

O(4)-Breaking Bubble

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False vacuum decay in scalar field theory is thought to be dominated by Coleman's O(4)-symmetric bounce, the minimum action nontrivial solution of the imaginary-time equations of motion. Beyond the bounce, nonconstructive existence proofs of O(4)-breaking solutions are available in the mathematics literature, but the solutions themselves, and their physics, have remained unknown. Considering the simple, bounded-below, scalar field potential $V(\phi) = (m^2/2)\phi^2 - (\lambda/4)\phi^4 + (g/6)\phi^6$, we construct a nonradial solution explicitly: two bubble tubes of opposite sign wrapping orthogonal rings, invariant under $O(2) \times O(2)$ rotations combined with a Z_2 symmetry that exchanges the rings. The solution admits valid Cauchy data for real-time evolution from a $t = 0$ slice, and supports an odd > 1 number of unstable deformation modes. Having found this first solution we proceed to discover additional examples, in 3D as well as other 4D theories, including a Z_2 -violating potential and scalar QED for which there were no previously known existence proofs or O(4)-dominance theorems.

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Introduction—The semiclassical theory of false vacuum decay rests on a Euclidean saddle point: the bounce, a finite-action solution of the Euclidean (imaginary-time) equations of motion that interpolates through the potential barrier [1,2]. For a single scalar field in flat four-dimensional spacetime, Coleman, Glaser, and Martin (CGM) proved that the solution of minimum action is O(4)-symmetric [3], and their proof was extended to multiple fields in [4]. The CGM theorem fixes the leading semiclassical decay rate, but it is a statement about the minimizer, not a uniqueness theorem, i.e., it leaves open whether the theory admits *any* other regular finite-action solutions of lower symmetry. Known examples of reduced-symmetry bounces rely on an external agent that breaks O(4) by hand, e.g., background fields [5,6], finite temperature [7], a de Sitter horizon [8], an already-existing bubble wall [9], or singular field configurations sustained by tuned potentials [10,11]. In the mathematics literature, by contrast, existence is settled under reasonably broad admissibility conditions. Bartsch and Willem [12] proved that the Euclidean scalar field equation admits finite-action sign-changing solutions invariant under $O(2) \times O(2)$ and odd under the exchange of two orthogonal planes, and

Mederski [13] and Jeanjean and Lu [14] extended the existence to the optimal (Berestycki-Lions) class of nonlinearities. These arguments permitted potentials growing no faster than quartically. However, a steeper, confining potential—such as the one we study below—is nonetheless brought into this class by truncation [15], a modification that leaves bounded finite-action solutions unchanged. These proofs were nonconstructive: they characterized a minimizer within the symmetry class and bounded its action from below, but the solution itself—its profile, fluctuation spectrum, and physical role—was not exhibited. Constructive and spectral results do exist for neighboring classes: necklaces of alternating-sign bumps with dihedral symmetry [16], which are explicitly not of the present type, and, for the critical conformal equation, sign-changing solutions concentrating on Hopf links and Clifford tori [17]. The Bartsch-Willem class itself has, to our knowledge, never been exhibited concretely.

In this Letter we construct a solution and read off its physics. After doing so first for the Bartsch-Willem class, we extend our method and discover other solutions that go beyond currently known existence proofs, such as $O(2) \times O(2)$ without parity symmetry, as well as solutions in scalar quantum electrodynamics (QED), where no CGM-like O(4)-dominance theorem exists.

For concreteness, we start with the following theory:

$$V(\phi) = \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 + \frac{g}{6}\phi^6, \quad (1)$$

with $m^2, \lambda, g > 0$, bounded from below. Rescaling $\phi \rightarrow (m/\sqrt{\lambda})\phi$ and $x \rightarrow x/m$ brings the Euclidean action to

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$$S = \frac{1}{\lambda} \int d^4x \left[\frac{1}{2} (\partial\phi)^2 + \frac{1}{2} \phi^2 - \frac{1}{4} \phi^4 + \frac{\bar{g}}{6} \phi^6 \right], \quad (2)$$

with $\bar{g} = gm^2/\lambda^2$. At the leading semiclassical order, the theory depends on the single dimensionless combination $\bar{g}$, and the overall $1/\lambda$ cancels in any ratio of actions, in particular in $S_{\text{odd}}/S_{\text{O}(4)}$ below. We work with the rescaled form from here on and write g for $\bar{g}$. The false vacuum sits at $\phi = 0$; for $0 < g < 3/16$, a deeper vacuum exists at $\phi \neq 0$, becoming degenerate with the origin as $g \rightarrow 3/16$ (the thin-wall limit).

The solution—Group the four Euclidean coordinates into two planes and define $r = \sqrt{x_0^2 + x_1^2}$, $s = \sqrt{x_2^2 + x_3^2}$. We impose the ansatz

$$\phi = \phi(r, s), \quad \phi(r, s) = -\phi(s, r), \quad (3)$$

i.e., invariance under $O(2) \times O(2)$ rotations of the two planes, together with a parity P that exchanges the planes and simultaneously flips $\phi \rightarrow -\phi$ [a symmetry of Eq. (1), which is even in ϕ]. This is precisely the symmetry class of Refs. [12–14] (the space $X_\tau \cap H_{\mathcal{O}_2}^1$ in the notation of [13,14], with $N = 4$, $M = 2$), which inspired the ansatz; a finite-action solution minimizing the action among all nontrivial solutions in this class is guaranteed to exist for our potential [13,14]. We solve the Euclidean equation $\nabla^2 \phi = V'(\phi)$ on the (r, s) half-plane with a globalized Newton method starting from a seed function in the symmetry class. The solution converges on multiple grids, to machine precision [see Supplemental Material (SM) [18] for details].

Figure 1 shows the constant-field surfaces of the solution on the slice $x_0 = 0$ for $g = 0.1$. The structure is transparent: a lump of positive field centered on the s axis at $(r, s) = (0, R_*)$, and its P mirror image, with negative sign, at $(R_*, 0)$. Restoring the angles, the positive lump is a “bubble tube” whose core traces a ring of radius R_* in the (x_2, x_3) plane, while the negative tube wraps the orthogonal ring in the (x_0, x_1) plane. The rings are mutually orthogonal in $\mathbb{R}^4$.

The solution is even in x_0 , so this slice carries time-symmetric Cauchy data $(\phi, \dot{\phi} = 0)$, and conservation of the Euclidean-time energy along x_0 , together with $\phi \rightarrow 0$ as $x_0 \rightarrow -\infty$, fixes the Lorentzian energy to zero. As for Coleman’s bounce, the configuration would materialize at rest. Its Euclidean development and real-time evolution are computed in SM [18].

Action—At $g = 0.1$ we find, after Richardson extrapolation in the grid spacing,

$$S_{\text{odd}} = 2849.8, \quad \frac{S_{\text{odd}}}{S_{\text{O}(4)}} = 7.260, \quad (4)$$

with $S_{\text{O}(4)} = 392.517$ the action of the $O(4)$ bounce computed in the same units. The virial identity $S = K/2$,

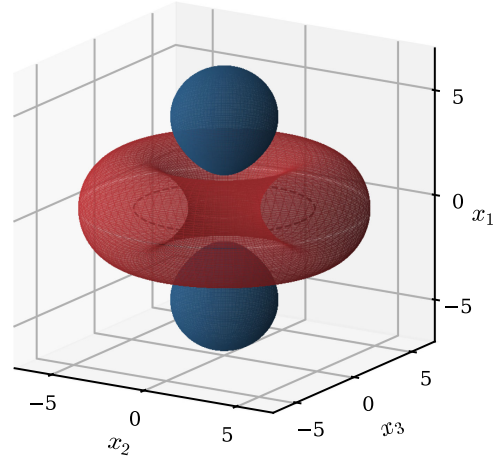


FIG. 1. Constant-field surfaces $|\phi| = 1$ of the nonradial solution at $g = 0.1$, on the slice $x_0 = 0$. The positive ring appears as a torus in the (x_2, x_3) plane (core dashed); the negative ring, which lies in the (x_0, x_1) plane, pierces the slice in two antisign bubbles on the torus axis at $x_1 = \pm R_*$, with $R_* \simeq 4.15$. [The solution on the reduced (r, s) half-plane is shown in SM [18], Fig. S1, left panel.] Because the solution is even in x_0 , the $x_0 = 0$ slice carries valid Cauchy data for real-time development.

with K the gradient term, is satisfied to six digits by the bounce and at the level of the $O(h^2)$ discretization error by the grid solution. We track the solution across $0.02 \leq g \leq 0.135$; the action ratio, shown in Fig. 2, decreases monotonically from 9.7 to 6.4. The CGM [3] bound, $S_{\text{odd}}/S_{\text{O}(4)} \geq 1$, holds with room to spare, and so does the stronger variational bound $S_{\text{odd}} \geq 2 S_{\text{O}(4)}$, which every nontrivial solution of the symmetry class Eq. (3) must obey [13,14]. The latter bound has important consequences for physics: since nucleation rates scale as e^{-S} , any nonradial bubble in the symmetry class—including the solution constructed here—is less likely to nucleate than a pair of independent Coleman bubbles.

The tube cross section is, to good accuracy, the $O(3)$ -symmetric bounce of the same potential: at $g = 0.1$ the core value is $\phi = 2.950$ against $\phi_3(0) = 2.906$ for the three-dimensional bounce, whose action sets a tension $\sigma_3 = 48.05$. A thin-tube estimate of the action, two rings of length $2\pi R_*$ each,

$$S_{\text{odd}} \approx 2 \times 2\pi R_* \sigma_3, \quad (5)$$

reproduces the true action to $O(10\%)$ at $g = 0.1$. The ring radius grows from $R_* = 1.5$ to 6.8 across the same range, tracking the thin-tube estimate. Toward $g \rightarrow 0$ the rings shrink and approach each other, and the tube picture degrades; toward the thin-wall limit the numerics lose the solution, and we do not know whether the branch persists all the way to $g = 3/16$.

Fluctuation spectrum—The physical role of a saddle is fixed by the spectrum of its fluctuation operator,

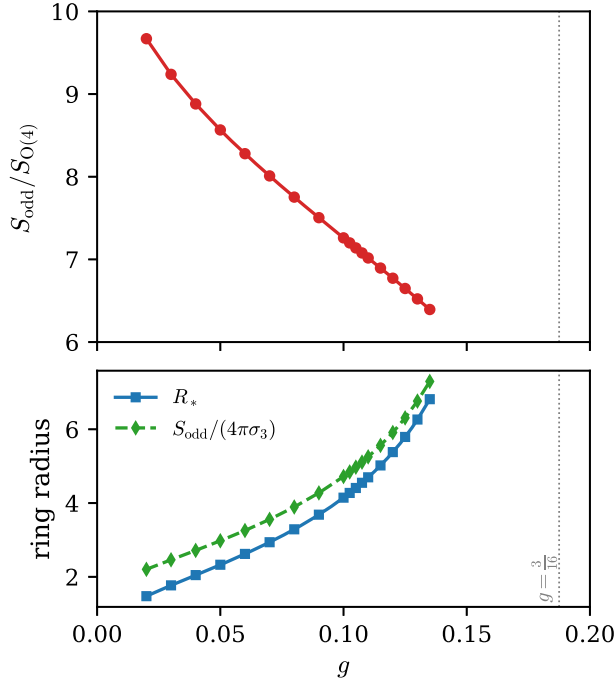


FIG. 2. Top: action of the nonradial solution in units of the $O(4)$ bounce action, across the coupling range of the model. The ratio stays well above unity (the CGM bound), and indeed above the stronger bound $S_{\text{odd}} \geq 2S_{O(4)}$ obeyed by every solution in the class. Bottom: ring radius R_* (squares), compared with the effective radius $S_{\text{odd}}/(4\pi\sigma_3)$ inferred from the thin-tube model [Eq. (5), diamonds]. Numerical continuation fails for $g \gtrsim 0.14$, where the thin-wall limit makes the Newton basin narrow; the trend of R_* is consistent with the solution growing large and soft as $g \rightarrow 3/16$ (dotted line).

$\mathcal{H} = -\nabla^2 + V''(\phi)$. Decomposing fluctuations as $\delta\phi = \psi(r, s)e^{i(m\theta_1 + n\theta_2)}$, with $\theta_{1,2}$ the angles of the two planes, $\mathcal{H}$ block-diagonalizes into sectors labeled by (m, n) , with sectors (m, n) and (n, m) degenerate by P . Table I summarizes the low-lying spectrum at $g = 0.1$.

The zero modes count the broken symmetries and close the bookkeeping: four translations, plus four rotations, because the solution breaks the six generators of $O(4)$ down to the two of $O(2) \times O(2)$. We verified that the $(1, 1)$ quadruplet of zero modes are proportional to the combination $r\partial_s\phi - s\partial_r\phi$ to numerical precision.

The negative modes have an equally clean origin. A straight, isolated bubble tube has a single localized negative mode, its cross-sectional collapse or expansion, with $\omega_0^2 = -0.561$ at $g = 0.1$. Bending the tube into a ring of radius R_* and Fourier decomposing along it gives the dispersion estimate

$$\omega_n^2 \approx \omega_0^2 + \frac{n^2}{R_*^2}, \quad (6)$$

which predicts instability for $n \leq 3$ and stability for $n \geq 4$ at $g = 0.1$ —exactly as found in Table I, where the $n = 3$

TABLE I. Low-lying fluctuation spectrum at $g = 0.1$. Degeneracies include the angular multiplicity $(2 - \delta_{m0})(2 - \delta_{n0})$ and the $(m, n) \leftrightarrow (n, m)$ exchange. At this coupling there are 15 negative and 8 zero modes. All ω^2 carry an $O(h^2)$ discretization uncertainty below 10^{-5} , estimated by Richardson extrapolation to vanishing grid spacing.

Sector	ω^2	Degeneracy	Interpretation
(0,0)	-0.492	1	Breathing, P -even
(0,0)	-0.315	1	Breathing, P -odd (dilation)
(0,0)	-0.090	1	Ring-ring seesaw
(0,1)	-0.349	4	Tube bending, $n = 1$
(0,2)	-0.200	4	Tube bending, $n = 2$
(0,3)	-0.0078	4	Tube bending, $n = 3$
(0,1)	0	4	Broken translations
(1,1)	0	4	Broken $O(4)$ rotations
(0,4)	+0.233	4	First stable bending mode

mode sits barely below zero and the $n = 4$ mode safely above. [The estimate Eq. (6) reproduces the eigenvalues themselves only to within a factor of 2 for the lower modes; its robust output is the location of the zero crossing.] The two $n = 0$ breathing modes of the individual rings hybridize into the P -even or P -odd pair at the bottom of the $(0, 0)$ sector; the P -odd combination overlaps at the 93% level with the dilation direction $x \times \nabla\phi$, i.e., it is the “would-be bounce” negative mode of this saddle. The soft third $(0, 0)$ mode is a seesaw that grows one ring at the expense of the other.

Following the spectrum along the branch (SM [18] Fig. S2) turns the mode count into a function of g . The dispersion Eq. (6) makes the trend intuitive: as g grows, R_* grows and $|\omega_0^2|$ shrinks, so bending modes are stabilized one quadruplet at a time. The $n = 4$ quadruplet crosses zero at $g \simeq 0.034$ and the $n = 3$ quadruplet at $g \simeq 0.104$; the count of negative modes steps through $19 \rightarrow 15 \rightarrow 11$, while the eight zero modes persist unchanged, as they must. We checked that the marginal eigenvalues are stable against box size and grid spacing at the crossings (SM [18]).

What this saddle is and is not—The $O(4)$ -breaking solution does not contribute to the imaginary part of the false-vacuum energy at leading semiclassical order, as its action is well above the $O(4)$ bounce’s action [2] and, moreover, it has multiple fluctuation eigenmodes [19]. It does populate the subleading structure of the path integral, related to the large-order behavior of the theory [20].

Whether the saddle contributes to vacuum decay at *any* order depends, in Picard-Lefschetz terms [21–23], on whether its upward-flow cycle intersects the integration cycle of the false-vacuum path integral. Two facts argue that it does. First, the negative-mode count k is odd everywhere on the branch [19, 15, or 11—structurally: three modes from the $(0, 0)$ sector plus quadruplets], so the fluctuation factor i^k is imaginary and could feed into $\text{Im } E$. Second, the gradient flow through real-field configuration space along the P -odd

dilation mode connects the false vacuum on one side to unbounded true-vacuum growth on the other (SM [18]): the saddle behaves like a genuine transition state, with the same flow topology that underlies the bounce's contribution. The other two (0,0) negative modes connect escape to escape; we did not flow the remaining 12 negative modes, as this would require three-dimensional flows.

Gravitational waves—Coleman's bubble is spherical and radiates gravitationally only through quantum fluctuations [24]. The nonradial bubble radiates already at leading classical order. We can make a rough estimate of the emission using the quadrupole formula. The wall tension is $\sigma \sim m^3/\lambda$, arising from the field swinging by $\Delta\phi \sim m/\sqrt{\lambda}$ across a thickness $\sim 1/m$; the nucleated configuration is an oblate ring of this wall, of radius R_* and energy $M \sim \sigma R_*^2$, whose mass quadrupole $Q \sim MR_*^2$ changes over the light-crossing time $t_* \sim R_*$ as the bubble rounds out. Then

$$E_{\text{GW}} \sim G \frac{M^2}{R_*} \sim G \sigma^2 R_*^3 \sim \left(\frac{m}{M_{\text{Pl}}} \right)^2 \lambda^{-2} m, \quad (7)$$

where $G = M_{\text{Pl}}^{-2}$. The radiated energy is a fraction $E_{\text{GW}}/M \sim G \sigma R_* \sim (m/M_{\text{Pl}})^2/\lambda$ of the bubble's energy budget. Against the fluctuation-induced emission of a *radial* bubble [24], which scales as a single power of σ where ours carries σ^2 , the classical signal is parametrically enhanced by $1/\lambda$. A direct calculation of E_{GW} using the numerical real-time evaluation is a feasible task, in principle, but is beyond the scope of this work. We expect that, weighted by the nucleation probability $e^{-(S_{\text{odd}} - S_{\text{O}(4)})}$, the nonradial population's emission should be negligible in any stochastic background.

Other solutions—

Parity-violating potential: The choice of Eq. (1) is only representative: the same machinery allows us to construct nonradial solutions in other potentials, including solutions for which, to our knowledge, there have not been earlier proofs of existence. Consider removing the Z_2 symmetry of Eq. (1), using the potential

$$V_\kappa(\phi) = \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4 + \frac{g}{6}\phi^6 + \kappa\phi^3. \quad (8)$$

For $\kappa = g = 0.1$, starting with the $\kappa = 0$ nonradial solution as seed, we find a new solution. The $x_0 = 0$ slice of this solution is shown in Fig. 3 (also SM [18] Fig. S1, right panel). The main features of this new saddle are consistent with a deformation of the Z_2 -symmetric case: at the chosen values of g and κ it has 13 negative modes, and action larger than that of both $O(4)$ -symmetric saddles that the theory admits.

Scalar QED: Here,

$$S = \int d^4x \left[\frac{1}{4e^2} F_{\mu\nu} F_{\mu\nu} + |D_\mu \Phi|^2 + V(\rho) \right], \quad \Phi = \frac{\rho}{\sqrt{2}} e^{i\theta}, \quad (9)$$

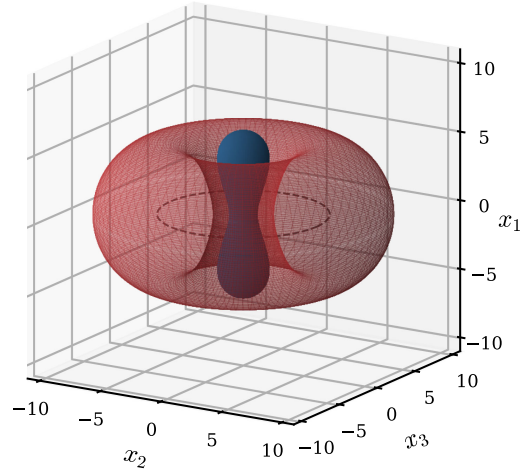


FIG. 3. Constant-field surfaces $|\phi| = 1$ on the slice $x_0 = 0$ of the nonradial, parity-violating solution obtained for the potential of Eq. (8) at $\kappa = g = 0.1$.

with $V(\rho) = \frac{1}{2}\rho^2 - \frac{1}{4}\rho^4 + (g/6)\rho^6$ and $D_\mu = \partial_\mu - iA_\mu$. Defining $(x_0, x_1) = r(\cos \alpha, \sin \alpha)$ and $(x_2, x_3) = s(\cos \beta, \sin \beta)$, we find $O(4)$ -breaking saddles starting from the ansatz

$$\Phi_B = \frac{\rho(r, s)}{\sqrt{2}} e^{i(m\alpha + n\beta)}, \quad A_B = a(r, s)d\alpha + b(r, s)d\beta, \quad (10)$$

with integer m, n . (Reference [25] used a similar construction to find infinite-action solutions of the magnetic Ginzburg-Landau equations in 4D.) The lowest-action branch we found is $(m, n) = (1, 0)$ or $(0, 1)$, demonstrated in Fig. 4 for $g = 0.1$ and $e = 1$. Here, we can reduce Eq. (10) to $\Phi_B = [\rho(r, s)/\sqrt{2}]e^{i\alpha}$, $A_B = a(r, s)d\alpha$. Interestingly, for scalar QED there is no $O(4)$ -dominance theorem. The solution's action decreases with increasing gauge coupling e : for $e = 1$ we find $S_{(1,0)}(e = 1)/S_{\text{O}(4)} \approx 2.120$, while for $e = 4$ we find $S_{(1,0)}(e = 4)/S_{\text{O}(4)} \approx 1.708$. Thus, the $S_{\text{odd}} \geq 2 S_{\text{O}(4)}$ bound of the pure scalar theory [13,14] indeed does not apply. Increasing e causes the radius of the null tube threading $\rho(r, s)$ and $a(r, s)$ (see

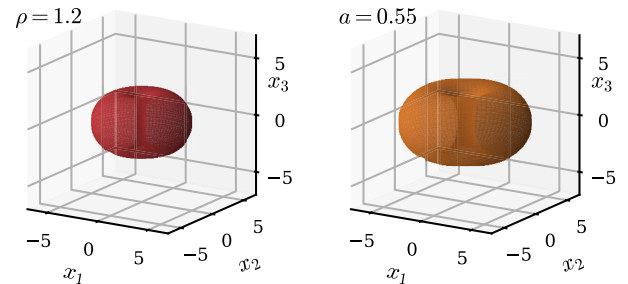


FIG. 4. Constant-field surfaces on the slice $x_0 = 0$ of the nonradial scalar QED (1,0) saddle for $g = 0.1$ and $e = 1$.

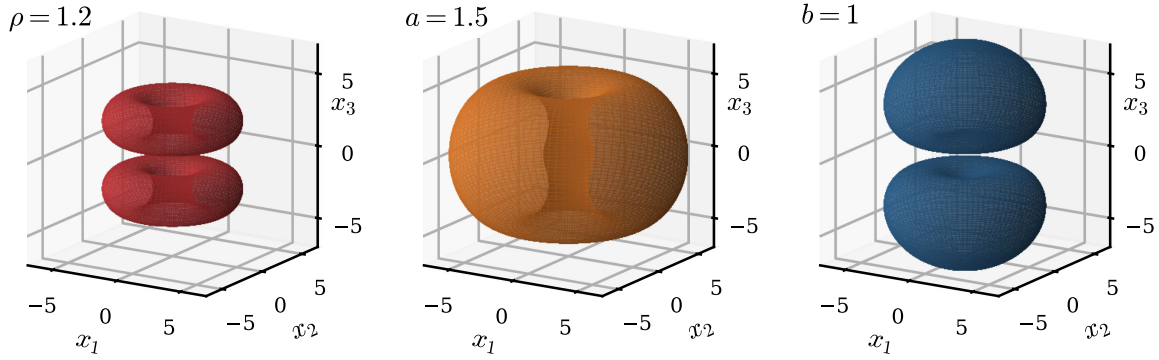


FIG. 5. Constant-field surfaces on the slice $x_0 = 0$ of the $(m, n) = (3, 2)$ nonradial scalar QED saddle for $g = 0.1$ and $e = 1$.

Fig. 4) to shrink, which requires numerical care and we did not pursue it beyond $e = 4$. As another example, Fig. 5 shows the $x_0 = 0$ slice of an $(m, n) = (3, 2)$ saddle with $S_{(3,2)}(e = 1)/S_{O(4)} \approx 10.74$. For the scalar QED solutions we did not attempt to count the number of negative modes N_- , but for all examples we were able to establish $N_- \geq 3$.

We note an interesting difference between the pure scalar and the scalar QED scenarios. The existence of a pure scalar finite-action nonradial saddle seems to require at least three potential local minima: the false vacuum and two other minima between which the solution interpolates; Eqs. (1) and (8) were minimal examples in this sense. The reason for this is the Gidas, Ni, and Nirenberg theorem [26], which states that decaying solutions that do not change sign must be radial. Scalar QED does not adhere to this rule: the phase of the scalar field is gauge-dependent. Indeed, although we do not expand on this here, we were able to find a nonradial solution of the $(1, 0)$ type in scalar QED with the double-well potential $V(\rho) = \frac{1}{2}\rho^2 - \frac{1}{3}\rho^3 + (\lambda/4)\rho^4$ for $\lambda = 0.15$ and $e = 1$.

3D: It is of interest to explore nonradial saddles in different numbers of dimensions. We demonstrate an $O(3)$ -breaking saddle in $d = 3$ with the potential [Eq. (1)]; this

could refer to the tunneling state in $d = 2 + 1$ (where, incidentally, the ϕ^6 theory is renormalizable), or to the high-temperature dimensionally reduced theory starting from $d = 3 + 1$. With intuition gained in the 4D analysis, we construct a seed configuration made of positive and negative linked tubes with tube cross section set by the $O(2)$ -symmetric bounce of the same potential. For $g = 0.1$ this seed converges into the saddle shown in Fig. 6. Interestingly, the tubes partially fragment into a linked five-bead pattern. The action (energy in the high-T interpretation) of this saddle is $S_{\phi(3)}/S_{O(3)} \approx 10.238$ and it has $N_- > 1$.

Outlook—The first solution we presented gives concrete form to the nonradial solutions whose existence was established variationally in [12–14], demonstrating that flat-space, zero-temperature scalar field theory admits regular Euclidean saddles beyond the $O(4)$ bounce, with action an order-one multiple of the bounce action. Using the same machinery we also presented, in lesser detail, new classes of solutions that go beyond previous existence proofs. This is likely the tip of the iceberg of reduced-symmetry saddles, the role of which has been speculated for five decades [1,3,4]. It would be interesting to pursue generalizations to different numbers of dimensions, compact or periodic boundary conditions, other potentials, and multifield models, where the landscape of saddles is essentially unexplored. It would also be interesting to extend the investigation of scalar gauge theories, and especially to understand if $O(4)$ -dominance theorems [3,4,13,14] apply in that arena. Our methods should also be useful for parallel investigations in condensed matter physics; see, e.g., Refs. [27,28].

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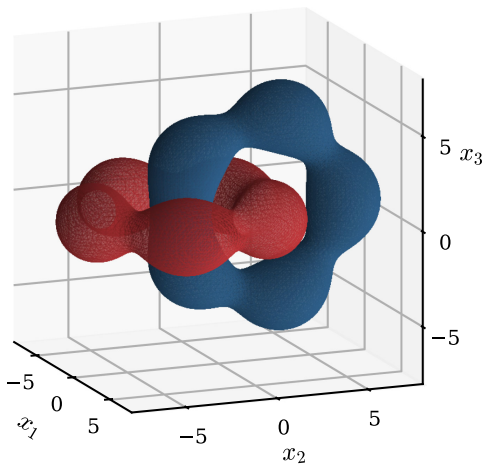


FIG. 6. Constant-field $|\phi| = 1$ surfaces of the $O(3)$ -breaking saddle in $d = 3$ Euclidean dimensions.

Data availability—There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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- [1] S. R. Coleman, The fate of the false vacuum. I. Semi-classical theory, *Phys. Rev. D* **15**, 2929 (1977); **16**, 1248(E) (1977).
 - [2] C. G. Callan, Jr. and S. R. Coleman, The fate of the false vacuum. II. first quantum corrections, *Phys. Rev. D* **16**, 1762 (1977).
 - [3] S. R. Coleman, V. Glaser, and A. Martin, Action minima among solutions to a class of Euclidean scalar field equations, *Commun. Math. Phys.* **58**, 211 (1978).
 - [4] K. Blum, M. Honda, R. Sato, M. Takimoto, and K. Tobioka, $O(N)$ invariance of the multi-field bounce, *J. High Energy Phys.* **05** (2017) 109; **06** (2017) 60.
 - [5] S. Hassan, J. March-Russell, and G. Obied, Chern-Simons bubbles: Lopsided false vacuum decay in axion electrodynamics, [arXiv:2402.00119](https://arxiv.org/abs/2402.00119).
 - [6] S. Hassan, Topological bubble nucleation, Ph.D. thesis, Oxford University, 2024.
 - [7] Y. Shoji and M. Yamaguchi, Symmetry of bounce solutions at finite temperature, [arXiv:2511.05950](https://arxiv.org/abs/2511.05950).
 - [8] A. Masoumi and E. J. Weinberg, Bounces with $O(3) \times O(2)$ symmetry, *Phys. Rev. D* **86**, 104029 (2012).
 - [9] B. Czech, Barnacles—a novel Channel for Vacuum Decay, *Phys. Lett. B* **713**, 331 (2012).
 - [10] M. Sasaki, V. Yingcharoenrat, and Y.-l. Zhang, Beyond Coleman’s instantons, *Classical Quantum Gravity* **42**, 235017 (2025).
 - [11] M. Sasaki, V. Yingcharoenrat, and Y.-l. Zhang, Singular instantons with finite action, *Int. J. Mod. Phys. A* **40**, 2540010 (2025).
 - [12] T. Bartsch and M. Willem, Infinitely many nonradial solutions of a Euclidean scalar field equation, *J. Funct. Anal.* **117**, 447 (1993).
 - [13] J. Mederski, Nonradial solutions of nonlinear scalar field equations, *Nonlinearity* **33**, 6349 (2020).
 - [14] L. Jeanjean and S.-S. Lu, Nonlinear scalar field equations with general nonlinearity, *Nonlinear Anal.: Real World Appl.* **190**, 111604 (2020).
 - [15] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations, I: Existence of a ground state, *Arch. Ration. Mech. Anal.* **82**, 313 (1983).
 - [16] M. Musso, F. Pacard, and J. Wei, Finite-energy sign-changing solutions with dihedral symmetry for the stationary nonlinear Schrödinger equation, *J. Eur. Math. Soc.* **14**, 1923 (2012).
 - [17] M. del Pino, M. Musso, F. Pacard, and A. Pistoia, Torus action on S^n and sign changing solutions for conformally invariant equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **12**, 209 (2013).
 - [18] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/pxds-lev1> for additional plots and details on the numerical procedure.
 - [19] S. R. Coleman, Quantum tunneling and negative eigenvalues, *Nucl. Phys.* **B298**, 178 (1988).
 - [20] G. Basar, G. V. Dunne, and M. Unsal, Resurgence theory, ghost-instantons, and analytic continuation of path integrals, *J. High Energy Phys.* **10** (2013) 041.
 - [21] E. Witten, Analytic continuation of Chern-Simons theory, *AMS/IP Stud. Adv. Math.* **50**, 347 (2011).
 - [22] Y. Tanizaki and T. Koike, Real-time Feynman path integral with Picard-Lefschetz theory and its applications to quantum tunneling, *Ann. Phys. (Amsterdam)* **351**, 250 (2014).
 - [23] A. Behtash, G. V. Dunne, T. Schäfer, T. Sulejmanpasic, and M. Ünsal, Toward Picard-Lefschetz theory of path integrals, complex saddles and resurgence, *Ann. Math. Sci. Appl.* **2**, 95 (2017).
 - [24] K. Blum and M. Mirbabayi, A single-bubble source for gravitational waves in a cosmological phase transition, *J. Cosmol. Astropart. Phys.* **08** (2024) 039.
 - [25] Y. Liu, X.-N. Ma, J. Wei, and W. Wu, Entire solutions of the magnetic Ginzburg–Landau equation in $\mathbb{R}^4$, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **26**, 757 (2025).
 - [26] B. Gidas, W.-M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle, *Commun. Math. Phys.* **68**, 209 (1979).
 - [27] Z. Zhang, G. Cui, K. Jiang, A.-C. Shi, P. Zhang, J. Yin, and L. Zhang, Exploring transition pathways in the Landau-Brazovskii model, *Phys. Rev. Res.* **8**, 013196 (2026).
 - [28] A. Benfenati, A. Maiani, F. N. Rybakov, and E. Babaev, Vortex nucleation barrier in superconductors beyond the Bean-Livingston approximation: A numerical approach for the sphaleron problem in a gauge theory, *Phys. Rev. B* **101**, 220505 (2020).