right)^{j_k-1}}{(j_k - 1)!} \frac{1}{(-\eta)^{j_n}} \left[\frac{(u_{n-1})^{\delta_{n,\text{even}}}}{1 - u_{n-1}} \right]^{\eta}, \end{aligned} \quad (24)$$

where $\mathcal{N} = \Gamma(1 + \eta)(-2i)^{-\eta}$. The z integrals produce regularized HPLs that evaluate to multi-zeta values [69], and the auxiliary regulator η drops out once we return to the incomplete integrals $\mathcal{I}_{\dots}(z)$ using the simple identity $\int_r^{\infty} = \int_0^{\infty} - \int_0^r$ (the far-zone solution is UV finite). Examples are given in Supplemental Material [50].

Note that all integrals (22) are IR finite. IR divergences in the far-zone Born series (11) occur when the innermost integrand is of the form $(\log^n r/r)$ for some integer n . In this

TABLE I. Scalar black hole Love numbers $C_{\ell,n}$ in dimensional regularization, where $R_s = 2GM$ and $\bar{\mu}^2 = \mu^2 4\pi e^{-\gamma_E}$.

	$\ell = 0$	$\ell = 1$	$\ell = 2$
G	0	...	...
G^2	$4\pi R_s^2$	...	...
G^3	$4\pi R_s^3[(1/4\epsilon) + \log R_s \bar{\mu} + \frac{19}{12} + \gamma_E]$	0	...
G^4	$\frac{44}{3}\pi R_s^4[(1/4\epsilon) + \log R_s \bar{\mu} - \frac{13}{66} - (\pi^2/11) + \gamma_E]$	$\pi(R_s^4/3)$	...
G^5	$\frac{11}{6}\pi R_s^5\{(1/4\epsilon^2) + (2/\epsilon)[\log R_s \bar{\mu} + \gamma_E + \frac{56}{33}]$ $+ (2 \log R_s \bar{\mu} + 2\gamma_E + \frac{167}{66})^2 - \frac{338611}{21780} - (8\pi^2/3) - (48\zeta_3/11)\}$	$\frac{1}{6}\pi R_s^5[(1/4\epsilon) + 2 \log R_s \bar{\mu} - \frac{19}{30} + 2\gamma_E]$	0

case we trivially integrate up to a hard-cut-off $r < R$ until all leftover integrals take the form (22). We then express R in terms of the dimensional regulator ϵ (see Supplemental Material [50]).

Near-zone Born series and renormalization group—In the near-zone, we expand in G and in ω (i.e., a small r expansion). The potential that is iterated by the Born series (11) in the near zone is $\omega^2 + V_{\text{Grav}}(r)$. The Green's function is given by

$$G^{\text{near}}(r, r') = \frac{r^{\ell-\ell'}(r')^{1+\ell-\epsilon} - (r \leftrightarrow r')}{1 + 2\ell - 2\epsilon} \Theta(r - r') \quad (25)$$

with $\Psi_0(r)$ given by (16). All integrands in the near zone are simple monomials and can be integrated trivially. A simple power counting argument may then be used to predict when UV divergences can first appear. For instance, from the first Born iteration $\int_0^r dr r^{2+2\ell-2\epsilon} V_{\text{Grav}}(r) \sim \int_0^r dr r^{2+2\ell-n+2\epsilon(n-1)}$ we see that a $1/\epsilon$ pole appears when $n = 3 + 2\ell$ and, indeed, the first UV divergence shows up at order $\mathcal{O}(G^3)$ [32]. This also suggests an alternative way of regulating UV divergences by shifting ℓ away from integer values, as noticed in [42,72].

The UV divergences are, of course, tightly related to a renormalization group (RG) equation which predicts logarithmic running of Love numbers. All UV divergences in our approach originate from the boundary condition (16) becoming degenerate when $\epsilon = 0$. By a change of basis $(B_{\text{reg}}, B_{\text{irr}})^T = Z(\bar{B}_{\text{reg}}, \bar{B}_{\text{irr}})^T$ we can thus ensure that the solution expressed in terms of the *renormalized coefficients* $\bar{B}$ remains finite as $\epsilon \rightarrow 0$. According to standard renormalization theory, the price to pay is that coefficients become scale dependent and satisfy a renormalization group equation: $[\mu(d/d\mu) + \gamma^{(\ell)}](\bar{B}_{\text{reg}}, \bar{B}_{\text{irr}})^T = 0$. The matrix Z and anomalous dimension in a minimal subtraction scheme are recorded in Supplemental Material [50]. The renormalized Love numbers $c_\ell \bar{F}_\ell = \bar{B}_{\text{irr}}/\bar{B}_{\text{reg}}$ can then be interpreted as a boundary condition in the region $GM \ll r \ll \omega^{-1}$, providing a precise dictionary between the renormalization scheme used in the worldline EFT and the GR definition of Love numbers. The renormalized Love numbers satisfy the RG equation:

$$\mu \frac{d}{d\mu} \bar{F}_\ell(\omega) = c_\ell \gamma_{1,2}^{(\ell)} \bar{F}_\ell^2 + (\gamma_{1,1}^{(\ell)} - \gamma_{2,2}^{(\ell)}) \bar{F}_\ell - \frac{\gamma_{2,1}^{(\ell)}}{c_\ell}. \quad (26)$$

We note the presence of a quadratic term, which starts contributing at $\mathcal{O}(G^5)$ in the black hole case. To our knowledge it had not been identified before.

Results—In Table I we show the Love numbers $C_{\ell,n}$ determined by matching our answer to BHPT, for $\ell = 0, 1, 2$ up to $\mathcal{O}(G^5)$. Phase shifts, RG equations, and Love numbers up to $\mathcal{O}(G^7)$ are listed in Supplemental Material [50]. We agree with the $\mathcal{O}(G^3)$ results of [32] and make new predictions through $\mathcal{O}(G^7)$.

Conclusion—We developed a method to compute the long-wavelength scattering off compact objects using iterated Born integrals. It addresses both a conceptual issue; namely, the gauge-invariant definition of dynamical Love numbers, and a technical challenge, which is the systematic computation of the perturbative expansion in the EFT. Although our method was tested on scalar wave scattering, it establishes a general framework for computing gravitational Love numbers, which arise only at higher-loop order [starting at $\mathcal{O}(G^5)$] in the EFT expansion.

Let us briefly discuss the extension to the physically relevant case of tensor perturbations, for Schwarzschild and spinning (Kerr) black holes. For the former, the effective wave equation in spherical harmonics is structurally similar to the scalar case and admits vacuum solutions similar to (20). We thus expect the small-frequency expansions to be governed by the same basis of iterated integrals. Additional contact terms in the EFT, arising from the gravitational backreaction of the worldline, are conceptually well understood [63,73] and should not pose technical difficulties. The extension to spinning (Kerr) black holes is expected to be more intricate but still tractable. While a separable generalization of the Teukolsky equation is not known in $d \neq 4$, the near-zone EFT solution will continue to admit a power-law Born expansion as discussed below (25), allowing for the extraction of Love numbers in dimensional regularization.

In the end, combined with known near-horizon expansions of black hole perturbation theory in terms of harmonic polylogarithms [43], our far-zone construction

provides an end-to-end framework for computing the scattering of long-wavelength perturbations off black holes, and generic compact objects within the worldline EFT approach.

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