

Does Hot QCD Have a Conformal Manifold in the Chiral Limit?

Shi Chen^{*} and Aleksey Cherman[†]

School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455, USA

Robert D. Pisarski[‡]

Physics Department, Brookhaven National Laboratory, Upton, New York 11973-5000, USA

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Recent lattice evidence may suggest the chiral phase transition in QCD is second order for $N_f \geq 2$ massless flavors. We constrain conformal field theory (CFT) descriptions of a critical line in temperature T and imaginary baryon chemical potential $\theta_B = i\mu_B/T$. An 't Hooft anomaly at general θ_B constrains the transition even at $\theta_B = 0$, leaving only three minimal scenarios. The best-motivated scenario for $N_f \geq 3$, and perhaps also $N_f = 2$, is beyond Ginzburg-Landau, featuring a conformal manifold of θ_B -dependent universality classes with an exactly marginal operator related to baryon density.

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Introduction—Understanding the phase diagram of quantum chromodynamics (QCD) as a function of temperature T and baryon chemical potential μ_B is a central challenge in high-energy physics, with implications for heavy-ion collisions, astrophysics, and the early Universe. At $\mu_B = 0$, increasing T restores the spontaneously broken approximate chiral symmetry through a crossover at $T \sim 155$ MeV [1,2]. If N_f quarks are massless, chiral symmetry becomes exact, and the crossover becomes a genuine phase transition. The standard approach to this nonperturbative phase transition appeals to Landau universality. A Ginzburg-Landau analysis based on the chiral condensate $\Phi = \bar{\psi}_L \psi_R$ [3] suggests a generic first-order transition for $N_f \geq 3$, and a second-order transition for $N_f = 2$, in the $O(4)$ Wilson-Fisher universality class.

Early lattice QCD simulations seemed to agree with this prediction [4–11], but recent improved simulations [12–21] do not support first-order chiral transitions in the chiral limit for any N_f from $N_f = 2$ up to the onset of the conformal window, at $\mu_B = 0$, and along a line to imaginary μ_B . These simulations can, thus, be interpreted as suggesting that the chiral transition is either second order or perhaps very weakly first order for all N_f . Both of these scenarios would need an explanation beyond the original Ginzburg-Landau analysis.

Various theoretical interpretations of the recent lattice results have been proposed [22–26], but compelling solutions remain elusive. Here we systematically constrain the 3D conformal field theories (CFTs) that can appear on a natural critical line of chiral transitions in the θ_B - T phase diagram, with real $\theta_B = i\mu_B/T$. We find only three minimal scenarios compatible with the 't Hooft anomaly [27–29], high- T perturbative calculations [30], and low- T QCD phenomenology, shown in Fig. 1.

In both the Landau (left) and Landau-DQCP (middle) scenarios, the critical line flows to a single Ginzburg-Landau CFT for generic θ_B , but the Landau-DQCP scenario also requires an analog of a condensed-matter deconfined quantum critical point (DQCP) [31–33] to appear at $\theta_B = \pi$. In the conformal-manifold scenario (right), the critical line is a continuum of distinct beyond-Landau 3D CFTs. The Landau and Landau-DQCP scenarios require a natural fixed point in the Ginzburg-Landau description of Ref. [3] that has not been found despite extensive searches for $N_f \geq 3$ and, thus, are disfavored for generic N_f . In contrast, we find that the conformal manifold scenario is not excluded by existing constraints, and the existence of a suitable 3D CFT is supported by a $(2 + \epsilon)$ D expansion and existing conformal bootstrap bounds for $N_f \geq 2$. So the answer to the title's question could be yes.

Symmetry and anomaly—We work in Euclidean signature throughout. Massless QCD, a 4D $SU(N_c)$ gauge theory for $N_c \geq 3$ with N_f massless fundamental Dirac fermions, has continuous global symmetry,

$$G = \frac{U(1)_B \times SU(N_f)_L \times SU(N_f)_R}{\mathbb{Z}_{N_f}}, \quad (1)$$

^{*}Contact author: s.chern.phys@gmail.com

[†]Contact author: acherman@umn.edu

[‡]Contact author: pisarski@bnl.gov

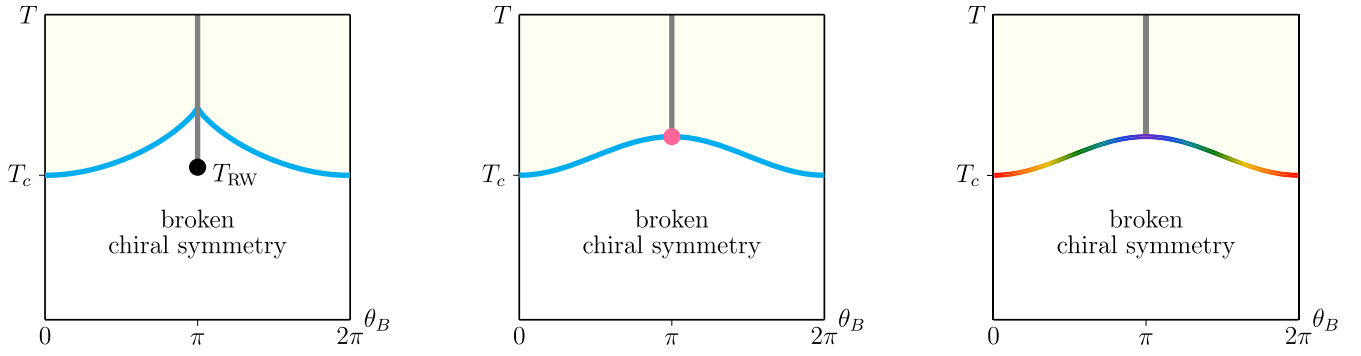


FIG. 1. Minimal phase diagrams of massless QCD consistent with a natural second-order line of chiral phase transitions. The gray vertical line represents Roberge-Weiss phase transitions of first order, while T_c is the chiral phase transition temperature at $\theta_B = 0$. Left: Landau scenario, where T_{RW} is the Roberge-Weiss Ising critical point. Middle: Landau-DQCP scenario, where the Roberge-Weiss line terminates at a beyond-Landau critical point (pink dot) analogous to a DQCP. Right: Conformal-manifold scenario, where the rainbow line represents a continuous family of CFTs. The conformal-manifold scenario is plausible for all $N_f \geq 2$, while the Landau and the Landau-DQCP scenarios are only possible for $N_f = 2$ due to the absence of candidate Ginzburg-Landau CFTs for $N_f > 2$.

where $\mathbb{Z}_{N_f}$ sets $(e^{2\pi i N_c/N_f}, e^{-2\pi i/N_f} \mathbf{1}_{N_f}, e^{-2\pi i/N_f} \mathbf{1}_{N_f})$ to the identity. Here, $U(1)_B = U(1)_V/\mathbb{Z}_{N_c}$ is the baryon symmetry, while $SU(N_f)_L \times SU(N_f)_R$ is the chiral symmetry. The $U(1)_A$ symmetry of the classical theory is destroyed by the axial anomaly, and so it is not part of G . The discrete global symmetry acts nontrivially on G and is generated by charge conjugation $\mathcal{C}$ and geometric reflections, including time reversal.

The symmetry G has an 't Hooft anomaly [34]. Turning on background gauge fields A_B , A_L , and A_R for $U(1)_B$, $SU(N_f)_L$, and $SU(N_f)_R$, respectively, leads to an anomalous background gauge variation. Mathematically, this can be described by a 5D invertible field theory [35] (see Refs. [36–40]), with partition function,

$$\mathcal{I}_{5D}(A_B, A_L, A_R) = \mathcal{J}_{5D}^{N_c}(A_L) \mathcal{J}_{5D}^{-N_c}(A_R) \times \exp \left[\frac{i}{8\pi^2} \int_{5D} A_B \text{tr}(F_L^2 - F_R^2) \right], \quad (2)$$

where $F_{L,R}$ is the field strength of $A_{L,R}$. For $N_f \geq 3$, $\mathcal{J}_{5D}(A)$ is the 5D Chern-Simons partition function [41], while for $N_f = 2$ it is the 5D spin topological invariant associated with the Witten anomaly [42,43].

As a renormalization group (RG) invariant, the 't Hooft anomaly [Eq. (2)] ensures massless excitations [44,45], but their nature and interactions depend on N_c and N_f . Below the conformal window (very roughly $N_f \lesssim \frac{8}{3} N_c$ [46–51]), the continuous symmetry G is spontaneously broken to

$$H = \frac{U(1)_B \times SU(N_f)_V}{\mathbb{Z}_{N_f}}, \quad (3)$$

while the discrete symmetries remain intact. The gapless IR physics is described via a nonlinear scalar field,

$$U(x) \in G/H \simeq SU(N_f). \quad (4)$$

In this phase, $\mathcal{J}^{N_c}(A_L) \mathcal{J}^{-N_c}(A_R)$ in Eq. (2) is matched by the Wess-Zumino-Witten action [52] for $N_f \geq 3$ and by the discrete spin θ -angle [52] for $N_f = 2$. The A_B term is matched by identifying the Skyrme current {from $\pi_3[SU(N_f)] = \mathbb{Z}$ } with the baryon current [53].

Circle compactification—We now put massless QCD on $\mathbb{R}^3 \times S^1$ with circumference β . An S^1 -homogeneous gauge field splits into a gauge field on $\mathbb{R}^3$ and a holonomy scalar field on $\mathbb{R}^3$. In particular, the $U(1)_B$ holonomy gives a 2π -periodic scalar field θ_B [54]. For constant θ_B , the circle compactification has two Hamiltonian interpretations, and all physical observables, including the free energy density,

$$f_\beta(\theta_B) \equiv -\lim_{V \rightarrow \infty} \frac{1}{\beta V} \ln Z_\beta(\theta_B), \quad (5)$$

must be *simultaneously* interpretable in both pictures: (1) (Thermal) Hamiltonian evolution along S^1 yields a grand canonical ensemble on $\mathbb{R}^3$ with temperature $T = 1/\beta$ and baryonic chemical potential $\mu_B = -iT\theta_B$. Then $-f_\beta(\theta_B)$ is the thermal pressure. (2) (Quantum) Hamiltonian evolution along $\mathbb{R}$ yields a zero-temperature quantum system on $\mathbb{R}^2 \times S^1$ with circumference β and a baryonic Aharonov-Bohm phase $e^{i\theta_B}$. Then, $f_\beta(\theta_B)$ is the Casimir energy density. Therefore, with real $\theta_B \neq 0$, we lose reflection positivity reflecting S^1 but preserve reflection positivity reflecting an $\mathbb{R} \subset \mathbb{R}^3$. The imaginary baryon density at zero Matsubara frequency,

$$\rho_B(x) \equiv \frac{i}{\beta} \int_0^\beta d\tau j_B^0(x, \tau), \quad (6)$$

is anti-Hermitian in the thermal picture but is Hermitian in the quantum picture.

The quantum picture is convenient for describing 3D infrared (IR) effective theories at length scales $\ell \gg \beta$. These theories are unitary 3D quantum field theories (QFTs) for any real θ_B , and ρ_B flows to a Hermitian operator thereof. When the 3D effective theory becomes scale invariant at a critical point, unitarity is expected to enhance scale invariance to full conformal invariance; see, e.g., Refs. [55–60].

Upon S^1 compactification, the 5D invertible theory [Eq. (2)] reduces to the 4D invertible theory,

$$\mathcal{I}_{4D}(\theta_B, A_L, A_R) = \exp \left[\frac{i}{8\pi^2} \int_{4D} \theta_B \text{tr}(F_L^2 - F_R^2) \right]. \quad (7)$$

It is well defined on closed 4D oriented manifolds but is not invariant under $\theta_B \rightarrow \theta_B + 2\pi$ if the 4D manifold has a 3D boundary. Hence, with S^1 -homogeneous $A_{L,R}$, the partition function of S^1 -compactified massless QCD satisfies [27–29]

$$\begin{aligned} Z_\beta(\theta_B + 2\pi, A_L, A_R) \\ = Z_\beta(\theta_B, A_L, A_R) \mathcal{J}_{3D}(A_L) \mathcal{J}_{3D}^{-1}(A_R), \end{aligned} \quad (8)$$

where $\mathcal{J}_{3D}(A)$ is the 3D Chern-Simons partition function,

$$\mathcal{J}_{3D}(A) = \exp \left[\frac{i}{4\pi} \int_{3D} \text{tr} \left(\text{Ad}A - i \frac{2}{3} A^3 \right) \right]. \quad (9)$$

This is a mixed anomaly between $SU(N_f)_{L,R}$ and a 2π -periodic coupling [61–66], or a “ (-1) -form $U(1)$ symmetry” [67–70].

Upon S^1 compactification, the discrete symmetry is generated by $\mathcal{C}$, the S^1 -reflection $\mathcal{S}$, and $\mathbb{R}$ -reflections $\mathcal{R}$ for $\mathbb{R} \subset \mathbb{R}^3$. They act on S^1 -homogeneous 3D background fields as

$$\begin{aligned} \theta_B(x) &\xrightarrow{\mathcal{C}} -\theta_B(x), & A_{L,R}(x) &\xrightarrow{\mathcal{C}} -A_{R,L}^*(x), \\ \theta_B(x) &\xrightarrow{\mathcal{S}} -\theta_B(x), & A_{L,R}(x) &\xrightarrow{\mathcal{S}} A_{R,L}(x), \\ \theta_B(x) &\xrightarrow{\mathcal{R}} \theta_B(\bar{x}), & A_{L,R}(x) &\xrightarrow{\mathcal{R}} \bar{A}_{R,L}(\bar{x}), \end{aligned} \quad (10)$$

where the bars indicate an $\mathbb{R}$ reflection. $\mathcal{C}$, $\mathcal{S}$, and $\mathcal{R}$ exchange L and R . When $N_f = 2$, $\mathcal{CS}$ can be undone by $SU(N_f)_{L,R}$.

A constant $\theta_B \neq 0, \pi$ preserves only the subgroup generated by $\mathcal{CS}$ and $\mathcal{R}$, whereas $\mathcal{S}$, $\mathcal{C}$, $\mathcal{SR}$, and $\mathcal{CR}$ are restored at $\theta_B = \pi$ by the 2π periodicity of θ_B . Therefore, at $\theta_B = \pi$, Eq. (8) gives a mixed anomaly between $SU(N_f)_{L,R}$ and $(\mathbb{Z}_2)_{\mathcal{SR}}$ [27–29]. Equivalently, this is an anomaly of $SU(N_f)_{L,R} \rtimes (\mathbb{Z}_2)_{\mathcal{CR}}$ or of $[SU(N_f)_L \times SU(N_f)_R] \rtimes (\mathbb{Z}_2)_{\mathcal{S}}$ or $\rtimes (\mathbb{Z}_2)_{\mathcal{C}}$. The 4D invertible theory [Eq. (7)] captures this because, at $\theta_B = \pi$, Eq. (7) can be defined on unorientable manifolds or with twists that exchange L and R .

Low- T and high- T phases—Although it does not guarantee gapless modes, the 3D anomaly [Eq. (7)] excludes θ_B -uniform trivially gapped phases, as well as a trivially gapped phase at $\theta_B = \pi$. The 3D IR physics at length scales $\ell \gg \beta$ must see θ_B and the symmetry,

$$\frac{SU(N_f)_L \times SU(N_f)_R}{\mathbb{Z}_{N_f}} \begin{cases} \rtimes (\mathbb{Z}_2)_{\mathcal{CS}, \mathcal{R}}, & \theta_B = 0, \pi \\ \rtimes (\mathbb{Z}_2)_{\mathcal{CS}, \mathcal{R}}, & \theta_B \neq 0, \pi \end{cases}, \quad (11)$$

where discrete symmetry needs to quotient out $(\mathbb{Z}_2)_{\mathcal{CS}}$ for $N_f = 2$. The operator ρ_B of Eq. (6) is neutral under the second line of Eq. (11) but is charged under the first. In contrast, the 3D baryon symmetry $U(1)_B$ and chiral holonomies $\theta_{L,R}$ need not appear in the IR.

At sufficiently low T , the symmetry G is expected to remain broken to H just as at $T = 0$, and the IR physics is described by the 3D reduction of the nonlinear scalar field $U(x)$ of Eq. (4). The 3D symmetry [Eq. (11)] acts on it by $U(x) \rightarrow LUR^{-1}$ for $L, R \in SU(N_f)$ and

$$U(x) \xrightarrow{\mathcal{C}} U^T(x), \quad U(x) \xrightarrow{\mathcal{S}} U^\dagger(x), \quad U(x) \xrightarrow{\mathcal{R}} U^\dagger(\bar{x}). \quad (12)$$

When $N_f = 2$, $\mathcal{CS}$ is absorbed into $SU(2)_{L,R}$. The low- T 3D IR effective QFT is given by [27,71]

$$\int \mathcal{D}U \exp \left\{ -\frac{1}{g^2} \int d^3x \text{tr} \partial_a U \partial^a U^\dagger + i \int \theta_B \frac{\text{tr}(U^\dagger dU)^3}{24\pi^2} \right\}. \quad (13)$$

To verify anomaly matching in Eq. (13), one can turn on background gauge fields $A_{L,R}$ and study their interplay with the Skyrme threeform [27,52,71]. In this low- T gapless phase, varying θ_B does not induce phase transitions since baryons are heavy.

At sufficiently high T , perturbative evaluations of some quantities are trustworthy; see, e.g., Refs. [72,73]. They imply that chiral symmetry is restored and there is a mass gap for all θ_B [30]. Roberge and Weiss [30] pointed out a first-order phase transition at $\theta_B = \pi$. This is associated with discrete symmetry breaking at $\theta_B = \pi$,

$$(\mathbb{Z}_2)_{\mathcal{CS}, \mathcal{R}}^3 \rightarrow (\mathbb{Z}_2)_{\mathcal{CS}, \mathcal{R}}^2, \quad (14)$$

which needs to quotient out $(\mathbb{Z}_2)_{\mathcal{CS}}$ for $N_f = 2$. The order operator is ρ_B with

$$\langle \rho_B \rangle = \frac{\partial f_\beta(\theta_B)}{\partial \theta_B}. \quad (15)$$

In this high- T gapped phase, continuous chiral symmetry flows to an emergent twoform $U(1)$ symmetry [74], and the 3D anomaly [Eq. (7)] is matched via symmetry transmutation [75]; we explain this high- T IR effective theory in Supplemental Material [76] [Sec. S1].

In summary, chiral symmetry is spontaneously broken in the low- T phase and restored in the high- T phase, while the discrete symmetry at $\theta_B = \pi$ is spontaneously broken [77] in the high- T phase and restored in the low- T phase. This complementarity of symmetry realizations is a fingerprint of mixed anomalies.

Consistent phase diagrams—The low- T and high- T phases are separated by a critical line $\theta_B \mapsto T_c(\theta_B)$. Motivated by recent lattice results [13–21], we postulate that this line generically consists of natural second-order points with emergent scale invariance. The IR physics of such a natural critical point is a 3D unitary CFT whose neutral sector under the symmetry [Eq. (11)] has precisely one relevant primary scalar operator $\mathcal{O}_E$ to which the T perturbation flows. The operator ρ_B also flows to $\mathcal{O}_E$ at $\theta_B \neq 0, \pi$, or at $\theta_B = 0, \pi$ if spontaneous symmetry breaking [Eq. (14)] occurs.

The 't Hooft anomaly [Eq. (7)] prohibits the critical line from flowing to a single CFT for all θ_B . First, to produce the behavior [Eq. (8)], the IR physics must change as we smoothly vary θ_B to $\theta_B + 2\pi$. Second, it must differ at $\theta_B = 0$ and π since $\theta_B = \pi$ has an additional mixed anomaly between $SU(N_f)_{L,R}$ and discrete symmetries, which are absent at $\theta_B = 0$. Therefore, the IR physics must change from $\theta_B = 0$ to $\theta_B = \pi$, either abruptly or gradually. Abrupt changes correspond to phase transitions at special θ_B , like in the high- T phase. Gradual changes, as in the low- T phase, correspond to smooth variations of the CFT data along the critical line.

If the changes in the IR physics are abrupt, a neighborhood of $\theta_B = 0$ on the critical line only cares about the breaking $G \rightarrow H$ and should fit the Landau paradigm, i.e., be described by a natural fixed point of the Ginzburg-Landau model of Ref. [3]. Minimally, it suffices to have one phase transition at $\theta_B = \pi$ and a Ginzburg-Landau description for the rest of the critical line. If the transition at $\theta_B = \pi$ is first-order (Landau scenario, Fig. 1, left), discrete symmetry breaking [Eq. (14)] with $\langle \rho_B \rangle \neq 0$ occurs at $\theta_B = \pi$. There must then be a window at $\theta_B = \pi$ where chiral and discrete symmetries are broken, with an Ising transition at its lower edge [78]. If instead the transition at $\theta_B = \pi$ is second order (Landau-DQCP scenario, Fig. 1, middle), it must be a beyond-Landau critical point analogous to condensed-matter DQCPs [31–33]. The operator ρ_B flows to a relevant (or marginally relevant) operator at this DQCP.

If the changes in the IR physics are gradual, the CFT data instead vary smoothly with θ_B on the critical line (conformal-manifold scenario, Fig. 1, right). This scenario requires a conformal manifold—a continuum of distinct CFTs—generated by an exactly marginal operator $\mathcal{O}_B$ [81]. The operator ρ_B flows to $\mathcal{O}_B$ at $\theta_B = 0, \pi$ and to a linear combination of $\mathcal{O}_B$ and $\mathcal{O}_E$ at $\theta_B \neq 0, \pi$. The presence of $\mathcal{O}_B$ makes this scenario intrinsically beyond Ginzburg-Landau.

One can imagine more complicated scenarios with phase transitions away from $\theta_B = \pi$, but they generically require unnatural multicritical points. In any case, the three scenarios above are the minimal ones under the assumption of a second-order critical line.

Landau scenarios—The Landau and Landau-DQCP scenarios posit that the critical line at $\theta_B \neq \pi$ is described [83] by a natural fixed point of the 3D linear sigma model of Ref. [3]:

$$\int \mathcal{D}\Phi \exp \left\{ - \int d^3x \left[\text{tr} \partial_a \Phi \partial^a \Phi^\dagger + \mu \text{tr} \Phi \Phi^\dagger + \lambda_1 (\text{tr} \Phi \Phi^\dagger)^2 + \lambda_2 \text{tr} (\Phi \Phi^\dagger)^2 - \lambda_3 (\det \Phi + \text{c.c.}) \right] \right\}, \quad (16)$$

where $\Phi(x) \sim \bar{\psi}_L \psi_R$ is an N_f -by- N_f complex matrix field, with the symmetry [Eq. (11)] acting on $\Phi(x)$ in the same way as $U(x)$. The question is whether this model hits a natural critical point as μ varies [84].

When $N_f = 2$, the model [Eq. (16)] at mean-field level undergoes an $O(4)$ second-order transition at $\mu = |\lambda_3|$. This critical point survives fluctuations, and flows [92] to the $O(4)$ Wilson-Fisher universality class [3,93]. Hence, the Landau and Landau-DQCP scenarios are both in principle plausible for $N_f = 2$.

At mean-field level, the λ_3 term drives the transition first order for $N_f = 3$ but allows an $O(2N_f^2)$ second-order transition at $\mu = 0$ for $N_f \geq 4$. Unfortunately, beyond mean-field level, extensive searches have found no stable fixed points in the model [Eq. (16)] for all $N_f \geq 3$ [3,22,24,93–97]. We, thus, consider the Landau and Landau-DQCP scenarios to be disfavored for $N_f \geq 3$.

Conformal manifold—The only option not yet disfavored for $N_f \geq 3$ is the conformal-manifold scenario, which might work for $N_f = 2$ as well. We denote the CFT at θ_B on the critical line by $SU(N_f)_{\theta_B}$. We obtain a nonconformal field theory by adding the deformation,

$$\mu \int d^3x \mathcal{O}_E, \quad (17)$$

to the $SU(N_f)_{\theta_B}$ CFT. This $SU(N_f)_{\theta_B}$ field theory must describe the 3D IR physics of finite- T massless QCD for all T . It should flow to a gapped phase for $\mu > 0$ and a Nambu-Goldstone phase for $\mu < 0$.

Whether the $SU(N_f)_{\theta_B}$ CFT exists is a nontrivial question since nonsupersymmetric conformal manifolds are rare in 3D [98–103]. We now argue for the existence of the $SU(N_f)_0$ CFT that generates the conformal manifold via $\mathcal{O}_B$ deformations.

If we can arbitrarily “contract” the $\mu < 0$ RG flow into its IR end by smoothly deforming the $SU(N_f)_0$ field theory, we can study a deformed version of the $SU(N_f)_0$ CFT via loop expansions around the Gaussian IR fixed point.

The IR tail of the $\mu < 0$ RG flow is described by the 3D $SU(N_f)$ principal chiral model (PCM) of Eq. (13). Conveniently, loop expansions in the $(2 + \epsilon)$ D $SU(N_f)$ PCM with arbitrarily small $\epsilon > 0$ are known to exhibit an arbitrarily short RG flow from an interacting ultraviolet (UV) fixed point to the Gaussian IR fixed point [104]. It is tempting to view this UV fixed point as a weakly coupled avatar of the $SU(N_f)_0$ CFT.

The $(2 + \epsilon)$ D UV fixed point should contain the avatar of the exactly marginal operator $\mathcal{O}_B$. The key observation [105–108] is that a d -dimensional QFT with $d \in \mathbb{R}_+$ is like an infinite-dimensional QFT since it is formally an analytic continuation of a family of n -dimensional QFTs for all $n \in \mathbb{Z}_+$. The $(2 + \epsilon)$ D $SU(N_f)$ PCM, thus, sees the topology of the $SU(N_f)$ target space in all finite dimensions, including $\pi_3[SU(N_f)] = \mathbb{Z}$. As a result, at the $(2 + \epsilon)$ D UV fixed point, the closed Skyrme threeform has protected scaling dimension exactly 3 and generates a short conformal multiplet [107]. The $(2 + \epsilon)$ D Skyrme threeform is, thus, a promising avatar for $\mathcal{O}_B$.

A large- N analysis [107] indicates that, if the 3D $SU(2)_0$ CFT exists, it might be close to the 3D $O(4)$ Wilson-Fisher CFT, so existing lattice measurements [109] might not distinguish them. The idea that fixed points in $2 + \epsilon$ and $4 - \epsilon$ dimensions could have different extrapolations to 3D has recently been highlighted by De Cesare and Rychkov [107,108].

To see whether the $(2 + \epsilon)$ D UV fixed point extrapolates to $\epsilon = 1$, we study the scaling dimensions of $\mathcal{O}_E$ and the lowest-dimension primary scalar $\mathcal{O}_U$ in the $N_f \otimes \bar{N}_f$ representation using the three-loop results of Refs. [110–112] and the four-loop results of Ref. [113]:

$$\Delta_E = 2 - \frac{1}{2}\epsilon^2 - \frac{1}{4}\epsilon^3 - \left(\frac{5}{16} + \frac{9\zeta_3}{4N_f^2}\right)\epsilon^4 + \mathcal{O}(\epsilon^5), \quad (18a)$$

$$\Delta_U = \left(1 - \frac{1}{N_f^2}\right)\left[\epsilon - \frac{1}{2}\epsilon^2 + \frac{1}{2}\epsilon^3\right] + \mathcal{O}(\epsilon^4), \quad (18b)$$

where ζ_3 is Apéry’s constant. We estimate Δ_E and Δ_U in the 3D $SU(N_f)_0$ CFT using Padé-Borel resummation. We summarize our 3D estimates with small N_f in Table I, which are consistent with unitarity $\Delta_E, \Delta_U \geq \frac{1}{2}$ and relevance $\Delta_E, \Delta_U < 3$.

We present our resummation procedure in Supplemental Material [76] [Sec. S4]. We estimate the resummation uncertainties by imaginary parts of the Borel singularities and variations under different Padé approximants. The alternating truncated series Δ_U has a sizably smaller resummation error bar than Δ_E because the Borel singularities in the Padé approximants associated with Δ_U are away from the Borel integral contour. In any ϵ expansion, there is no reliable way to estimate the systematic errors in an extrapolation to $\epsilon = 1$ from a realistic number of

TABLE I. Estimates for Δ_E and Δ_U in the 3D $SU(N_f)_0$ CFT obtained from Padé-Borel resummations of the known $(2 + \epsilon)$ D expansions.

N_f	2	3	4	5
Δ_E	1.94 ± 0.56	1.76 ± 0.67	1.64 ± 0.71	1.56 ± 0.72
Δ_U	0.54 ± 0.02	0.64 ± 0.02	0.67 ± 0.02	0.69 ± 0.02

expansion terms, so our results should only be viewed as suggestive.

There are also some conformal bootstrap [114] studies for 3D CFTs with symmetry $[SU(N_f)_L \times SU(N_f)_R]/\mathbb{Z}_{N_f}$. For $N_f = 2$, Ref. [115] found an island with $1.64 \lesssim \Delta_E \lesssim 1.68$ and $0.517 \lesssim \Delta_U \lesssim 0.521$. For $N_f = 3$, Ref. [116] found a rough kink at $\Delta_U = 0.62 \pm 0.01$. We view the apparent agreement at $N_f = 2$ and $N_f = 3$ between our $(2 + \epsilon)$ D estimates and the bootstrap estimates as encouraging.

Gathering all the evidence, we consider it promising to conjecture [117] the existence of 3D $SU(N_f)_{\theta_B}$ CFTs, though further tests are needed. Assuming that these CFTs exist, it is natural to further conjecture that QCD flows to these CFTs at the chiral transition and, thus, realizes the conformal-manifold scenario in the chiral limit from $N_f = 2$ to the lower edge of the conformal window.

Outlook—Recent lattice results [13–21] on the QCD chiral transition motivated us to consider that QCD flows to a natural second-order critical line for all N_f below the conformal window. We found that only three scenarios (Fig. 1) are consistent with this possibility. Lattice QCD simulations should distinguish them by measuring the leading IR behaviors of ρ_B at $\theta_B = 0, \pi$ on the critical line, in Table II.

The conformal-manifold scenario is the only option not disfavored for $N_f \geq 3$ by current evidence. However, it requires the existence of the $SU(N_f)_{\theta_B}$ universality class, which is an open question of independent interest. Demonstrating it will require improved conformal bootstrap bounds on 3D CFTs with the symmetry [Eq. (11)] and further development of the $(2 + \epsilon)$ D expansion. Lattice Monte Carlo simulations of the 3D $SU(N_f)$ PCM using Lüscher’s admissibility method [120–123] (see also Ref. [124]) might be valuable.

While we assume a second-order critical line, a *weakly* first-order transition is not excluded by Refs. [13–21].

TABLE II. The leading IR behaviors of the operator ρ_B at $\theta_B = 0, \pi$ on the critical line.

θ_B	Landau	Landau-DQCP	Conformal manifold
0	Irrelevant	Irrelevant	Exactly marginal
π	$\langle \rho_B \rangle \neq 0$	Relevant	Exactly marginal

If such transitions are controlled by nearby nonunitary complex CFTs [125,126], such as the $(2 + \epsilon)D$ fixed points, much of our discussion would carry over.

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