

Entanglement Structure and Matrix Inequalities from Isotypic Measurements

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We detect entanglement partitions of multipartite quantum systems by exploiting their inherent symmetries. Structures like genuinely multipartite entanglement, m -separability, and entanglement depth are detected as very special cases. This formulation enables us to characterize all entanglement partitions of all three- and four-partite states and witnesses with unitary and permutation symmetry, which we denote Schur-Weyl isotypic witnesses. In particular, we find and parametrize a complete set of bound entangled states therein. For larger systems, we provide a large family of analytical witnesses detecting multipartite states of arbitrary size where none of the parties is separable from the rest. The proposed method relies on weak Schur sampling with projective measurements onto isotypic components and can be implemented in a quantum computer. Beyond physics, our results extend to the mathematical literature: we establish new inequalities between matrix immanants, which constitute an open problem, and characterize the set of such inequalities for matrices of sizes three and four.

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Introduction—Over the past decades, large improvements in the control over quantum systems of increasing size have been achieved [1,2]. This advances the development of quantum devices, for which a key feature is the presence of quantum entanglement [3]. Yet, multiple systems can be entangled in different structures, each beneficial for different applications. The most resourceful case is arguably genuinely multipartite entanglement, where no subsystem is separable from the rest, and a variety of methods have been derived to detect it [4–6]. However, detecting finer structures is a notorious challenge [7–12].

At the same time, rapid progress in quantum control is increasingly approaching the physical realization of quantum computers [13–15]. Beyond enabling efficient algorithms for classically difficult problems [16–19], quantum computers are also powerful tools for analyzing quantum systems, for instance, in many-body simulation [20,21] and quantum chemistry [22,23]. In particular, quantum hardware is a natural platform to certify the correlations of quantum systems. While well-established techniques such as randomized measurements are effective for bipartite

systems [24–27], less is known for more involved entanglement structures.

In this Letter, we develop a technique to detect the entanglement structure of a multipartite system, which is tractable to implement in a quantum computer. By exploiting entanglement symmetries under permutations and unitaries (Proposition 1), we detect the entanglement partition of a quantum system (Definition 1). We characterize the entanglement partitions of three- and four-partite invariant systems (Proposition 2 and Table I), and identify the families of bound entangled states therein. Exploiting symmetries further, we derive analytical criteria to certify that no local parties are separable from the rest in arbitrary system sizes (Theorem 1). Different methods for practical estimation of the criteria introduced are compared in Table II. Besides contributing to physics, the results presented lead to new symmetric matrix inequalities, a long-standing problem in the mathematics community (Proposition 3). In particular we characterize the set of immanant inequalities for positive semidefinite matrices of sizes three and four.

Main goal—Consider a three-partite quantum system. A state is fully separable if it can be written as $Q_{ABC} = \sum_i p_i q_A^i \otimes q_B^i \otimes q_C^i$. It is biseparable if it can be written as

$$Q_{ABC} = \sum_{i,j,k} p_i q_A^i \otimes q_{BC}^j + p_j q_B^j \otimes q_{AC}^k + p_k q_{AB}^k \otimes q_C^k, \quad (1)$$

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TABLE I. Numerical characterization of four-partite ($n = 4$) κ -separability witnesses W_i with unitary and permutation symmetry, defined by a linear combination of the projections Π_λ with the given coefficients. Here d denotes the smallest local dimension for which none of the projectors involved vanish. Witnesses W_1 to W_5 arise from immanant inequalities [28]. Witnesses W_6 to W_8 are to our knowledge new and give rise to previously unknown matrix inequalities. The coefficients α_κ are the minimum expectation values over states with separability partition κ , and $\alpha_{1^4}^{PPT}$ is the minimal expectation value over PPT states across all possible bipartitions (detection of PPT entanglement is marked by a negative value in the last column). We denote $W_i^{\epsilon_i} := W_i - \epsilon_i \mathbb{1}$, where the precisions ϵ_6 , ϵ_7 , and ϵ_8 and the methodology are given in SM [29].

W	c_4	$c_{3,1}$	$c_{2,2}$	$c_{2,1^2}$	c_{1^4}	d	$\alpha_{1 3}$	$\alpha_{2 2}$	$\alpha_{1 1 2}$	$\alpha_{1^4}^{PPT}$
W_1	9	-1	0	0	0	2	-1	-1	-1	0
W_2	0	9	0	-9	0	3	-1	-1/2	-0.31	0
W_3	0	0	0	1	-9	4	-9/4	-3/2	-1/2	0
W_4	4	0	-1	0	0	2	-1/2	-1	-1/2	0
W_5	0	0	1	0	-4	4	-1	-1/3	-1/6	0
$W_6^{\epsilon_6}$	8	0	-4	1	-1	4	-2	-4	-2	-4/7
$W_7^{\epsilon_7}$	12	-1	-3	1	0	3	-2	-3	-2	-3/20
$W_8^{\epsilon_8}$	0	2	-3	-1	3	4	-1	-3	-1/2	-9/86

where $p_i, p_j, p_k \geq 0$ and $\sum_{i,j,k} p_i + p_j + p_k = 1$. And it is genuinely multipartite entangled if it is neither fully separable nor biseparable (see Fig. 1). The case of larger systems is more involved as separability structures could be distributed in multiple ways. Here we consider the following entanglement structures, which fine-grain the entanglement properties of the state in hand [38–43].

Definition 1—Given a partition $\kappa = [k_1 | \dots | k_m]$ with $k_1 \geq \dots \geq k_m$ and $k_1 + \dots + k_m = n$, an n -partite state ρ is κ -separable, written $\rho \in \kappa$ -SEP, if

TABLE II. Scaling of different methods for the weak Schur sampling for local dimension d . The left two columns implement the full quantum Schur transform, which outputs all three registers, λ , M_λ and T_λ . The two known methods for this are the Bacon-Chuang-Harrow algorithm [30,31] and the revised Kroví's algorithm [31]. The two right columns avoid some redundant information (there is no information about M_λ) and obtain only the registers λ and T_λ , thus allowing for easier circuits: these are the original Kroví algorithm [32] and the generalized phase estimation (GPE) [33]. Both the Kroví and GPE approaches are based on quantum Fourier transform for the symmetric group. For the case of qubits ($d = 2$), a recent weak Schur sampling method [34] achieves $\tilde{O}(n)$ scaling.

Registers	$\lambda, M_\lambda, T_\lambda$		λ, T_λ		λ
Method	[30,31,35,36]	[31]	[31,32]	[33,37]	[34]
Depth	$\tilde{O}(\min(n^5, nd^4))$	$\tilde{O}(n^{3.5})$	$\tilde{O}(n^{3.5})$	$\tilde{O}(n^4)$	$\tilde{O}(n)$
Space	$\tilde{O}(\min(n^2, d^2))$	$\tilde{O}(n^2)$	$\tilde{O}(n^2)$	$\tilde{O}(n^2)$	$\tilde{O}(n)$

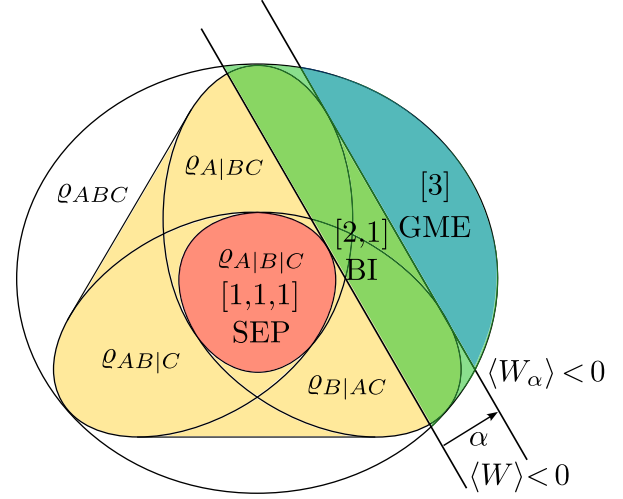


FIG. 1. Structures of three-partite separability. Three-partite states can be fully separable (SEP), with separability partition $\kappa = [1|1|1]$; biseparable ($Q_{AB|C}$, $Q_{A|BC}$, $Q_{AC|B}$ and their convex combination), with partition $\kappa = [2|1]$; or genuinely multipartite entangled (GME), with $\kappa = [3]$. We obtain κ -separability witnesses by adding white noise to isotypic witnesses W detecting biseparable states (green area) with parameter α_κ , which induces a shift in the state space. In the three-partite case this allows for the detection of genuinely multipartite entanglement (blue area), while further structures arise when more parties are involved.

$$\rho = \sum_i p_i \rho_{K_1^i}^i \otimes \dots \otimes \rho_{K_m^i}^i, \quad (2)$$

where each factor $\rho_{K_r^i}^i$ is shared in a disjoint collection $K_r^i \subseteq \{1, \dots, n\}$ of $|K_r^i| = k_r$ local parties.

For example, the state $\rho_A \otimes \rho_{BCD}$ is $[3|1]$ -separable and the state $\rho' = (\rho_{AB} \otimes \rho_{CD} + \rho_{AC} \otimes \rho_{BD})/2$ is $[2|2]$ -separable.

Determining the separability partition κ of a state is crucial for multipartite quantum protocols such as distributed computing and network communication. In particular, two paradigmatic notions of multipartite entanglement stem from it: k_1 -producibility or *entanglement depth*, namely the largest number k_1 of systems among the terms decomposing ρ [44], and m -separability, namely the number m of composite systems in a product state [10]. These forms of entanglement are dual to each other [9] and have been considered in several forms [45] for practical applications. While a variety of methods have been long studied to detect both non- m -separability [4,6,8,10–12] and non- k_1 -producibility [46–48], much less is known about the more general characterization of Definition 1 [38–41].

Methodology—To detect κ -separability, we exploit two symmetries of the problem:

Permutation symmetry: It follows from Eq. (2) that the separability partition κ of a quantum state ρ cannot be affected by the action V_π of a permutation $\pi \in S_n$,

$$V_\pi \rho V_\pi^\dagger \in \kappa\text{-SEP} \Leftrightarrow \rho \in \kappa\text{-SEP}. \quad (3)$$

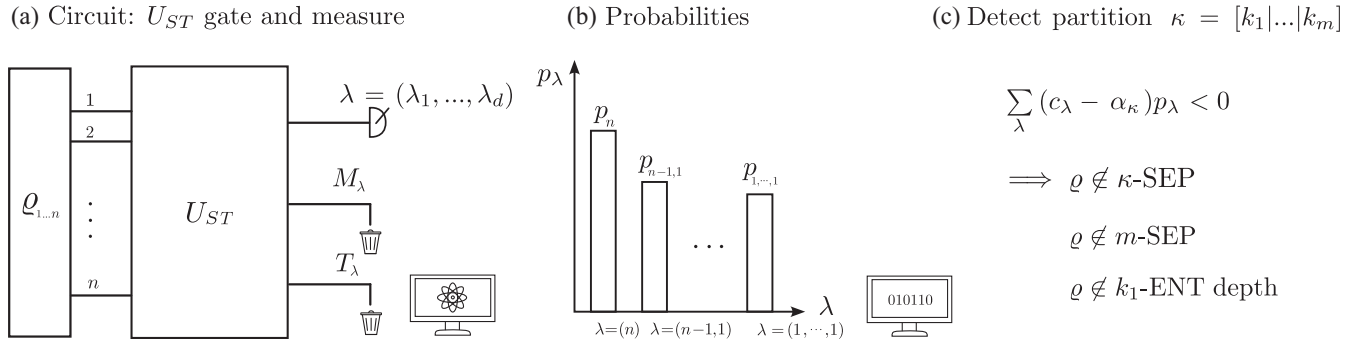


FIG. 2. Detection of entanglement partitions κ in n -partite systems. (a) We perform weak Schur sampling on the n -partite state describing the system, for instance, implementing the Schur transform gate (U_{ST}) and measuring the register λ while discarding the registers M_λ and T_λ . This step is tractable in a quantum computer, and efficiency can be gained with more direct methods of measuring λ such as generalized phase estimation (see Table II). (b) We repeat the procedure multiple times, and store in a classical computer the probabilities p_λ of obtaining each outcome λ . These probabilities are the projections of q onto each irreducible subspace labeled by λ . (c) Based on the probabilities p_λ , we detect states that are not in certain separability partitions κ . In particular, the first component k_1 of κ determines the entanglement depth of q , and the number of components m in κ determines its separability length.

Unitary symmetry: It is a defining property of entanglement that it cannot be affected by the action of local unitary gates. Here we use entanglement symmetry under the diagonal action of the unitary group,

$$U^{\otimes n} q U^{\otimes n \dagger} \in \kappa\text{-SEP} \Leftrightarrow q \in \kappa\text{-SEP}. \quad (4)$$

Extensive work has been done in the study of entanglement exploiting separately unitary [49] and permutation [50] symmetries, and detecting states with these properties is relevant in both theory and experiment [51–53]. Here we use the combination between both symmetries above, which are exactly those of the Hilbert space decomposition through Schur–Weyl duality,

$$(\mathbb{C}^d)^{\otimes n} \simeq \bigoplus_{\lambda \vdash_d n} \mathcal{U}_\lambda \otimes \mathcal{S}_\lambda, \quad (5)$$

where $\lambda = (\lambda_1, \dots, \lambda_d)$ is a partition of n in the sense that $\lambda_1 + \dots + \lambda_d = n$ and $\lambda_1 \geq \dots \geq \lambda_d \geq 0$. The notation $\lambda \vdash_d n$ means that $\ell(\lambda) \leq d$, where λ is a partition of n elements and $\ell(\lambda)$ is the number of nonzero components. Here $\mathcal{U}_\lambda$ is an irreducible representation (irrep) of the unitary group $U(d)$, and $\mathcal{S}_\lambda$ is an irrep of the symmetric group S_n . The projectors Π_λ onto isotypic components $\mathcal{U}_\lambda \otimes \mathcal{S}_\lambda$ are called *Young projectors*. The dimension d_λ of the irrep $\mathcal{S}_\lambda$ is given by evaluating the symmetric group character at the identity, $d_\lambda = \chi_\lambda(id)$, and it is given by the *hook-length* formula. The basis change unitary matrix of Eq. (5) is called *Schur transform*, and it can be implemented efficiently on a quantum computer [30–32,54]. The measurement of Young projectors is called *weak Schur sampling*, and it can be done efficiently via the Schur transform or via generalized phase estimation [33,55]. The scheme to perform weak Schur sampling is depicted in Fig. 2.

We consider the detection of κ -separable states using the linear combination $\sum_{\lambda \vdash_d n} c_\lambda p_\lambda$, where $c_\lambda \in \mathbb{R}$, of probabilities $p_\lambda = \text{tr}(q \Pi_\lambda)$ of obtaining different partitions in measurement outcomes, namely, we look at witnesses of the form $W = \sum_{\lambda \vdash_d n} c_\lambda \Pi_\lambda$. Since each projector Π_λ represents an isotypic component, we denote these witnesses as *Schur–Weyl isotypic (SW-isotypic)*. Following the notation of [56], each witness W is defined by a tuple $\vec{c} = (c_\lambda)_{\lambda \vdash_d n}$. The ansatz of SW-isotypic witnesses is motivated to generalize a wide supply of full-separability witnesses involving projections onto irreducible subspaces, of the form $c_\lambda \Pi_\lambda - c_\mu \Pi_\mu$ [28,56], available from long-studied so-called *matrix immanant inequalities* [57]. Immanants generalize the notion of the determinant and permanent of $n \times n$ square matrices and are given by $\text{Im}_\lambda(G) = \sum_{\pi \in S_n} \chi_\lambda(\pi) \prod_{i=1}^n G_{i, \pi^{-1}(i)}$ for each irrep $\lambda \vdash n$ for each irrep λ . The connection between immanant inequalities and witnesses can be seen by considering the immanants of Gram matrices and writing them as expectation values of operators in Hilbert spaces. Both existing results in this regard and their extension are detailed in Supplemental Material (SM) [29].

For a given W , by considering the minimum expectation value $\alpha_\kappa = \min_{q' \in \kappa\text{-SEP}} \text{tr}(q' W)$ over κ -SEP and the fact that Π_λ resolve to the identity, κ -separability criteria are given by

$$\begin{aligned} \text{tr}(W_{\vec{c}, \kappa} q) &< 0, \quad \Rightarrow q \notin \kappa\text{-SEP}, \\ W_{\vec{c}, \kappa} &:= \sum_{\lambda \vdash_d n} (c_\lambda - \alpha_\kappa^\lambda) \Pi_\lambda. \end{aligned} \quad (6)$$

Here we write in full generality the separability value as α_κ^λ to denote both the partition κ and the vector $\vec{c}$ defining the witness. This notation serves to establish a general framework, although it will be simplified when the context

allows (see Fig. 1). The problem is thus to find the optimal value α_κ . To this end, we use the permutation invariance of the projectors Π_λ .

Proposition 1—The value α_κ is given by

$$\alpha_\kappa = \min_{|\psi\rangle=|\psi\rangle_{k_1}\otimes\cdots\otimes|\psi\rangle_{k_m}} \langle\psi|W|\psi\rangle, \quad (7)$$

where $|\psi\rangle_{k_r}$ is shared among parties $k_{r-1} + 1, \dots, k_r$.

Proof—The value α_κ is by assumption the smallest expectation value of W over κ -separable states. By convexity, we need only consider pure states $|\psi\rangle = |\psi\rangle_{k_{n_1}} \otimes \cdots \otimes |\psi\rangle_{k_{n_m}} \in \kappa\text{-SEP}$. Since the projectors Π_λ are permutation invariant, the expectation value $\langle\psi|W|\psi\rangle$ does not change if all terms decomposing the state $|\psi\rangle$ are permuted to be of the ordered form $|\psi\rangle_{k_1} \otimes \cdots \otimes |\psi\rangle_{k_m}$, which completes the proof. ■

Let us consider quantum states ϱ_{sym} that are invariant under permutations and copies of unitaries,

$$V_\pi \varrho_{\text{sym}} V_\pi^\dagger = U^{\otimes n} \varrho_{\text{sym}} U^{\dagger \otimes n} = \varrho_{\text{sym}}, \quad (8)$$

for all $\pi \in S_n$ and $U \in U(d)$ [58,59]. These can be written as a convex combination of the projectors Π_λ and are not κ -separable if and only if they are detected by some witnesses of the form (6). This is because the minimization of $\text{tr}(W\varrho)$ over κ -separability witnesses can be restricted without loss of generality to isotypic witnesses, which can be written as (6). For each system size n and d , the number of such witnesses that are independent is finite since the state space is given by a polytope.

Entanglement characterization for invariant systems—By exploiting the permutation symmetry and unitary symmetry of projectors Π_λ , we characterize the set of three-partite witnesses with the permutation and unitary symmetries considered above.

Proposition 2—For three-partite systems, all nontrivial (i.e., excluding $p_\lambda \geq 0$) criteria of the form (6) for separability partitions are

$$\varrho \in \text{SEP} \Rightarrow \begin{cases} p_{(2,1)} - 4p_{(1,1,1)} \geq 0, \\ 3p_{(3)} - p_{(2,1)} + p_{(1,1,1)} \geq 0, \end{cases} \quad (9)$$

$$\varrho \in [2|1]\text{-SEP} \Rightarrow p_{(2,1)} - 2p_{(1,1,1)} \geq 0, \quad (10)$$

and convex combinations thereof. These criteria are necessary and sufficient for κ -separability of tripartite states with symmetry (8).

This characterizes the three-partite SW-isotypic witnesses; the proof is given in SM [29]. The first witness in Eq. (9) and $4p_{(3)} - p_{(2,1)} \geq 0$ [which is a convex combination of the two witnesses in Eq. (9)] are known and arise from immanant inequalities [28]. To our knowledge, the second witness in Eq. (9), as well as Eq. (10), cannot be derived using existing immanant inequalities,

although in principle they may be derived independently from the results in [49]. In fact, Eq. (9) detects three-partite states with positive partial transpositions (PPT), while the other ones cannot. We thereby identify the distinguished one-parameter family of states:

$$\varrho_p = p \frac{\Pi_{(3)}}{\text{tr}\Pi_{(3)}} + (1-p) \frac{\Pi_{(2,1)}}{\text{tr}\Pi_{(2,1)}}. \quad (11)$$

Using Proposition 2, we verify that all such states are $[2|1]$ -separable for $0 \leq p \leq 1$, NPT entangled for $0 \leq p < 1/5$, bound entangled for $1/5 < p \leq 1/4$, and fully separable for $1/4 < p \leq 1$.

A numerical characterization of the SW-isotypic witnesses for $n = 4$ -partite κ -separability is given by eight extremal witnesses $W_1, \dots, W_8$. These are explicitly written in Table I, and the techniques used are further explained in SM [29]. For a larger number of parties $n > 4$, the semi-definite programming variable sizes become significantly larger. However, using a desktop computer and approximating separability by the PPT relaxation, this framework allows us to distinguish some κ -separability structures within seven-qubit systems. As a simple example, from the two-rowed partitions immanant inequality $d_{(4,3)} \text{per}(G) \geq \text{Im}_{(4,3)}(G)$ with $d_{(4,3)} = 14$, we consider the seven-qubit full-separability witness

$$W = 14^2 \Pi_{(7)} - \Pi_{(4,3)}. \quad (12)$$

By minimizing the expectation value $\langle \text{tr}_1(|0\rangle\langle 0|W) \rangle$ (where W is trace normalized) over PPT states across the desired partitions and numerical inspection on the results, we find that the associated witness $W' = W + \alpha \Pi_{(4,3)}$ with $\alpha \approx 24 / \binom{33}{5}$ detects states that are not a convex combination of $\varrho_1 \otimes \varrho_2 \otimes \varrho_{3,4,5,6,7}$ and permutations thereof (i.e., $[5|1|1]$ -separable) or a convex combination of $\varrho_1 \otimes \varrho_{2,3,4} \otimes \varrho_{5,6,7}$ and permutations thereof (i.e., $[3|3|1]$ -separable).

Multipartite systems—In large multiparticle systems [60–63], it is often particularly important to ensure that no single local system is separable from the rest, rather than genuinely multipartite entanglement which is difficult to produce in practice. Certifying that no local parties are disentangled ensures that all local parties have access to multipartite teleportation [64], distributed quantum computing [65], conference key agreement [66], and quantum secret sharing [67]. Formally, local users have access to these schemes only if the global n -partite state cannot be written as

$$\varrho = \sum_{i=1}^n p_i \sum_j q_j \varrho_i^j \otimes \varrho_{1\dots n \setminus i}^j. \quad (13)$$

Examples of states that decompose in this way are *semi-separable* [68], and specific methods to detect states

outside their set have been explored for certain system sizes [69,70]. Here we use the symmetries of the criteria introduced, to find witnesses for states that are not of the form (13) for any local dimension and number of parties. For that we find analytically the exact value of $\alpha_{[n-1|1]}$.

Theorem 1—Let $W = \sum_{\lambda \vdash_{d^n}} c_\lambda \Pi_\lambda$ with $c_\lambda \in \mathbb{R}$. Then

$$\alpha_{[n-1|1]} = \min_{\mu \vdash_{d^{n-1}}} \min_{\nu \sqsubseteq \mu} \sum_{e_k \in \text{AC}_d(\mu)} c_{\mu+e_k} w_k^{(\mu,\nu)},$$

$$w_k^{(\mu,\nu)} := \frac{\prod_{i=1}^{d-1} (\nu_i - i - \mu_k + k - 1)}{\prod_{i=1, i \neq k}^d (\mu_i - i - \mu_k + k)}. \quad (14)$$

Here $\text{AC}_d(\mu)$ is the set of addable corners of the Young diagram μ , indexed by the row k , such that $\ell(\mu + e_k) \leq d$. The relation $\nu \sqsubseteq \mu$ means $\mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \dots \geq \mu_{d-1} \geq \nu_{d-1} \geq \mu_d$.

The proof is given in SM [29] and consists of finding the largest eigenvalue of $\text{tr}_1(|0\rangle\langle 0| \otimes 1^{\otimes n-1} \Pi_\lambda)$. This theorem provides a recipe to construct simple criteria to detect many-body states with $\kappa \neq [n-1|1]$. An explicit example is the following criterion for an arbitrary even number of qubits n :

$$p_{(n,0)} - \frac{p_{(n/2,n/2)}}{D^2} + \frac{1}{2D^2} < 0 \Rightarrow \kappa(q) \notin [n-1|1], \quad (15)$$

where $D := d_{(n/2,n/2)} = \binom{n}{n/2} - \binom{n}{n/2-1}$ is the so-called *Catalan number*. To see this, we first recall that the matrix inequality $\text{per}(G) - \text{Im}_{(n-r,r)}(G)/d_{(n-r,r)} \geq 0$ holds for any positive semidefinite matrix M [71,72]. In the Hilbert space representation, this inequality implies in particular that, if q is fully separable, then $p_{(n,0)} - p_{(n/2,n/2)}/D^2 \geq 0$. Theorem 1 shows that, for this witness, the smallest expectation value over $[n-1|1]$ -separable states is $\alpha_{[n-1|1]} = -1/2D^2$ for qubits (namely with local dimension $d=2$), and therefore by adding the asymptotically small shift $1/2D^2$ Eq. (15) holds. The exact derivation of this witness, more general families, dimension-free witnesses, and families of witnesses for high-dimensional systems are given in SM [29]. Concerning the families of witnesses presented in this Letter, we emphasize that, if the permanent-dominance conjecture is shown to be true (namely that the permanent is the largest normalized immanant on positive semidefinite matrices [72,73]), then any expression of the form $p_{(n)} - p_\lambda/d_\lambda^2$ is a full-separability witness and can be lifted to detect κ -inseparability with Theorem 1.

Although our main motivation here is to detect states where no single subsystem is separable from the rest, Theorem 1 can also be used for other bipartitions. For instance, if the system in hand is composed of an even number of subsystems $n = 2t$ of dimension d each, one can group them into t subsystems of dimension d^2 each. Then Theorem 1 provides families of criteria detecting $[n-2|2]$

entanglement in symmetric systems, namely it detects states where no pairs of local subsystems are disentangled from the rest.

Implementation—For numerical use of the witnesses found in this Letter, we provide a symbolic-algebraic SageMath code to generate the Young projectors Π_λ onto the irreducible subspaces $\mathcal{U}_\lambda \otimes \mathcal{S}_\lambda$ of $(\mathbb{C}^d)^{\otimes n}$, to be used in Python language [74]. These can be effectively generated and diagonalized to find α -values for nontrivial system sizes. Experimentally, the entanglement criteria derived in this Letter rely on performing projective measurements onto Young projectors. This is a special case of generalized phase estimation known as *weak Schur sampling*, and it can be done efficiently using a variety of techniques [30–35,55,75]. In standard methods, the key idea is to diagonalize the Hilbert space through the unitary Schur transform U_{ST} as

$$U_{\text{ST}}|i_1\rangle \dots |i_n\rangle = \sum_{\lambda, M_\lambda, T_\lambda} c_{\lambda, M_\lambda, T_\lambda} |\lambda\rangle |M_\lambda\rangle |T_\lambda\rangle, \quad (16)$$

where $c_{\lambda, M_\lambda, T_\lambda}$ are products of the Clebsch-Gordan coefficients and $\lambda \vdash_{d^n}$, $|M_\lambda\rangle \in \mathcal{U}_\lambda$ and $|T_\lambda\rangle \in \mathcal{S}_\lambda$, which can be done efficiently on quantum circuits [30–33]. Then the projection $\text{tr}(\Pi_\lambda \rho)$ of a state ρ onto the irreducible subspace $\mathcal{U}_\lambda \otimes \mathcal{S}_\lambda$ is obtained by measuring the first register $|\lambda\rangle$ onto the computational basis of $\mathbb{C}^{p(n,d)}$, where $p(n,d)$ is the number of possible partitions $\lambda \vdash_{d^n}$. Since we are interested in obtaining the label λ , this procedure can be further optimized avoiding a full diagonalization of the Hilbert space (see Table II).

Conclusions—We have developed a method to detect separability partitions κ by projecting onto invariant subspaces of the n -qudit Hilbert space. By exploiting the symmetries of separability structures, we have been able to analytically fully characterize the entanglement of three qudits with unitary and permutation symmetries and numerically characterize the case of four qudits. We have found and characterized a rich structure of κ -separability, genuinely multipartite entanglement, and bound entanglement for these system sizes, with few real parameters. The symmetry of the method allows us to compute the criteria introduced for larger system sizes. In particular, we provide a family of analytical criteria to detect many-body states where no single party is separable from the rest, which is of particular interest in distributed quantum protocols. Following from this result, we provide ready-to-use explicit families tailored to explicit local dimensions. The method introduced can be implemented in a quantum computer through weak Schur sampling. Besides physics, our results impact the long-studied theory of matrix immanants in mathematics. We explicitly provide new immanant inequalities, characterize their set for matrices of sizes three and four, and introduce rank-dependent inequalities.

This Letter opens several lines of further work. From the physics perspective, it would be interesting to understand the exact relation between the partitions λ corresponding to irreducible subspaces and the separability partitions κ . From the quantum computing perspective, our Letter highlights the importance of the study of the quantum Schur transform and phase estimation and its implementation. From the mathematical side, it remains open whether further immanant inequalities can be found from physical principles.

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Data availability—All code and data used in this manuscript are in the GitHub repository [74].

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End Matter

New matrix inequalities—This contribution extends to constructing new matrix inequalities involving irreducible representations. To date, several works oriented to quantum physics (including this contribution) have focused on using immanant inequalities to detect entanglement [28,56,76,77]. However, immanant inequalities are an intensively studied field of mathematics on their own, and characterizing their set constitutes a current open problem [73,78–80]. Here we use the correspondence in the converse direction to find new such inequalities. For that we need the following reformulation of the results of [28], slightly strengthened by additional rank constraints.

Proposition 3—Let $\{\text{Im}_\lambda\}_\lambda$ be immanants, and let $\{\Pi_\lambda\}_\lambda$ be Young projectors. The inequality

$$\sum_{\lambda \vdash n} a_\lambda \text{Im}_\lambda(G) \geq 0 \quad (\text{A1})$$

holds for all $n \times n$ positive semidefinite matrices G of rank at most d , if and only if

$$\sum_{\lambda \vdash n} \frac{a_\lambda}{d_\lambda} \text{tr}(\Pi_\lambda \varrho) \geq 0 \quad (\text{A2})$$

for all n -partite separable states in local dimension d .

The proof follows exactly the steps of [28], with additional rank constraints on the matrix G which translate into

constraints on the local dimension of the physical system (see SM [29]). This provides a mapping from the space $\mathbb{C}^n \times \mathbb{C}^n$ of $n \times n$ matrices with rank d to the n -qudit Hilbert space $(\mathbb{C}^d)^{\otimes n}$ of some homogeneous local dimension d , and vice versa. Here we use this relation in the converse direction, to find matrix inequalities from n -qudit new matrix inequalities. As an example, the only three-partite witnesses known from immanant inequalities were $\Pi_{(3)} - \Pi_{(2,1)}/4$, $\Pi_{(2,1)}/4 - \Pi_{(1,1,1)}$, and the linear combinations thereof, which we have shown to be decomposable. Yet, we found a linearly independent witness, $3\Pi_{(3)} - \Pi_{(2,1)} + \Pi_{(1,1,1)}$. Besides detecting bound entanglement, this additional witness provides the following inequality for positive semidefinite 3×3 matrices G :

$$3\text{per}(G) - 2\text{Im}_{21}(G) + \det(G) \geq 0. \quad (\text{A3})$$

Since these three witnesses fully characterize the set of three-qudit block-positive operators given as linear combinations of projectors Π_λ , it follows from Proposition 3 that the three corresponding inequalities constitute a complete, analytical characterization for 3×3 matrices. Numerically, we obtain a similar characterization for 4×4 matrices arising from Table I.