

Geometric Bookkeeping Guide to Feynman Integral Reduction and ϵ -Factorized Differential Equations

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We report on three improvements in the context of Feynman integral reduction and ϵ -factorized differential equations. First, we show that with a specific choice of prefactors, we trivialize the ϵ dependence of the integration-by-parts identities. Second, we observe that with a specific choice of order relation in the Laporta algorithm, we directly obtain a basis of master integrals, whose differential equation on the maximal cut is in Laurent polynomial form with respect to ϵ and compatible with a particular filtration. Third, we prove that such a differential equation can always be transformed to an ϵ -factorized form. This provides a systematic algorithm to obtain an ϵ -factorized differential equation for any Feynman integral. Furthermore, the choices for the prefactors and the order relation significantly improve the efficiency of the reduction algorithm.

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Introduction—Recent advancements in particle physics experiments at the Large Hadron Collider are based on precision measurements that in turn demand increasing precision of theoretical predictions. Given small couplings at high energies, perturbative quantum field theory is the suitable method to compute different processes. Thus, computing Feynman integrals is the core of any precision calculation and techniques for these computations are a topic of current research interest [1–17].

An effective way to compute Feynman integrals is the method of differential equations [18–21]. This method can be used analytically or numerically [22–28]. One utilizes integration-by-parts identities to derive a set of (non- ϵ -factorized) differential equations [29–31]. This step is algorithmic and involves only linear algebra. The only limitation is the availability of computing resources. For an analytic calculation one usually performs two additional steps. In the second step, one transforms the system of

differential equations to an ϵ -factorized form [32]. In the last step, one solves the ϵ -factorized differential equations order by order in ϵ in terms of iterated integrals [33]. The third step is also straightforward, and there are no conceptual issues, provided appropriate boundary values are given. Since the boundary values depend on one kinematic variable less, they are simpler to calculate. In fact, they can be recursively reduced to single-mass vacuum integrals [22–24]. Analytic calculations are the method of choice for Feynman integrals depending on a small number of kinematic variables [34]. Rapidly converging series expansions in variables suggested by the ϵ -factorized form provide fast numerical evaluation routines for these Feynman integrals.

There are two bottlenecks within this approach. The first bottleneck is the availability of computing resources for the required integration-by-parts reduction. A major source of expression swell in integration-by-parts reduction are spurious polynomials in the denominator, which depend on the dimensional regulator ϵ and the kinematic variables x . There are some heuristic methods that try to avoid the occurrence of this situation [35,36].

The second bottleneck is conceptual: can one always find a transformation to an ϵ -factorized differential equation? For families of Feynman integrals that evaluate to

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multiple polylogarithms a systematic procedure is known [37–39] and has been implemented in several computer programs [40–42]. For Feynman integral families that go beyond multiple polylogarithms we know examples where the transformation to an ε -factorized form has been constructed [15,43–55]. In addition, there are methods that under specific (limiting) assumptions (for example, restriction to a specific geometry [16,56,57], advance knowledge of the alphabet [58], or a good guess of the initial basis [59]) allow the construction of the required transformation.

In this Letter, we report on three significant improvements. (1) We show that with a particular choice of prefactors we can trivialize the ε dependence of the integration-by-parts identities. (2) We observe that an order relation inspired by geometry in the reduction algorithm leads to a basis of master integrals, whose differential equation is in a special form (which we call “ F^* compatible”). (3) We present an algorithm to convert an F^* -compatible differential equation into an ε -factorized differential equation.

In practical terms, these findings imply significant efficiency improvements for integration-by-parts reduction and a systematic algorithm to obtain an ε -factorized differential equation.

We are interested in dimensionally regulated Feynman integrals. We denote the dimensional regularization parameter by ε , the kinematic variables by $x = (x_1, \dots, x_{N_B})$, and a basis of master integrals by $I = (I_1, \dots, I_{N_F})$. The latter satisfies a system of first-order differential equations,

$$dI = \hat{A}(\varepsilon, x)I, \quad d = \sum_{j=1}^{N_B} dx_j \frac{\partial}{\partial x_j}, \quad (1)$$

where $\hat{A}(\varepsilon, x)$ is a $N_F \times N_F$ matrix, whose entries are differential one-forms, rational in ε and x . The task is to find an invertible $N_F \times N_F$ matrix $R(\varepsilon, x)$ such that the transformed basis $K = R^{-1}I$ satisfies

$$dK = \varepsilon A(x)K, \quad (2)$$

where $A(x)$ is independent of ε . A differential equation of the form Eq. (2) is said to be ε -factorized. It is sufficient to focus on the maximal cut, as the required transformation off the maximal cut can be obtained by solving a differential equation. We therefore restrict to the maximal cut and with a slight abuse of notation, N_F now denotes the number of master integrals on the maximal cut.

We utilize mathematical tools (twisted cohomology and Hodge theory) that allow us to treat any Feynman integral independent of a specific geometry. It is well-known that the integrands of the master integrals can be viewed as twisted cohomology classes [60,61]. From Hodge theory, we borrow the concepts of filtrations [62–66].

The filtrations induce a decomposition of the master integrals on the maximal cut into smaller sets. Furthermore,

we perform integration-by-parts reduction with an ordering criterion based on the filtrations. We observe that this leads to an intermediate basis J , such that the differential equation for J on the maximal cut is in a Laurent polynomial form

$$dJ = \sum_{k=k_{\min}}^1 \varepsilon^k A^{(k)}(x)J, \quad (3)$$

where $A^{(k)}(x)$ is independent of ε and the occurring powers of ε are restricted by one of the filtrations [the precise statement is given in Eq. (23)]. The basis J is related to the basis I by a transformation $J = R_1^{-1}I$, where $R_1(\varepsilon, x)$ is rational in ε and x .

In a second step we construct a matrix $R_2(\varepsilon, x)$, which leads to a basis $K = R_2^{-1}J$, such that the differential equation for K on the maximal cut is in ε -factorized form. The dependence of R_2 on ε is rather simple, however, it may involve transcendental functions of the kinematic variables x . These transcendental functions are defined as the solution of a system of ε -independent first-order differential equations.

Concepts and method—In this section, we introduce the main concepts and outline the method. A detailed description of the algorithm is given in a longer companion paper [67].

We consider the Feynman integrals on the maximal cut in either the democratic or a loop-by-loop Baikov representation [68–70]. Let N be the number of residues taken for the maximal cut and let $\mathcal{C}_{\text{maxcut}}$ be the corresponding contour. We denote the remaining Baikov variables by $(z_1, \dots, z_n)$. We introduce an arbitrary scale m , which we use to render the Feynman integrals, the Baikov variables, and the kinematic variables dimensionless. We further denote the number of loops by l , the number of space-time dimensions by D , Euler’s constant by γ_E , and the inverse propagators by σ_j . Within dimensional regularization, we will always set $D = D_{\text{int}} - 2\varepsilon$ with $D_{\text{int}} \in \mathbb{Z}$. We obtain the Baikov polynomials $p_i(z)$ from the Feynman integral with all propagators raised to the power 1,

$$\int_{\mathcal{C}_{\text{maxcut}}} \prod_{r=1}^l \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{1}{\prod_{j=1}^N \sigma_j} \sim \int d^n z \prod_{i \in I_{\text{all}}} [p_i(z)]^{\alpha_i}. \quad (4)$$

It is important to notice that the exponents α_i are always of the form

$$\alpha_i = \frac{1}{2}(a_i + b_i \varepsilon), \quad \text{with } a_i, b_i \in \mathbb{Z}. \quad (5)$$

We define I_{odd} as the set of indices for which a_i is odd and I_{even} as the set of indices for which a_i is even.

In order to capture possible singularities at infinity, we extend the affine space with coordinates $(z_1, \dots, z_n)$ to projective space $\mathbb{CP}^n$ with homogeneous coordinates

$[z_0 : z_1 : \dots : z_n]$. Let d_i be the degree of p_i and denote by P_i the d_i homogenization,

$$P_i(z_0, z_1, \dots, z_n) = z_0^{d_i} p_i\left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right). \quad (6)$$

We further set $P_0(z_0, z_1, \dots, z_n) = z_0$ and

$$a_0 = \begin{cases} 0 & \text{if } \sum_{i \in I_{\text{odd}}} d_i \text{ even,} \\ -1 & \text{if } \sum_{i \in I_{\text{odd}}} d_i \text{ odd,} \end{cases} \quad (7)$$

$$b_0 = -\sum_{i \in I_{\text{all}}} b_i d_i.$$

We can unify the notation by including the index 0 in I_{even} or I_{odd} , depending on a_0 being zero or (-1) , respectively. We denote the resulting index sets by I_{even}^0 , I_{odd}^0 , and I_{all}^0 .

Within twisted cohomology, we may always move integer powers of the Baikov polynomials between the twist function and the rational differential n form. We can therefore define a “minimal” twist function by requiring $a_i \in \{-1, 0\}$ for all i ,

$$U(z_0, z_1, \dots, z_n) = \prod_{i \in I_{\text{odd}}^0} P_i^{-\frac{1}{2} + \frac{1}{2} b_i \varepsilon} \prod_{j \in I_{\text{even}}^0} P_j^{\frac{1}{2} b_j \varepsilon}. \quad (8)$$

$U(z)$ is a homogeneous function. The even and the odd polynomials will play different roles in the following. If an even polynomial is present in the denominator of the rational differential n form, we may take a residue and reduce to a simpler problem with one Baikov variable less. The odd polynomials define a geometry, which, to a first approximation, is associated with the Feynman integral and given by

$$y^2 = \prod_{i \in I_{\text{odd}}^0} P_i(z) \quad (9)$$

in a suitable weighted projective space.

The central objects are differential forms, which can be written as

$$\Psi_{\mu_0 \dots \mu_{N_D}}[Q] = C U(z) \hat{\Phi}_{\mu_0 \dots \mu_{N_D}}[Q] \eta, \quad (10)$$

where $\mu_j \in \mathbb{N}_0$ and $\hat{\Phi}_{\mu_0 \dots \mu_{N_D}}[Q]$ is an ε -independent meromorphic function in z , given by

$$\hat{\Phi}_{\mu_0 \dots \mu_{N_D}}[Q] = \frac{Q}{\prod_{i \in I_{\text{all}}^0} P_i^{\mu_i}}. \quad (11)$$

Q is a homogeneous polynomial in the Baikov variables z . η is the standard n form defined by

$$\eta = \sum_{j=0}^n (-1)^j z_j dz_0 \wedge \dots \wedge \widehat{dz_j} \wedge \dots \wedge dz_n, \quad (12)$$

where the hat indicates that the corresponding term is omitted. The ε -dependent prefactor C is independent of z (but may depend on x). It is defined such that the overall normalization of $\Psi_{\mu_0 \dots \mu_{N_D}}[Q]$ by an ε -dependent function is consistent and such that factors of $(\alpha_i - \mu_i)$, associated with differentiation of $\Psi_{\mu_0 \dots \mu_{N_D}}[Q]$ with respect to z or x , are absorbed into the prefactor. The latter property trivializes the ε dependence of the integration-by-parts identities and is responsible for a significant efficiency improvement. The explicit definition of the prefactor is given in the companion paper [67]. We further set

$$|\mu| = \sum_{i \in I_{\text{all}}^0} \mu_i. \quad (13)$$

We denote the vector space spanned by the differential forms of Eq. (10) by Ω_{ω}^n . This is an infinite-dimensional vector space. It is customary to put a subscript $\omega = d \ln U$. We obtain a finite-dimensional vector space by modding out linear relations like integration-by-parts identities. The resulting finite-dimensional vector space is the twisted cohomology group H_{ω}^n .

We denote the vector space of Feynman integrals on the maximal cut modulo integration-by-parts identities by V^n , this is again a finite-dimensional vector space. There is an injective map,

$$\iota: V^n \hookrightarrow H_{\omega}^n, \quad (14)$$

obtained from expressing the maximal cut of the Feynman integral in the Baikov representation. In general, the map will not be surjective. There are two reasons for this. First of all, integration can lead to symmetries among Feynman integrals (elements in V^n), which are not symmetries of the integrands (elements in H_{ω}^n). Second, a polynomial $p_j(z)$ with $j \in I_{\text{even}}$ can simply be a factor $z_r = \sigma_l$, where σ_l is an uncut inverse propagator. In this case, H_{ω}^n will also contain the integrands of the sector where the exponent of this inverse propagator is positive. If this sector has additional master integrals, they will also appear in H_{ω}^n . We call such a sector a “supersector,” and we include it in the analysis. Taking these subtleties into account, we can entirely work in the space H_{ω}^n and convert back to V^n in the end.

For the integrands defined in Eq. (10) we have three types of linear relations: integration-by-parts identities, distribution identities, and cancellation identities. The integration-by-parts identities read

$$0 = \frac{1}{\varepsilon} \Psi_{\mu_0 \dots \mu_i \dots \mu_{N_D}} [\partial_{z_j} Q_+] + \sum_{i \in I_{\text{all}}^0} \Psi_{\mu_0 \dots (\mu_i+1) \dots \mu_{N_D}} [Q_+ \cdot (\partial_{z_j} P_i)], \quad (15)$$

where Q_+ is an ε -independent homogeneous polynomial of degree $\deg Q_+ = \deg Q + 1$. The distribution identities are

rather trivial and originate from writing a polynomial $Q = Q_1 + Q_2$ as a sum of two other polynomials,

$$\Psi_{\mu_0 \dots \mu_{N_D}}[Q] = \Psi_{\mu_0 \dots \mu_{N_D}}[Q_1] + \Psi_{\mu_0 \dots \mu_{N_D}}[Q_2]. \quad (16)$$

The cancellation identities originate from a cancellation of P_j in the numerator and the denominator. They read

$$\begin{aligned} & \Psi_{\mu_0 \dots (\mu_j+1) \dots \mu_{N_D}}[P_j \cdot Q] \\ &= \frac{1}{\varepsilon} \left(\frac{1}{2} a_j - \mu_j + \frac{b_j}{2} \varepsilon \right) \Psi_{\mu_0 \dots \mu_j \dots \mu_{N_D}}[Q]. \end{aligned} \quad (17)$$

Note that the prefactor C in Eq. (10) has been meticulously defined to trivialize the ε dependence of the integration-by-parts identities. In fact, we may reduce the subsystem formed by Eqs. (15) and (16) by setting $\varepsilon = 1$. This is a significant efficiency improvement, as we have one variable less. This is possible because in the integration-by-parts identities Eq. (15) and the distribution identities Eq. (16) the explicit ε factors are synchronized with $|\mu|$ and the ε dependence of the coefficients in the reduction is therefore always monomial.

In the full system, this is, however, spoiled by the cancellation identities. In these relations, the offending part comes from the bracket on the right-hand side of Eq. (17). Nevertheless, it is advantageous to reduce the subsystem formed by Eqs. (15) and (16) first and then combine this reduced system with the cancellation identities of Eq. (17).

To each object $\Psi_{\mu_0 \dots \mu_{N_D}}[Q]$ as in Eq. (10) we associate three integer numbers $(r, o, |\mu|)$, where $|\mu|$ has already been defined in Eq. (13). We let r to be the largest number such that the r -fold residue of $\Psi_{\mu_0 \dots \mu_{N_D}}^0[Q]$ is nonzero, where $\Psi_{\mu_0 \dots \mu_{N_D}}^0[Q]$ is defined by setting $\varepsilon = 0$ in the twist function. The integer o denotes the pole order of $\Psi_{\mu_0 \dots \mu_{N_D}}^0[Q]$. The pole order is the maximum of pole orders at individual points. For $\alpha > 0$, the pole order of $z^{-\alpha} dz$ at $z = 0$ is $\lfloor \alpha \rfloor$, where $\lfloor x \rfloor$ denotes the floor function. For normal-crossing singularities, the pole order is additive, i.e., the pole order of $dz_1/z_1 \wedge dz_2/z_2^2$ at $(z_1, z_2) = (0, 0)$ is 3. For non-normal-crossing singularities, we first need to perform a blowup.

In addition, we introduce the concept of localizations in which we consider differential forms where P_i with $i \in I_{\text{even}}^0$ appears in the denominator of $\hat{\Phi}_{\mu_0 \dots \mu_{N_D}}[Q]$ (i.e., $\mu_i > 0$). For those differential forms, we may take a residue at $P_i = 0$. We also say that we localize on $P_i = 0$. In Ref. [67] we discuss in detail, how the ε -dependent part of the exponent of P_i in the twist function is treated when taking the residue. The residue is then a differential $(n-1)$ form, and we consider the integration-by-parts identities of these forms on the variety defined by $P_i = 0$. The explicit formulas are worked out in Ref. [67] and have the property

that they do not introduce algebraic extensions. If the differential $(n-1)$ form has a further even polynomial in the denominator of $\hat{\Phi}_{\mu_0 \dots \mu_{N_D}}[Q]$, this process can be iterated. In this way we obtain $(n-2)$ forms, $(n-3)$ forms, ..., 0 forms. We define the fourth integer number a as

$$a = \begin{cases} -w, & \Psi_{\mu_0 \dots \mu_{N_D}}[Q] \text{ is a preferred candidate} \\ & \text{from the localizations,} \\ 0, & \text{otherwise.} \end{cases} \quad (18)$$

The purpose of the variable a is to give preference to the master integrands from localizations as master integrands of the current problem. The a value can be computed recursively; the details can be found in [67].

Now we define the order relation for the Laporta algorithm on the space Ω_{ω}^n as

$$(a, w, o, |\mu|, \dots), \quad (19)$$

where $w = n + r$, and the dots stand for further criteria needed to distinguish inequivalent integrands. The relation $a_1 < a_2$ implies $\Psi_1 < \Psi_2$, with ties broken by w , etc.

The three numbers $w, o, |\mu|$ also define three filtrations $W_{\bullet}$, $F_{\text{geom}}^{\bullet}$, and $F_{\text{comb}}^{\bullet}$ on the space Ω_{ω}^n . The weight filtration $W_{\bullet}$ is defined by

$$\Psi_{\mu_0 \dots \mu_{N_D}}[Q] \in W_w \Omega_{\omega}^n \quad \text{if } n + r \leq w. \quad (20)$$

The weight filtration is the standard weight filtration from Hodge theory. The filtration $F_{\text{geom}}^{\bullet}$ is defined by

$$\Psi_{\mu_0 \dots \mu_{N_D}}[Q] \in F_{\text{geom}}^p \Omega_{\omega}^n \quad \text{if } n + r - o \geq p. \quad (21)$$

At fixed weight w , the filtration $F_{\text{geom}}^{\bullet}$ is a filtration by the pole order o . The third filtration $F_{\text{comb}}^{\bullet}$ is defined by

$$\Psi_{\mu_0 \dots \mu_{N_D}}[Q] \in F_{\text{comb}}^{p'} \Omega_{\omega}^n \quad \text{if } n - |\mu| \geq p'. \quad (22)$$

The combinatorial filtration $F_{\text{comb}}^{\bullet}$ is a filtration by the quantity $|\mu|$. The general idea is that we always work modulo simpler terms, i.e., modulo terms with fewer residues, lower pole order, or a smaller sum of indices $|\mu|$.

We denote by $\Psi = (\Psi_1, \dots, \Psi_{N_F})^T$ the basis of master integrands obtained from this algorithm. In all examples we tested, we observed that the differential equation is of the form as in Eq. (3). Moreover, we always observed that if Ψ_i has $|\mu| = |\mu|_i$ and Ψ_j has $|\mu| = |\mu|_j$, then

$$A_{ij}(\varepsilon, x) = \sum_{k=-(|\mu|_i-|\mu|_j)}^1 \varepsilon^k A_{ij}^{(k)}(x). \quad (23)$$

We call a differential equation that satisfies Eq. (23) an $F_{\text{comb}}^{\bullet}$ -compatible differential equation for the filtration $F_{\text{comb}}^{\bullet}$.

An F^* -compatible differential equation implies Griffiths transversality [71]. It is, however, a stronger statement, as it requires the differential equation to be in Laurent polynomial form with restrictions on the occurring powers of ε .

We now prove that we may always construct an ε -factorized differential equation from an F^* -compatible differential equation. Let $J = (J_1, \dots, J_{N_F})^T$ be a basis with an F^* -compatible differential equation and assume that J is ordered according to the F^* filtration, i.e., $J_1 \in F^{p_{\max}} V^n$ and $J_{N_F} \in F^{p_{\min}} V^n$. The matrix A defined by

$$A = \sum_{k=-n}^1 \varepsilon^k A^{(k)}(x) \quad (24)$$

has then a block structure induced by the F^* filtration. It will be convenient to organize the matrix A as

$$A = \sum_{k=-n}^1 B^{(k)}(x), \quad (25)$$

with $B^{(1)}(x) = \varepsilon A^{(1)}(x)$. For $k < 1$ the matrices $B^{(k)}(x)$ are lower block-triangular. The blocks on the lower j th block subdiagonal are given by the terms of order ε^{n+k-j} of the corresponding blocks of A . We say that a term is of B -order k if the term appears in $B^{(k)}$.

We now construct the matrix R_2 , leading to the ε -factorized basis $K = R_2^{-1} J$. The matrix R_2 is given as

$$R_2 = R_2^{(-n)} R_2^{(-n+1)} \dots R_2^{(-1)} R_2^{(0)}. \quad (26)$$

All matrices $R_2^{(k)}$ are lower block-triangular. The matrix $R_2^{(-n)}$ is of B -order $(-n)$, the matrices $R_2^{(k)}$ with $-n < k \leq 0$ are given by

$$R_2^{(k)} = \mathbf{1} + T_2^{(k)}, \quad (27)$$

where $T_2^{(k)}$ is of B -order k , and $\mathbf{1}$ denotes the $N_F \times N_F$ unit matrix. We construct the matrices $R_2^{(k)}$ iteratively, starting from $k = -n$ and ending with $k = 0$. We set $\tilde{A}^{(-n)} = A$ and

$$\tilde{A}^{(k+1)} = \left(R_2^{(k)} \right)^{-1} \tilde{A}^{(k)} R_2^{(k)} - \left(R_2^{(k)} \right)^{-1} d R_2^{(k)}. \quad (28)$$

The matrix $R_2^{(k)}$ is determined by

$$\left[\left(R_2^{(k)} \right)^{-1} \tilde{A}^{(k)} R_2^{(k)} - \left(R_2^{(k)} \right)^{-1} d R_2^{(k)} \right]_k = 0, \quad (29)$$

where $|_k$ indicates that only terms of B -order k are taken. Equation (29) defines an ε -independent system of first-order differential equations for the unknown functions in the ansatz for $R_2^{(k)}$. There are as many equations as there are unknown functions. Equation (29) also ensures that $\tilde{A}^{(k+1)}$ only has terms of B -order $\{k+1, \dots, 1\}$. Thus, in every iteration step, we improve the B order. After transformation with $R_2^{(0)}$, the matrix $\tilde{A}^{(1)}$ only has terms of B -order 1.

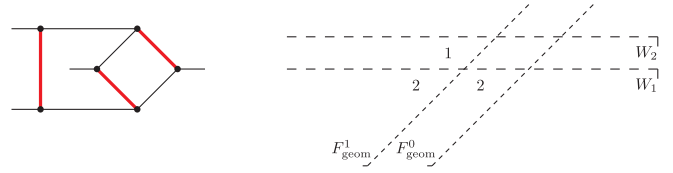


FIG. 1. A non-planar double-box integral with internal masses (indicated by red lines). The top sector has 5 master integrals, which decompose with respect to the filtrations as shown in the right figure.

This completes the construction of the ε -factorized differential equation on the maximal cut. The extension beyond the maximal cut is straightforward: Any offending term is strictly lower block-triangular and can be removed with an ansatz similar to R_2 .

Examples—We have tested the method on several known examples; see also in the longer companion paper [67]. In the End Matter, we give a pedagogical example of a Feynman integral with nontrivial filtrations. We construct the ε -factorized form without using any information on the specific geometry (an elliptic curve in this case).

With the method outlined in this Letter, we were able to compute previously unknown Feynman integrals through ε -factorized differential equations, including the parts beyond the maximal cut. One example is a nonplanar double box integral with internal masses as indicated in Fig. 1. This integral contributes to Møller scattering. It is known that the maximal cut of the top sector involves a genus two curve [72]. The top sector has five master integrals, and the F_{geom}^* filtration and the $W_{\bullet}$ filtration decompose this sector into $5 = 2 + 2 + 1$, as shown in Fig. 1.

An even more advanced example is the three-loop banana graph with four unequal masses, shown in Fig. 2. The geometry involves a K3 surface. The top sector consists of 11 master integrals and the F_{geom}^* filtration and the $W_{\bullet}$ filtration decompose this sector into $11 = 1 + 4 + 1 + 5$, as shown in Fig. 2.

We observe significant efficiency improvements. A measure, which is independent of the implementation of the algorithm, is the size of the differential equation. We observe for nontrivial systems reductions in size up to a factor of 200 after step 1 and a further reduction in size up to a factor of 10 after step 2. Clearly, reconstructing with finite field methods [73,74], rational functions that are significantly smaller, is more efficient. The factor 200 after step 1 is observed in the H graph of Ref. [75]. For the nonplanar double box integral in Fig. 1, it is a factor of 25. For the three-loop banana integral in Fig. 2, it is just a factor of 1. The improvement is explained by the avoidance of spurious polynomials in the denominator of our algorithm. For the three-loop banana integral the standard method does not introduce spurious polynomials, hence there is no significant improvement in the size of the differential equation.

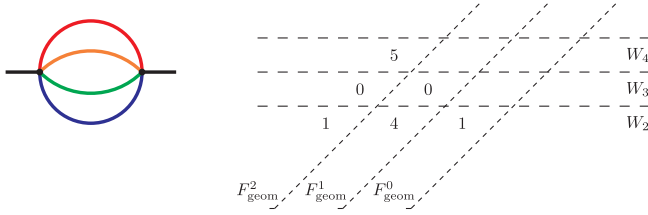


FIG. 2. The three-loop banana graph with unequal masses. The top sector has 11 master integrals, which decompose with respect to the filtrations as shown in the right figure.

Conclusions—In this Letter, we reported on a systematic algorithm to obtain an ϵ -factorized differential equation without relying on prior knowledge of the underlying geometry. It would be interesting to investigate whether the order relation proposed in this Letter always leads to an F_{comb}^* -compatible differential equation. We do not know about a counterexample, but this is currently a conjecture. Additionally, we expect that the proposed method for integration-by-parts reduction yields further efficiency improvements.

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Data availability—The data are not publicly available. The data are available from the authors upon reasonable request.

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End Matter

As a pedagogical example, we consider the Feynman integral named “sector 79” in Ref. [45]. This is one of the simplest examples with nontrivial filtrations. There are three master integrals in the top sector. The Feynman graph and the (final) decomposition in terms of the filtration F_{geom}^* and the weight filtration $W_\bullet$ are shown in Fig. 3. We follow the notation of Ref. [45]. The inverse propagators are

$$\begin{aligned}\sigma_1 &= -(k_1 + p_2)^2 + m^2, & \sigma_2 &= -k_1^2 + m^2, \\ \sigma_3 &= -(k_1 + p_1 + p_2)^2 + m^2, & \sigma_4 &= -(k_1 + k_2)^2 + m^2, \\ \sigma_5 &= -k_2^2, & \sigma_6 &= -(k_2 + p_3 + p_4)^2, \\ \sigma_7 &= -(k_2 + p_3)^2 + m^2, & \sigma_9 &= -(k_2 - p_2 + p_3)^2, \\ \sigma_8 &= -(k_1 + p_2 - p_3)^2 + m^2.\end{aligned}\quad (\text{A1})$$

We set $x_1 = s/m^2$ and $x_2 = t/m^2$. On the maximal cut we obtain, using the loop-by-loop approach, a minimal one-dimensional Baikov representation from an integrand

with a dot on either propagator 4 or 7. With $D = 4 - 2\epsilon$ and $z_1 = \sigma_8/m^2$ this yields

$$\begin{aligned}e^{2\epsilon\gamma_E} \int_{C_{\text{maxcut}}} \prod_{r=1}^2 \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{(m^2)^{2+2\epsilon}}{\sigma_1 \sigma_2 \sigma_3 \sigma_4^2 \sigma_7} \\ = C_{\text{Baikov}} \int \frac{dz_1}{2\pi i} [p_1(z)]^{-\frac{1}{2}} [p_2(z)]^{-\frac{1}{2}-\epsilon} [p_3(z)]^{-\frac{1}{2}-\epsilon}.\end{aligned}\quad (\text{A2})$$

We have

$$C_{\text{Baikov}} = \frac{2^{4+4\epsilon} \pi^4 e^{2\epsilon\gamma_E}}{[\Gamma(\frac{1}{2}-\epsilon)]^2 x_1^{1+\epsilon}} [(1-x_2)^2 + x_1 x_2]^\epsilon$$

and

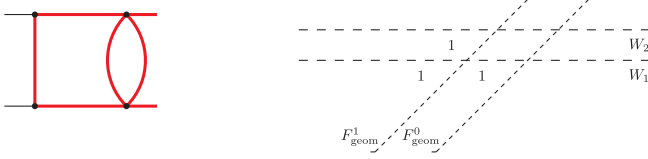


FIG. 3. A two-loop integral with masses (indicated by red lines). The top sector has 3 master integrals, which decompose with respect to the F^*_{geom} filtration and the W_* filtration as shown in the right figure.

$$\begin{aligned} p_1 &= z_1 - x_2, \\ p_2 &= z_1 + 4 - x_2, \\ p_3 &= (z_1 + 1)^2 - 4 \left[x_2 + \frac{(1 - x_2)^2}{x_1} \right]. \end{aligned} \quad (\text{A3})$$

Thus $I_{\text{even}} = \emptyset$ and $I_{\text{odd}} = \{1, 2, 3\}$. In this example we have $\dim V^1 = \dim H^1_{\omega} = 3$, hence, we do not need to worry about symmetries introduced by integration nor about supersectors.

The homogenizations are $P_1 = z_1 - x_2 z_0$, $P_2 = z_1 + (4 - x_2) z_0$ and

$$P_3 = (z_1 + z_0)^2 - 4 \left[x_2 + \frac{(1 - x_2)^2}{x_1} \right] z_0^2. \quad (\text{A4})$$

We further introduce $P_0 = z_0$ with $a_0 = 0$ and $b_0 = 6\epsilon$. We have $I^0_{\text{even}} = \{0\}$ and $I^0_{\text{odd}} = \{1, 2, 3\}$. The twist function is then given by

$$U(z_0, z_1) = P_0^{3\epsilon} P_1^{-\frac{1}{2}} P_2^{-\frac{1}{2}-\epsilon} P_3^{-\frac{1}{2}-\epsilon}. \quad (\text{A5})$$

It is easy to check that with $C_{\text{abs}} = \epsilon^3 x_1$ the product $C_{\text{abs}} C_{\text{Baikov}}$ is pure of transcendental weight zero.

In H^1_{ω} , we consider differential forms,

$$\Psi_{\mu_0 \mu_1 \mu_2 \mu_3}[\mathcal{Q}] = C_{\text{Baikov}} C_{\epsilon} U(z) \hat{\Phi}_{\mu_0 \mu_1 \mu_2 \mu_3}[\mathcal{Q}] \eta. \quad (\text{A6})$$

The twist function U is homogeneous of degree $d_U = -2$ and $\eta = z_0 dz_1 - z_1 dz_0$ is homogeneous of degree 2, hence $\hat{\Phi}$ has to be homogeneous of degree 0.

We start by looking at residues. As there is only one polynomial from the set $I^0_{\text{even}} = \{0\}$, the only possibility is a residue at $P_0 = 0$. For $(\mu_0, \mu_1, \mu_2, \mu_3) = (1, 0, 0, 0)$ we must have $\deg \mathcal{Q} = 1$. Thus, we consider

$$\hat{\Phi}_{1000}[z_1] = \frac{z_1}{z_0}. \quad (\text{A7})$$

In this case we have $C_{\text{rel}} = 3\epsilon$ and $C_{\text{clutch}} = \epsilon^{-1}$. At pole order 1, there are no further possibilities to construct differential forms with nonzero residues. Thus $H^{(1,1)}_{\text{geom}}$ is generated by

$$\Psi_2 = \Psi_{1000}[z_1] = 3\epsilon^3 x_1 U(z) \frac{z_1}{z_0} \eta. \quad (\text{A8})$$

We then consider the weight ($w = 1$) part. Within a given weight, our ordering criterion prefers differential forms of the lowest pole order. For pole order zero, we have $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$. In this case, we must have $\deg \mathcal{Q} = 0$ and therefore (up to irrelevant prefactors) $\mathcal{Q} = 1$. As this is the only possibility, $H^{(1,0)}_{\text{geom}}$ is generated by

$$\Psi_1 = \Psi_{0000}[1] = \epsilon^3 x_1 U(z) \eta. \quad (\text{A9})$$

For the last basis element, we consider pole order one. We have several possibilities, for example,

$$\frac{z_0}{P_1}, \quad \frac{z_0}{P_2}, \quad \frac{z_0^2}{P_3}. \quad (\text{A10})$$

Which form is picked will depend on the unspecified dots in the ordering criterion ($a, w, o, |\mu|, \dots$). The actual choice is not essential here, and for concreteness, let us assume that the algorithm picks z_0/P_1 . With this choice we have $C_{\text{rel}} = -1/2$, $C_{\text{clutch}} = 1/\epsilon$, and $H^{(0,1)}_{\text{geom}}$ is generated by

$$\Psi_3 = \Psi_{0100}[z_0] = -\frac{1}{2} \epsilon^2 x_1 U(z) \frac{z_0}{z_1 - x_2 z_0} \eta. \quad (\text{A11})$$

With $\Psi = (\Psi_1, \Psi_2, \Psi_3)^T$, we obtain a differential equation in Laurent polynomial form, compatible with the F^*_{comb} filtration,

$$d\Psi = [B^{(1)} + B^{(0)} + B^{(-1)}] \Psi, \quad (\text{A12})$$

with $B^{(-1)}$ and $B^{(0)}$ of the form

$$\begin{aligned} B^{(-1)} &= \left(\begin{array}{c|cc} B_{11}^{(-1)} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \frac{1}{\epsilon} B_{31}^{(-1)} & B_{32}^{(-1)} & B_{33}^{(-1)} \end{array} \right), B^{(0)} \\ &= \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline B_{21}^{(0)} & 0 & 0 \\ \hline B_{31}^{(0)} & 0 & 0 \end{array} \right). \end{aligned} \quad (\text{A13})$$

In these matrices, we made the ϵ dependence explicit and we indicated the block structure due to the F^*_{comb} filtration. We then rotate the system to an ϵ form with the rotation matrix

$$R_2 = R_2^{(-1)} R_2^{(0)}. \quad (\text{A14})$$

The general ansatz for $R_2^{(-1)}$ and $R_2^{(0)}$ is

$$\begin{aligned}
 R_2^{(-1)} &= \left(\begin{array}{c|cc} R_{11}^{(-1)} & 0 & 0 \\ \hline \frac{1}{\varepsilon} R_{21}^{(-1)} & R_{22}^{(-1)} & R_{23}^{(-1)} \\ \hline \frac{1}{\varepsilon} R_{31}^{(-1)} & R_{32}^{(-1)} & R_{33}^{(-1)} \end{array} \right), R_2^{(0)} \\
 &= \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline R_{21}^{(0)} & 1 & 0 \\ \hline R_{31}^{(0)} & 0 & 1 \end{array} \right). \quad (\text{A15})
 \end{aligned}$$

Again, we made the ε dependence explicit and we indicated the block structure due to the F_{comb}^* filtration. In this particular example, we find immediately that we may set $R_{21}^{(-1)} = R_{23}^{(-1)} = 0$ and $R_{22}^{(-1)} = 1$. For the other entries, we obtain a system of ε -independent first-order differential equations. An example is

$$d \ln R_{11}^{(-1)} = B_{11}^{(-1)} + \frac{R_{31}^{(-1)}}{R_{11}^{(-1)}} B_{13}^{(1)}. \quad (\text{A16})$$

It is not too difficult to solve this system. We then determine the preimages in V^1 , which map to the differential forms in H_ω^1 . It is sufficient to do this for (Ψ_1, Ψ_2, Ψ_3) . For Ψ_1 and Ψ_2 this is straightforward

$$\begin{aligned}
 \iota(\varepsilon^3 x_1 I_{111200100}) &= \Psi_1, \\
 \iota(3\varepsilon^3 x_1 I_{112001(-1)0}) &= \Psi_2. \quad (\text{A17})
 \end{aligned}$$

For Ψ_3 one first chooses a basis (I_1, I_2, I_3) in V^1 and determines coefficients c_1, c_2, c_3 from

$$\iota(c_1 I_1 + c_2 I_2 + c_3 I_3) = \Psi_3. \quad (\text{A18})$$

We then find

$$\begin{aligned}
 K_1 &= \frac{\varepsilon^3 x_1}{R_{11}^{(-1)}} I_{111200100}, \\
 K_2 &= 3\varepsilon^3 x_1 I_{1112001(-1)0} - R_{21}^{(0)} K_1. \quad (\text{A19})
 \end{aligned}$$

The explicit expression for K_3 is slightly more lengthy, and we refrain from reporting it here.

Note that the construction presented here does not use any information on a particular geometry. Of course, $R_{11}^{(-1)}$ is a period of an elliptic curve, but we do not need this information. We only need to know the filtrations, but not the specific geometry.