

Universal Critical Dynamics of Vortex Shedding in Quantum Fluids

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In classical fluid dynamics, the von Kármán vortex street behind a bluff body is the canonical example used to study vortex formation and dynamics. This is governed by a local Hopf bifurcation, which drives the continuous and asynchronous release of vortical structures in the wake. In superfluid systems, however, the quantization of vorticity fundamentally alters this picture. Several aspects of quantum vortex shedding remain unresolved, including the absence of a unified description spanning the pre- and post-threshold regimes. We show that the onset of vortex emission is governed by a global saddle node on invariant circle bifurcation. This mechanism produces a discrete transition characterized by a macroscopic bottleneck in phase space and a universal inverse-square-root divergence of the emission period near the critical velocity. These features are robust across obstacles of different shapes, revealing a universal mechanism underlying vortex shedding in quantum fluids.

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The dynamics of vortex formation and release is fundamental to describing the physics of quantum fluids, such as zero-temperature superfluid helium and atomic Bose-Einstein condensates, as it dictates how an otherwise dissipationless flow gives way to instability and energy transfer. Indeed, vortex nucleation marks the breakdown of superfluidity [1]. While a condensate can in principle sustain motion without drag, the appearance of quantized vortices at obstacles or through dynamical instabilities provides the primary pathway by which this dissipationless character is lost [2,3]. Moreover, vortex shedding governs energy and momentum redistribution: each nucleation event couples to the background flow, radiates phonons, and contributes to effective drag forces [2,4,5]. The regularity and timing of vortex emission encode the stability properties of the system and chart its transition from stationary flow to periodic and chaotic shedding [6–12]. As quantized vortices are topologically protected [4,6] and dissipation originates from mechanisms different from molecular viscosity, their shedding dynamics provides a unique insight into universal instability mechanisms that extend beyond the classical framework. Therefore, establishing the mechanisms that trigger vortex formation and

release is essential for developing a predictive theory of superfluid instabilities, thereby elevating the onset of shedding from a phenomenological threshold to a quantitatively predictive dynamical scenario.

In this Letter, we show that, although qualitative parallels can be drawn between vortex shedding in classical and quantum fluids, the nature of the underlying bifurcation is different. In the context of Newtonian fluids, the prototypical setup used to study vortex shedding is the uniform flow past a circular cylinder. In this configuration, the flow remains steady and symmetric up to a Reynolds number of approximately $Re \approx 47$ [13,14], beyond which periodic vortex shedding, commonly known as the von Kármán vortex street, emerges via a Hopf bifurcation. On the contrary, quantum fluids have no intrinsic viscosity, so that the classical Reynolds number is not directly applicable, although widely used generalizations can be found in the literature [8,15–18]. Moreover, vorticity is carried solely by quantized vortices, each with a fixed circulation quantum h/m . However, both experiments [7,19–23] and numerical simulations [2,3,24,25] reveal that superflow past an obstacle can destabilize via vortex nucleation, with the resulting quantized vortices often arranging into coherent wake patterns reminiscent of classical von Kármán streets.

Sasaki *et al.* [3], extending the seminal insights of Frisch *et al.* [2], demonstrated that an impenetrable laser-induced obstacle, moving through a Bose-Einstein condensate at speeds exceeding a critical velocity, generates long-lived vortex wakes. The value of the critical velocity is equal to a fraction of the speed of sound and it has been evaluated

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theoretically and experimentally [19,25–29]. The reason behind this is that locally around the surface of the obstacle the velocity of the flow exceeds the speed of sound, thus violating Landau’s criterion for superfluidity [1,2]. Three distinct shedding regimes were identified [3,19]: (i) just above the nucleation threshold, vortex-antivortex pairs detach symmetrically from the obstacle; (ii) at slightly higher velocities, pairs of like-signed vortices are shed alternately from either side, forming a staggered street with a geometric ratio $b/\ell \approx 0.28$, consistent with von Kármán’s stability criterion; (iii) at even higher velocities, vortex shedding becomes irregular. Despite these striking parallels, quantization fundamentally alters the underlying dynamics, thus permitting vortex detachment only when the vorticity exceeds the circulation quantum.

The onset of vortex shedding has been extensively studied in previous works. In one dimension, where vortex release is replaced by the periodic nucleation of one-dimensional solitons, a saddle-node bifurcation was analytically found by Hakim [30]. Saddle-node bifurcations have also been identified in two-dimensional superflow past a cylinder [29,31–34], primarily to describe the topology of stationary solutions branches. Stability is usually inferred from time-dependent simulations of computed stationary states, rather than being determined through a systematic linear stability analysis. Furthermore, several aspects remain unclear, including the absence of stationary solutions beyond the threshold, the emergence of dissipative signatures in the time-dependent evolution—most notably the non-Hamiltonian scaling of the vortex emission period [29,35]—and, more broadly, a unified description of the pre- and post-threshold regimes.

The dynamics of the system can be described at the mean-field level by the Gross-Pitaevskii equation (GPE) [6,36]. The modulus and phase gradient of the wave function $\psi(\mathbf{r}, t)$ can be used to define the macroscopic density and velocity fields. Starting from the equation written in the laboratory frame, we reformulate the problem as the scattering of a sinusoidal wave against the obstacle standing at the origin of a reference frame moving with velocity $-k$ on the x axis. Throughout this work, the following dimensionless form of the GPE will be adopted [37]

$$i\partial_t\psi = -k^2\psi - \Delta\psi + V(\mathbf{r})\psi + g(|\psi|^2 - 1)\psi. \quad (1)$$

The laser-induced obstacle is modeled by a Gaussian potential $V(\mathbf{r}) = V_0 e^{-r^2/\alpha^2}$. Given a steady solution $\phi(\mathbf{r})$ of (1), we linearize the system to analyze stability and reveal the vortex nucleation mechanism by positing [36]

$$\psi(\mathbf{r}, t) = [\phi(\mathbf{r}) + p(\mathbf{r})e^{i\omega t} + q^*(\mathbf{r})e^{-i\omega^* t}].$$

Retaining only linear terms in the perturbations p and q leads to the dimensionless Bogoliubov–de Gennes (BdG)

eigenproblem. The value of ω determines linear stability: any mode with $\text{Im } \omega < 0$ grows exponentially, signaling an instability of the base state.

We have applied state-of-the-art numerical continuation methods to find the stationary wake flows and linear stability tools to solve the BdG problem around each computed stationary solution. The temporal evolution of the system is simulated using a finite-difference discretization of the GPE. Sponge-type boundary conditions are implemented to avoid nonphysical reflections [37–40]. By combining the results of the continuation and linear stability procedures with those of the dynamical algorithm, we can determine precise stability thresholds and show the substantial differences between classical and quantum wake transitions, thereby establishing a bifurcation-based scenario that consistently links stationary branches to vortex shedding states within a unified dynamical systems framework.

To illustrate the dynamics of vortex emission, we summarize in Fig. 1 the temporal evolution of the wave function for speeds just above the critical value k_c . Figure 1(a) presents the time evolution of the density measured at a fixed spatial point located near the vortex detachment region. The signal displays a clear alternation of two characteristic time scales: a spike linked to the swift nucleation and release of vortices, during which the density goes to zero, and an oscillation separating consecutive shedding events, during which the wave function near the obstacle temporarily returns to a state closely resembling the steady, symmetric flow found for subcritical wave numbers. The amplitude of the signal remains of order one as $k \rightarrow k_c^+$, in open contrast with the classical von Kármán wake, where oscillations are generated with zero amplitude near the critical point. The sudden appearance of nonzero amplitude oscillations when the control parameter exceeds a threshold value is a well-documented phenomenon in different fields such as neuronal modeling, chemical oscillators, and laser dynamics and it is explained with the occurrence of a saddle node on an invariant circle (SNIC) bifurcation [41,42]. The SNIC is a global bifurcation in which a saddle and a node subcritical equilibrium points collide and annihilate when the control parameter is equal to the critical threshold on a one-dimensional invariant manifold that is topologically a circle. For parameter values greater than the critical one, this manifold becomes a periodic orbit while the subcritical fixed points no longer exist, see Fig. 3(b). This scenario contrasts with a local Hopf bifurcation [Fig. 3(a), see Refs. [42] and [43]], where an equilibrium loses stability as the control parameter crosses the threshold and a finite-period limit cycle emerges with small amplitude oscillations. The appearance of the limit cycle can be explained by linearizing the dynamics around the destabilized steady state. Crucially, the fixed point remains present for all values of the control parameter, even after the onset of the limit cycle.

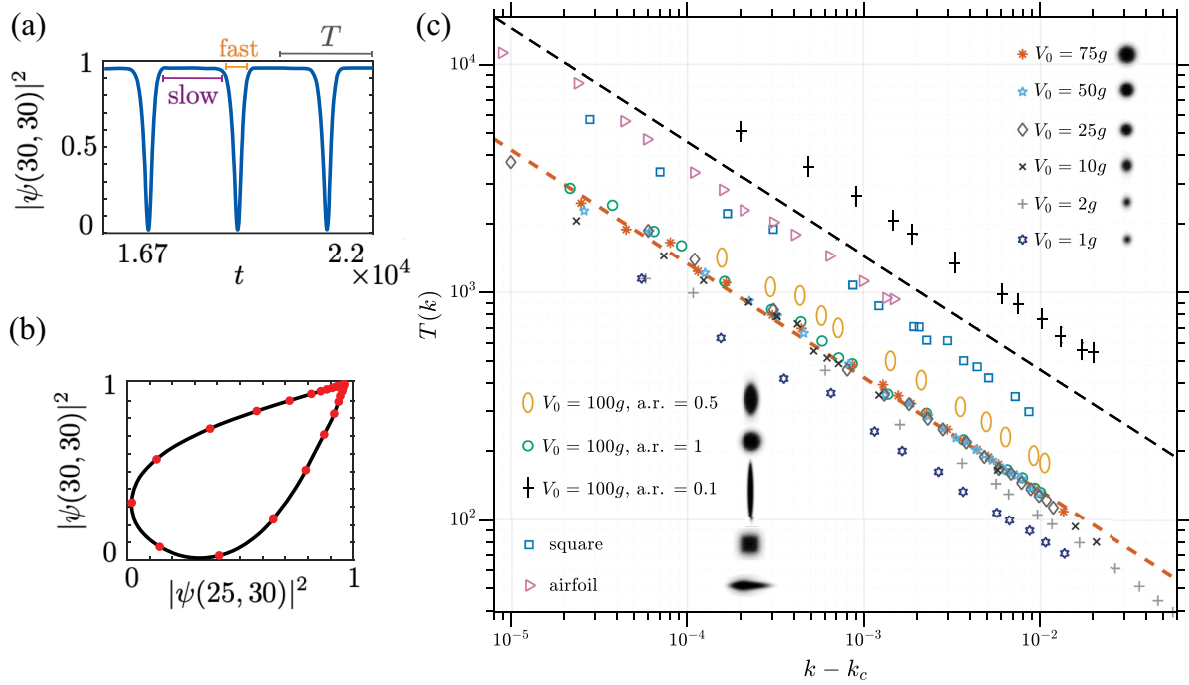


FIG. 1. (a) Supercritical time evolution for $g = 0.2073^2$, $V_0 = 100g$, $\alpha = 10$. During one period T a fast (vortex release) and a slow (vortex preparation) phase alternate. (b) Phase space cycle during a single period T . Red dots correspond to evenly spaced time instants, revealing a bottleneck in the evolution. (c) Log-log plot of the period as a function of $k - k_c$ for different impenetrable potential intensities and shapes. The orange dashed line corresponds to the normal form prediction (4) for $V_0 = 100g$, aspect ratio 1, while the black dashed line represents the $-1/2$ reference slope. The aspect ratio (a.r.) is defined as α_x/α_y , with $V(\mathbf{r}) = V_0 e^{-(x^2/\alpha_x^2 + y^2/\alpha_y^2)}$. The airfoil profile was defined in analogy with the one analyzed in [44].

In addition to the appearance of nonzero amplitude spikes, there are other criteria that are used in the literature [41,42] to characterize a SNIC bifurcation. The main ones are (i) the alternation in the supercritical evolution of a slow phase, reminiscent of the annihilated saddle-node equilibrium point, and a fast transition; (ii) the divergence of the period of the supercritical limit cycle with a $-1/2$ power law as the parameter approaches the threshold value. Concerning the first criterion for the system at hand, Fig. 1(b) highlights the difference in duration between the timescales reported in Fig. 1(a) by concentrating on a single period T . Sampling the signal at uniformly spaced time instants results in an accumulation near the values corresponding to the slow timescale. The system thus spends the majority of each period near the unperturbed density state that is not a steady solution of the problem when $k > k_c$, revealing a bottleneck [43] in the evolution in the phase space. This behavior highlights the discrete nature of quantum vortex shedding, as opposed to the continuous release that is typical of the classical von Kármán wake. In line with the second criterion, our numerical simulations confirm that the shedding period as a function of velocity, $T(k)$, exhibits critical behavior: as k approaches the critical threshold k_c , the slow envelope period increases, while the rapid nucleation spikes persist.

We observe the inverse square-root period divergence, $T(k) \propto 1/\sqrt{k - k_c}$, across a wide range of impenetrable laser profiles, including symmetric and nonsymmetric geometries [3,44–46], highlighting the universality of the scaling laws near the bifurcation threshold. More details about the parameter settings can be found in Supplemental Material [37]. Figure 1(c) shows the existence of this scaling law that successfully captures all numerical data, independent of the obstacle shape. The global nature and the scaling features of the bifurcation are in open contrast with the classical von Kármán wake, where a local Hopf bifurcation generates oscillations with zero amplitude and finite period at the critical point.

Taking advantage of the numerical tools developed to track stationary states and compute BdG modes, the bifurcation scenario can be validated and enriched unveiling the connections between the sub- and supercritical regions. Figure 2(a) illustrates the outcomes of the pseudoarclength continuation method (PACM) [37,38,47,48]. The turning point at $k = k_c$, beyond which no steady solution exists, confirms the presence of a saddle-node bifurcation. The value of the critical velocity for the physical configuration under consideration is $k_c \approx 0.0683$, in good agreement with the one reported in [3] for the same physical parameters. The analysis of the

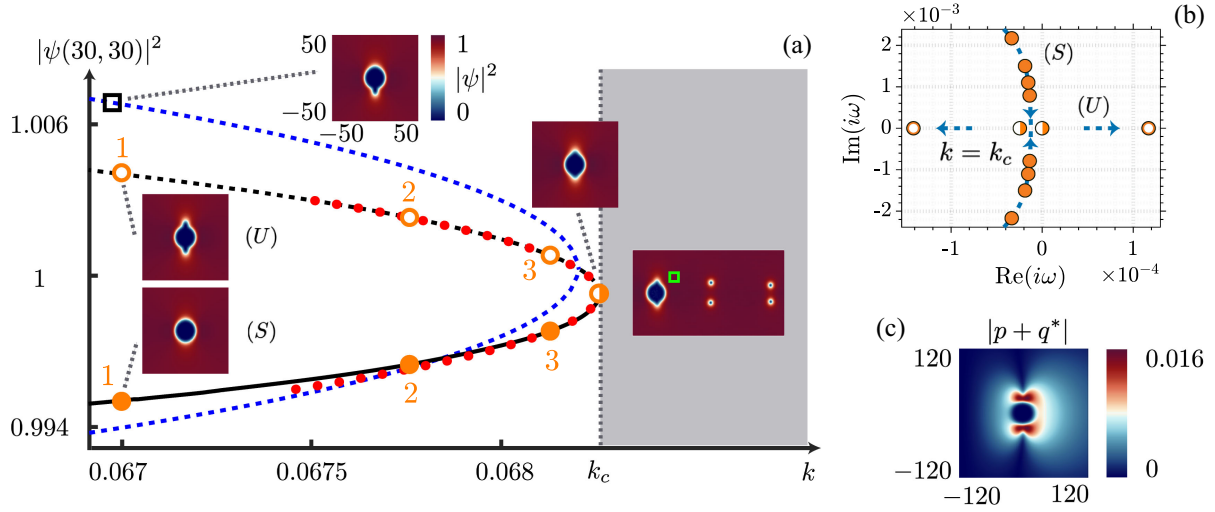


FIG. 2. (a) Output of the PACM for $g = 0.2073^2$, $V_0 = 100g$, $\alpha = 10$. The turning point at $k_c \approx 0.0683$, indicated with a yellow half-full marker, separates vortex-free stable steady states [solid black line, branch (S)] from symmetric unstable states [dashed black line, branch (U)]. The blue dashed line indicates an unstable branch associated with the breaking of vertical symmetry. Red dots represent the prediction of the normal form given in Eq. (2) while yellow circles correspond to the values reported in Fig. 3. For $k > k_c$ (gray region), the system exhibits vortex shedding, with the sampling point of Fig. 1(a) indicated with a green square. (b) Evolution of the leading eigenvalues as a function of k . Along branch (S) (orange full markers), the leading eigenvalues approach and split symmetrically on the negative real axis. At $k = k_c$ (half-full markers), one of the eigenvalues lies exactly at the origin. Along branch (U) (empty markers), the growth rate continues growing on the positive real axis. (c) Profile of the marginally stable eigenmode for $k = k_c$, showing a clear peak around the boundary of the obstacle.

BdG system reveals that along the vortex-free symmetric branch the dominant part of the spectrum, shown in Fig. 2(b), consists of a complex conjugate eigenpair whose imaginary parts move toward the real axis as the control parameter is increased in the direction of the turning point. In a neighborhood of the critical velocity on the symmetric branch, these two modes coalesce and split on the real axis. At $k = k_c$, one of these two eigenvalues crosses the origin with zero imaginary part and continues growing on the positive real axis when k decreases after the turn of the parabola. This behavior is fully coherent with the one outlined in Theorem 7.1 of [42] and characterizes the SNIC bifurcation. The spatial distribution of the marginally stable eigenmode is depicted in Fig. 2(c). It is concentrated around the boundary of the obstacle, in the same region responsible for vortex shedding in the supercritical domain. For lower values of the control parameter, the unstable branch further experiences a secondary bifurcation breaking the vertical symmetry of the flow. By plotting on the same graph the values of the stable and unstable solutions produced by the continuation algorithm for different subcritical velocities together with the cycles corresponding to supercritical periodic evolutions, we can illustrate the interplay between the saddle and the node equilibrium points that is characteristic of the SNIC bifurcation.

Figure 3(c) presents a two-dimensional phase portrait constructed by sampling the density of the wave function at two fixed points along the y axis located near the vortex detachment region. The phase diagram clearly reveals that

the stable and unstable solutions produced by the PACM collide and annihilate on a 1D manifold that coincides with the periodic cycles in the limit of infinitesimal proximity to the bifurcation point. More interestingly, guided by the information acquired from the phase diagram, we identified a single collective coordinate

$$Q_k(t) := |\psi(0, 25, t; k)|^2,$$

using which the system dynamics in a neighborhood of the bifurcation point can be described using a classic saddle-node normal form

$$\dot{Q}_k(t) = c(k - k_c) + a(Q_k - Q_c)^2, \quad (2)$$

with Q_c being the value of the collective coordinate at $k = k_c$. Following the approach developed in [41] for a neuronal dynamics model, the coefficients a and c can be fitted using the properties of the solution near the saddle node. We have checked that choosing different points in the definition of the collective coordinate Q_k leads to equivalent results [37], confirming the global nature of the bifurcation. Figure 2(a) illustrates the good agreement between the equilibrium curve of the reduced system (2) and the output of the PACM.

Even if the obtained normal form is only valid locally around the saddle-node point in the subcritical region and cannot reproduce supercritical limit cycles, it can capture the most important features of both regimes. In fact, the

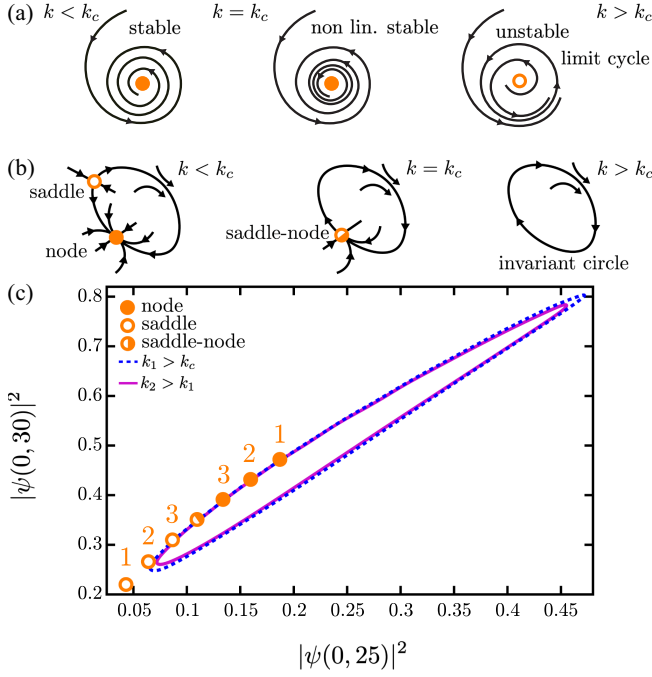


FIG. 3. Schematic representation of (a) the supercritical Hopf bifurcation and (b) the SNIC bifurcation (see Fig. 7.2 of [42] for further details). Panel (c) shows a reduced phase diagram corresponding to the same physical configuration illustrated in Fig. 2(a). The stable (filled circles) and unstable (open circles) steady states associated with different values of k collide in a saddle-node bifurcation along a one-dimensional manifold as $k \rightarrow k_c^+$. The dashed blue and solid purple curves represent the time evolution of two supercritical solutions ($k_2 > k_1 > k_c$). In the limit $k \rightarrow k_c^+$, the numerical trajectories approach and eventually coincide with the one-dimensional manifold. Point 1 corresponds to $k \approx 0.067$, point 2 to $k \approx 0.0677$, and point 3 to $k \approx 0.0681$. The same points are also indicated in Fig. 2(a).

bottleneck experienced by the periodic cycle allows us to estimate the frequency of the large-amplitude limit cycle attractor by considering a small neighborhood of the saddle-node point. The time required by the system to transition between any values Q_- and Q_+ can be estimated by integrating (2):

$$T_{Q_- \rightarrow Q_+} = \int_{Q_-}^{Q_+} \frac{ds}{c(k - k_c) + a(s - Q_c)^2}. \quad (3)$$

Choosing Q_- and Q_+ , respectively, as the starting and ending points of the slow phase, we obtain the following estimate:

$$T_{\text{bottleneck}}(k - k_c) \sim \frac{\pi}{\sqrt{ac}} \frac{1}{\sqrt{k - k_c}}, \quad (4)$$

in the limit $(k - k_c) \rightarrow 0^+$. The striking agreement between this theoretical prediction and the numerical simulations is shown in Fig. 1(c).

In conclusion, we have developed a theoretical and numerical framework to rigorously explain the bifurcation occurring in a laser-excited 2D quantum fluid bath at the critical velocity. Despite their phenomenological similarities, our study shows substantial differences between classical viscous wakes, affected by local bifurcations of Hopf type, and their quantum counterpart, characterized by a global SNIC bifurcation. The key factor behind these differences is the quantized circulation, which does not allow for a continuous release of vortices in the proximity of the critical velocity and induces the system to behave in a discrete fashion. The results shown in this Letter highlight robust features of the system that are not dependent on a precise choice of the experimental setup but that rather reveal a universal mechanism behind vortex shedding. Moreover, they provide a unified description of the onset of vortex nucleation and a solid background for the stability analysis of quantum fluid wakes.

Data availability—There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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