

Fu et al. Reply: In Ref. [1], the authors comment that the real-space spin Chern number is not a new invariant since it equals the Euler characteristic, and view the result in our Letter [2] as an application of the Chern-Gauss-Bonnet theorem. We emphasize that the real-space spin Chern number is new from the physics perspective, and it must be consistent with established mathematical theorems. The main contribution of our Letter lies in applying this invariant to elucidate how the topology of metallic structures inherently constrains the global distributions of optical polarizations in the near field.

The real-space spin Chern number is defined using the optical polarization fields on the structures' surfaces, thereby characterizing the global topological properties of these fields. In our Letter, we showed that this invariant equals the Euler characteristic via the Poincaré-Hopf theorem, which is essentially equivalent to the Chern-Gauss-Bonnet theorem in this context, as explained on Page 4: “*In this homogeneous circular polarization limit, the normal projection of the spin Berry curvature at each point is exactly identical with the local Gaussian curvature Ω_{Gauss} of the sphere (i.e., $\mathbf{n} \cdot \boldsymbol{\Omega}_{\text{spin}} \rightarrow \Omega_{\text{Gauss}}$), and thus $\mathbb{C}_{\text{spin}}$ is determined by the integral of Ω_{Gauss} , i.e., the Euler characteristic of the structure.*” We frame the discussion using Poincaré-Hopf theorem because it directly connects the structure's topology to the topological characteristics of surface polarization singularities, which explains the emergence of the structured light fields outside the surface, as illustrated in Figs. 1 and 2, where the surface polarization singularities act as sources or sinks of C lines in the surrounding light fields. On the other hand, the Chern-Gauss-Bonnet theorem does not provide this explicit connection.

The Comment provides a mathematical proof for the special case of orthonormal tangent frames ($\mathbf{e}_A, \sigma\mathbf{e}_B$), which corresponds to the complex tangent field $\mathbf{e} = \mathbf{e}_A + i\mathbf{e}_B$ of *uniform circular polarization* on the surface. This proof is essentially identical to our explanation cited in the above paragraph. While the Comment argues that the connection $\tilde{\omega}_{BA}$ defined with general tangent fields differs from the connection ω_{BA} of this special case by a globally defined 1-form, leading to $\int_M d\omega_{BA} = \int_M d\tilde{\omega}_{BA} = \chi(M)$, this reasoning implicitly depends on Stokes' theorem. In contrast, our Letter provides an explicit proof based on Stokes' theorem that applies to optical fields of *arbitrary polarization*. Thus, the Comment reformulates our results using differential form notation, but the underlying mathematical principles remain the same.

The Comment approaches our results from a mathematical standpoint, whereas our motivation and contribution lie in the physical context. We introduce the real-space spin Chern number to explain the relationship between near-field polarization singularities, optical geometric phases, and Euler characteristic of structures. Following established conventions in physics, topological invariants are often named to reflect their physical application (e.g., the topological invariant of two-dimensional spin textures is widely known as the Skyrmion number, although mathematically it represents the Brouwer degree of $S^2 \rightarrow S^2$). The real-space spin Chern number is a new concept in the physical sense: it is constructed *in real space using the magnetic fields of nonperiodic structures*, which is different from conventional (spin) Chern number constructed *in momentum space using the eigenstates of periodic structures* (see Table I in our Letter). This real-space invariant, naturally arising from the conventional Berry connection and optical spin (helicity), determines the global topological properties of near-field polarization, which is both experimentally measurable and physically transparent. Our Letter enables the manipulation of optical near fields through the geometric topology of optical structures, with potential applications in optical metrology and sensing.

We further note that the real-space spin Chern number does not unconditionally apply to any surface magnetic field, as noted in our Letter: “*The V points with a vanished field norm can also emerge on the structure's surface under certain symmetry, rendering the spin Berry curvature ill-defined at these points.*” In such cases, the relationship between the real-space spin Chern number and Euler characteristic becomes indeterminate. Our Letter thus elucidates the physical constraints on applying this topological invariant to real optical fields on surfaces.

Finally, the Comment authors note that the mathematical notions of connection, curvature and Chern class apply to any parameter space, and therefore view the extension of these notions to real space of physical systems as a misconception. From a physics perspective, however, the crucial point is identifying the appropriate physical context in which we can apply these notions to discuss physics. The embodiment of abstract mathematical notions in concrete physical contexts is essential to physics, and this is what distinguishes it from pure mathematics.

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- [1] D. Felbacq and E. Rousseau, preceding Comment, [Phys. Rev. Lett. **136**, 149301 \(2026\)](#).
- [2] T. Fu, R.-Y. Zhang, S. Jia, C. T. Chan, and S. Wang, [Phys. Rev. Lett. **132**, 233801 \(2024\)](#).

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