

Resurgence of $T\bar{T}$ -Deformed Partition Function

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We study nonperturbative effects of torus partition function of the $T\bar{T}$ -deformed 2D conformal field theory (CFT) by resurgence in this Letter and a companion paper. The deformed partition function can be written as an infinite series of the deformation parameter λ . We develop highly efficient methods to compute perturbative coefficients in the λ expansion. To exemplify, the first 600 coefficients for the $T\bar{T}$ -deformed free boson and free fermion are computed. The method can be generalized to any rational CFTs; we present the key idea in this Letter while relegating details and results to the companion paper. Equipped with the large order perturbative data, we provide convincing numerical evidence that the λ series is asymptotic and not Borel resumable. We extract the nonperturbative contribution by resurgence and propose that they originate from new complex saddle points after analytically continuing the modular parameters in the integral representation of the partition function. The proposal is checked by comparing the predicted asymptotic behavior of the coefficients and large order perturbative data, which match nicely. The implications of these nonperturbative contributions for the Stokes phenomenon, which relates the positive and negative signs of λ , as well as possible comparisons with holography, is also discussed.

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Introduction— $T\bar{T}$ deformation [1,2] offers us a valuable chance to study nonlocal quantum field theories (QFT) in depth. As an irrelevant deformation, the ultraviolet behavior of the deformed theory is significantly altered and exhibit features of gravity [3–7] and string theory [8–13]. However, contrary to generic irrelevant deformations that are plagued by ambiguities, $T\bar{T}$ deformation is solvable. As a result, a variety of interesting physical quantities can be computed exactly in deformed theory, based on the knowledge of their undeformed counterparts.

An important class of solvable QFTs in $1+1$ dimensions are the conformal field theories (CFTs). $T\bar{T}$ deformation of CFTs are particularly interesting due to their holographic interpretations [11,14–20]. A fundamental quantity of a 2D CFT is the torus partition function. The $T\bar{T}$ -deformed partition function has been investigated in [3,5,21–24], where it was proven that the deformed partition function is modular invariant. However, unlike the deformed spectrum that takes a simple closed form, the deformed partition function can only be given in a more

formal way, as an infinite series in the deformation parameter λ , or as an integral transform of the CFT partition function [5,12,23]. This is partly due to the complexity of the torus partition function. A direct computation of higher order contributions to the deformed partition function soon becomes very tedious. As a result, the investigation of the partition function is still preliminary. Fundamental questions such as the convergence of the λ series is still unclear and the analytic properties of the deformed partition function is largely unexplored.

In this Letter and a companion paper [25], we introduce powerful tools to study the $T\bar{T}$ -deformed torus partition function. First, we develop highly efficient methods to compute the coefficients in λ series, based on the theory of almost holomorphic forms (see, for instance, Ref. [26]). This allows us to obtain perturbative results at very high orders. For example, for free CFTs (both bosonic and fermionic), we obtain analytic results up to 600 orders. Using these large order perturbative data, we find convincing evidence that the λ series is asymptotic. In addition, the asymptotic series is not the usual Gevrey-1 type, but of a novel type, as we will show below. To analyze the asymptotic series, we apply the resurgence method [27] (see, e.g., Refs. [28–30] for an introduction). The resurgence theory allows us to explore the nonperturbative effects and analytic properties of the deformed torus partition function (See Refs. [31,32] for a resurgence analysis for the $T\bar{T}$ -deformed 2D Yang-Mills theory.), which turns out to be quite rich. One surprising observation

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is that the naive simple saddle point does not correspond to the nonperturbative contribution obtained from large order perturbative calculations. To explain the result from large order perturbation theory, we need to take into account the influence from singularities of the analytically continued CFT partition function, which shifts the saddle points. We obtained these new nonperturbative contributions and derive the asymptotic behaviors of the expansion coefficients, which match nicely with large order perturbation results.

$T\bar{T}$ -deformed torus partition function—The $T\bar{T}$ -deformed partition function is defined by

$$\mathcal{Z}(\tau, \bar{\tau}|\lambda) = \sum_n e^{2\pi i \tau_1 R P_n - 2\pi \tau_2 R \mathcal{E}_n(\lambda)}, \quad (1)$$

where $\mathcal{E}_n(\lambda)$ is the $T\bar{T}$ -deformed energy [1,2]. We can expand $\mathcal{Z}(\tau, \bar{\tau}|\lambda)$ as a formal series in λ

$$\mathcal{Z}(\tau, \bar{\tau}|\lambda) = \sum_{k=0}^{\infty} Z_k(\tau, \bar{\tau}) \lambda^k, \quad (2)$$

where $Z_0(\tau, \bar{\tau})$ is the undeformed CFT torus partition function. It is known that Z_k satisfies the following recursion relation [3,21]

$$Z_{k+1} = \frac{1}{2} \left(\frac{2\tau_2}{k+1} D^{(k)} \bar{D}^{(k)} - \frac{k}{2\tau_2} \right) Z_k, \quad k \geq 0, \quad (3)$$

where $D^{(k)}$ and $\bar{D}^{(k)}$ are the Ramanujan-Serre (RS) derivatives

$$D^{(k)} = \frac{\partial}{\partial \tau} - \frac{ik}{2\tau_2}, \quad \bar{D}^{(k)} = \frac{\partial}{\partial \bar{\tau}} + \frac{ik}{2\tau_2}. \quad (4)$$

The deformed partition function also allows an integral representation [5,23]

$$\mathcal{Z}(\tau, \bar{\tau}|\lambda) = \frac{\tau_2}{\pi \lambda} \int_{\mathcal{H}_+} \frac{d^2 \zeta}{\zeta_2^2} e^{-\frac{|\zeta - \tau|^2}{\lambda \zeta_2}} Z_{\text{CFT}}(\zeta, \bar{\zeta}). \quad (5)$$

The recursion relation (3) provides us an efficient way to compute higher order contributions in λ expansion while the integral representation (5) is instrumental for the saddle-point analysis.

Large order perturbative results—To compute $Z_k(\tau, \bar{\tau})$ in (2), we exploit the recursion relation (3) as well as the modularity properties. First of all, as argued in [21], the RS derivatives and thus the recursion relation (3) preserve the modularity of the partition function, and, consequently, the coefficients Z_k are modular forms of weights (k, k) . Second, the two RS derivatives $D^{(k)}$ and $\bar{D}^{(k)}$ act almost exclusively on the holomorphic and antiholomorphic sectors, and as a result, all Z_k factorize to (almost) holomorphic and (almost) antiholomorphic characters, similar to the

undeformed partition function Z_0 . Finally, it is observed that the (almost) holomorphic characters of Z_k obtained by repeatedly applying the recursion relation (3) are elements of a finitely generated differential ring, where the RS derivative $D := D^{(k)}$ serves as the differential operator. The differential relations between generators in the ring allow efficient calculation of Z_k .

Take the free boson for an example. The undeformed partition function is

$$Z_0^{\text{B}}(\tau, \bar{\tau}) = \frac{1}{\sqrt{\tau_2}} |\chi^{\text{B}}|^2, \quad (6)$$

where the holomorphic part is $\chi^{\text{B}}(\tau) = 1/\eta(\tau)$. After repeatedly applying (3), one finds Z_n also factorizes

$$Z_n^{\text{B}} = \sum_{\ell=0}^n c_{n,\ell}^{\text{B}}(\tau_2) |\chi_{\ell}^{\text{B}}|^2, \quad (7)$$

where the (almost) holomorphic characters are

$$\chi_{\ell}^{\text{B}} = D^{\ell} \chi^{\text{B}}, \quad (8)$$

which are all elements of the ring generated by

$$\{\eta^{-1}, \tilde{E}_2(\tau, \bar{\tau}), E_4(\tau), E_6(\tau)\}. \quad (9)$$

$\tilde{E}_2$ is the almost holomorphic Eisenstein series defined by

$$\tilde{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{3}{\pi \tau_2}, \quad (10)$$

and $E_4(\tau)$, $E_6(\tau)$ are the usual Eisenstein series. Well-known differential relations between the generators (9) lead to the differential recursion relations

$$D \bar{D} (|\chi_{\ell}^{\text{B}}|^2) = |\chi_{\ell+1}^{\text{B}}|^2 - \frac{4\ell^2 - 2\ell - 1}{8} |\chi_{\ell}^{\text{B}}|^2 + \left(\frac{\ell(2\ell - 3)}{8} \right)^2 |\chi_{\ell-1}^{\text{B}}|^2, \quad (11)$$

which allow us both to prove the factorization property (7) and to efficiently calculate the coefficients $c_{n,\ell}^{\text{B}}$, which are simple functions of τ_2 .

Similarly in the $T\bar{T}$ -deformed free fermion theory, the coefficients Z_n factorize and the holomorphic characters belong to a finitely generated differential ring. Simple differential relations between the ring generators are found to allow efficient calculations of Z_n .

We calculated the coefficients Z_n for both the free boson and the free fermion up to $n = 600$, and evaluate them for $\tau = i\tau_2$. We find that the λ series of the $T\bar{T}$ -deformed partition function is indeed an asymptotic series, in the sense that the coefficients Z_n grow factorially fast. But

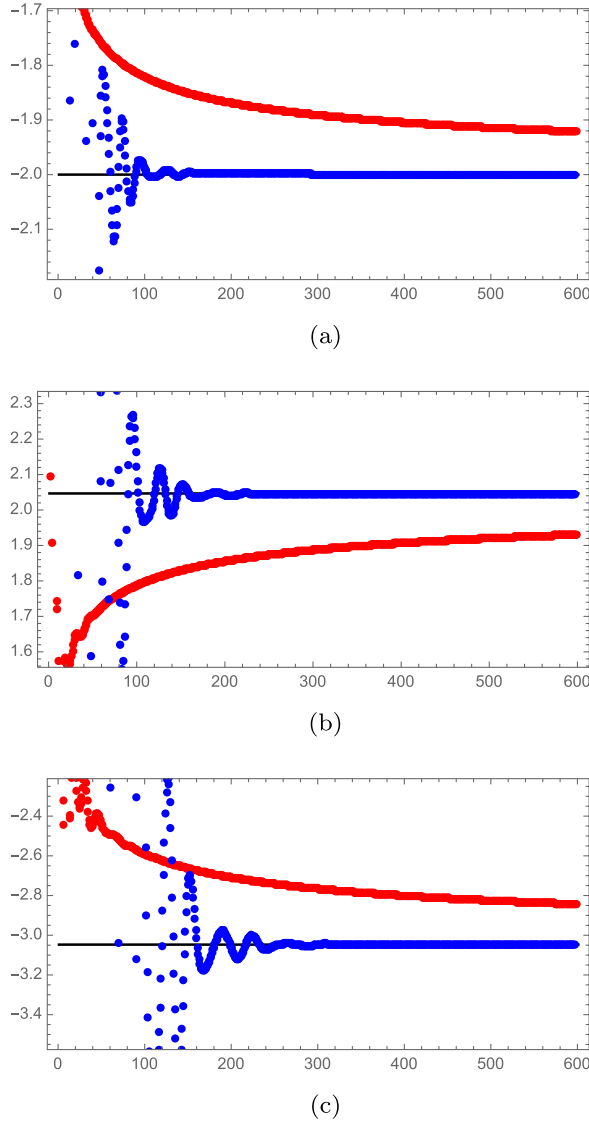


FIG. 1. The sequence of, respectively, s_n , $s'_n = \sqrt{n}(s_n - A)$, and $s''_n = n[s_n - A - (B/2)\sqrt{A/n}]$ (red dots), their Richardson transforms of order-1/2 (blue dots), and their asymptotic values (black line), for the example of $T\bar{T}$ deformed free boson at $\tau = i$. (a) s_n . (b) s'_n . (c) s''_n .

these coefficients display a slightly unusual asymptotic behavior

$$Z_n \sim \frac{\Gamma(n + \nu)}{A^{n+\nu}} \exp\left(B\sqrt{\frac{n + \nu}{A}}\right). \quad (12)$$

This asymptotic behavior can be verified through computing the auxiliary sequence $s_n = nZ_n/Z_{n+1}$, which should have the large n behavior

$$s_n = \frac{nZ_n}{Z_{n+1}} \sim A - \frac{B}{2}\sqrt{\frac{A}{n}} + \frac{B^2 - 8A\nu}{8n} + \dots \quad (13)$$

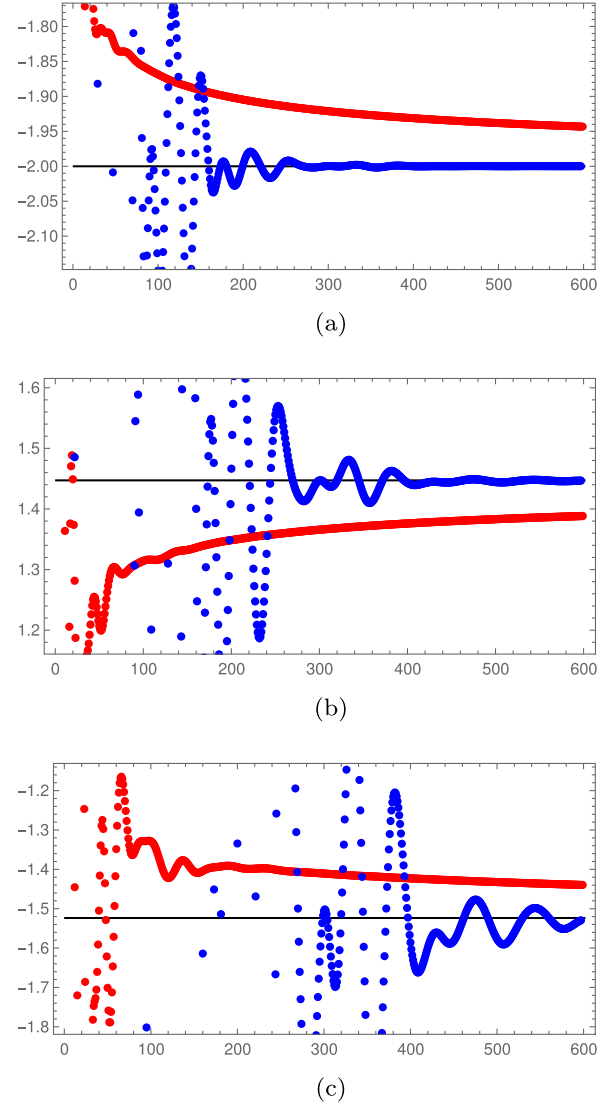


FIG. 2. The sequence of, respectively, s_n , $s'_n = \sqrt{n}(s_n - A)$, and $s''_n = n[s_n - A - (B/2)\sqrt{A/n}]$ (red dots), their Richardson transforms of order 1/2 (blue dots), and their asymptotic values (black line), for the example of $T\bar{T}$ deformed free fermion at $\tau = i$. (a) s_n . (b) s'_n . (c) s''_n .

We check that this is indeed the case, as showcased in Figs. 1 and 2, and find the values of the parameters to be

$$A = -2\tau_2, \quad B = i(1 + \tau_2)\sqrt{\frac{2\pi c}{3}}, \quad (14)$$

where $c = 1$ or $c = 1/2$ is the central charge of the undeformed free CFT, while $\nu = -1$ for free boson and $\nu = -1/2$ for free fermion.

The efficient algorithm for calculating Z_n can be extended to interacting and rational CFTs, where holomorphic characters satisfy modular linear differential equations (MLDEs) [33], which can be used to construct finitely generated rings to which all Z_n belong, and to write down

key differential recursion relations similar to (11). The example of the Lee-Yang model is concretely worked out in the companion paper [25], and the same asymptotic behavior of Z_n is observed.

The asymptotic behavior (12) implies the nonperturbative corrections of the type

$$Z^{\text{np}} \sim e^{-\frac{A}{\lambda} - \frac{B}{\sqrt{\lambda}}\lambda^{-\nu}} \sum_{k=0}^{\infty} b_k \lambda^{k/2}, \quad (15)$$

according to the resurgence theory, whose details can be found in the companion paper [25]. We will see in the next section that $T\bar{T}$ -deformed CFTs such as free boson and free fermion indeed have nonperturbative corrections of this type, with the correct parameters.

Saddle-point analysis—The asymptotic nature of the perturbative expansion in small λ naturally hints at nonperturbative contributions to the deformed partition function. Most importantly, it suggests that they contribute at the order of $\mathcal{O}(e^{2\tau_2/\lambda})$. In this section we identify their origins. These should be captured from the integral representation (5). The coupling constant λ only appears explicitly at the leading order in small λ through the integral kernel, which admits a pair of saddle points for $\zeta^* = \zeta_1^* + i\zeta_2^*$ at

$$\zeta_1^* = 0, \quad \zeta_2^* = \pm\tau_2. \quad (16)$$

By plugging them into the leading order contribution $\exp(-\lambda^{-1}|\zeta - \tau|^2/\zeta_2)$, the “physical” saddle at $(\zeta_2^* = \tau_2)$ contributes at the order $\mathcal{O}(\lambda^0)$, and expanding near it gives the perturbation series in (III). A first guess for the origin of the nonperturbative correction is the other saddle at $(\zeta_2^* = -\tau_2)$. However its contribution is of order $\mathcal{O}(e^{4\tau_2/\lambda})$, inconsistent with the large order behavior. (This saddle point is not excluded from the resurgence analysis. However, its contribution to the asymptotic coefficients Z_n scales like $n!(-4\tau_2)^{-n}$ and so is negligible to the numerical observation at large orders.)

We point out another mechanism for new saddle-points to emerge. The free energy $F_{\text{CFT}} = -\ln Z_{\text{CFT}}$ of the seed CFT, though sub-leading in small λ , could be enhanced near its singularities if present. There it could play a role in the leading order saddle-point equation:

$$\frac{1}{\lambda} \partial_{\zeta} \left(\frac{|\zeta - \tau|^2}{\zeta_2} \right) + \partial_{\zeta} F_{\text{CFT}}(\zeta, \bar{\zeta}) = 0, \quad (17)$$

and produce additional solutions. For the examples we considered, the seed partition functions do contain UV/IR divergences for ζ and $\bar{\zeta}$ independently. This is a property of rational 2D CFTs, due to its finite sum over characters. The leading order behavior of these divergences is universal by conformal symmetry. We can approximate

$$F_{\text{CFT}} \sim \frac{i\pi c_L}{6} (\zeta - \zeta^{-1}) - \frac{i\pi c_R}{6} (\bar{\zeta} - \bar{\zeta}^{-1}) \quad (18)$$

if both ζ and $\bar{\zeta}$ are approaching either 0 or ∞ . For nonchiral CFTs with $c_L = c_R = c/2$, and still considering only $\tau = i\tau_2$, plugging (18) into (17) gives a total of eight additional saddle-point solutions:

$$\zeta_1^* = \pm \sqrt{\frac{6\tau_2^2 \pm \pi c \tau_2 \lambda}{\pi c \lambda}}, \quad \zeta_2^* = \pm i\tau_2 \sqrt{\frac{6}{\pi c \lambda}}, \quad (19)$$

whose small λ expansion: [There exists another branch of solutions that simply interchanges $\zeta^* \leftrightarrow \bar{\zeta}^*$, whose contributions are identical to those in (19).]

$$\zeta^* = \pm \sqrt{\frac{\pi c \lambda}{24}} + \dots, \quad \bar{\zeta}^* = \pm \sqrt{\frac{24}{\pi c \lambda}} \tau_2 + \dots \quad (20)$$

are self-consistent with the approximation (18).

These saddle points occur in the “Regge” regime of the modular parameters ζ and $\bar{\zeta}$, analytically continued from the Euclidean configuration $\bar{\zeta} = \zeta^*$ [34]. Furthermore, their contributions to $\mathcal{Z}(\tau, \bar{\tau}|\lambda)$, after including the F_{CFT} term, take the following form:

$$\mathcal{Z}(\tau, \bar{\tau}|\lambda) \sim \exp \left(\frac{2\tau_2}{\lambda} + i \sqrt{\frac{2\pi c}{3\lambda}} (\pm\tau_2 \pm 1) \right) \quad (21)$$

This matches precisely with the extracted semiclassical behavior of (15), confirming the role these saddle points play in driving the resurgence behavior discussed. We therefore conclude that the $T\bar{T}$ -deformed partition function receives nonperturbative contributions from the interplay between the $T\bar{T}$ coupling λ and the Regge singularity of the seed partition function.

Stokes’ phenomena—The resurgence analysis only identifies the nonperturbative corrections that can potentially contribute to $\mathcal{Z}(\tau, \bar{\tau}|\lambda)$. In general, they are only present in certain wedges of the complex λ plane. The Stokes phenomenon refers to the discontinuous jump of the nonperturbative corrections across the wedge boundaries. Using the approximation (18) as a toy model that captures the essential nonperturbative effects, we study the Stokes phenomena as λ rotates in the complex plane from the positive to negative real axis, relating both signs of the $T\bar{T}$ coupling. The goal is to shed some light on the physics of the $\lambda < 0$ theories.

For $\lambda > 0$, the contribution from the physical saddle point $(\zeta_1^* = 0, \zeta_2^* = \tau_2)$ takes the form

$$\mathcal{Z}(\tau, \bar{\tau}|\lambda) \sim e^{\frac{2\tau_2}{\lambda} - \frac{\sqrt{(6\tau_2^2 - \pi c \lambda)(6 - \pi c \lambda)}}{3\lambda}}. \quad (22)$$

It contains a singularity at the critical temperature $\tau_2^c = \sqrt{\pi c \lambda}/6$. By restricting to lower temperatures,

i.e., $\tau_2 > \tau_2^c$, it can be shown that the contour $\mathcal{H}_+$ coincides with the Lefschetz thimble through the physical saddle. As a result, the other saddle-point contributions are absent. As $\lambda = |\lambda|e^{i\theta}$ rotates in the complex λ plane, it can be checked that a Stokes phenomenon is encountered at $\theta = \pi$, where the saddle points from (19) begin to contribute to $\mathcal{Z}(\tau, \bar{\tau}|\lambda)$. Taking this into account, the partition function for $\lambda = -|\lambda| < 0$ looks like

$$\mathcal{Z}(\tau, \bar{\tau}|-|\lambda|) \sim e^{-\frac{2\tau_2}{|\lambda|} + \frac{\sqrt{(6\tau_2^2 + \pi c|\lambda|)(6 + \pi c|\lambda|)}}{3|\lambda|}} + e^{-\frac{2\tau_2}{|\lambda|} + \sqrt{\frac{2\pi c}{3|\lambda|}} + \dots} \quad (23)$$

It is nonsingular for all temperatures $\tau_2 > 0$. We analyze the physical implications of (23) by assuming its validity can be extended to include $\tau_2 < \tau_2^c$.

Interestingly, each of the terms in (23) alone can be explained by distinct behaviors of the theory at $\lambda < 0$. The first term alone can be explained by a complex spectrum—the analytic continuation of the $\lambda > 0$ spectrum via $\lambda \rightarrow -|\lambda|$. On the other hand, the second term alone can be explained by imposing a UV cutoff at $\Lambda = 2|\lambda|^{-1}$:

$$\mathcal{Z}(\tau, \bar{\tau}|-|\lambda|) \sim \int^\Lambda dE e^{-\tau_2 E + S(E)} \sim e^{-\frac{2\tau_2}{|\lambda|} + S(2/|\lambda|)} + \dots \quad (24)$$

In the second line we performed an integration by parts, and collected the leading-order boundary term. For a Cardy-like entropy $S(E) = \sqrt{\pi E c/3}$, it explicitly matches the second term of (23).

There are two options for treating the spectral flow equation at $\lambda < 0$: (i) allowing complex energies; and (ii) imposing a UV cutoff. Our analysis suggests that they could be dynamically represented by distinct saddle-point contributions to the partition function at $\lambda < 0$. However, we emphasize that a coherent resolution is beyond the scope of our result. One obstacle is the singularity at $\tau_2 = \tau_2^c$ in (22) which is reminiscent of the Hagedorn singularity in string theory [35–37]. Its impact to the Stokes phenomenon is unclear, e.g., it may invalidate (23) at high temperatures. Its resolution possibly involves all winding sectors [38–41]. They were not considered when deriving (5) in [5,12,23] but could play important roles [42]. Another challenge is to consider the effects from other saddle points. They include the one at $\zeta_2^* \approx -\tau_2$, whose contribution is also absent at $\lambda > 0$ and enters at $\lambda < 0$ via the Stokes’ phenomenon, where it is negligible to both terms found in (23). They also include additional saddle points near the modular transformed images of the singularities we found. We leave these to future investigations.

Discussions—In this Letter, we studied nonperturbative effects of the torus partition of $T\bar{T}$ -deformed 2D CFTs.

By developing highly efficient methods for computing large order perturbative results, we provide convincing evidence that the series is asymptotic. We identify novel nonperturbative contributions for the deformed partition function which matches nicely with numerical predictions.

Establishing the fact that the λ series is asymptotic is a solid step towards a deeper understanding of the deformed partition function. If the series were convergent, one would expect the physics does not depend on the sign of λ within the convergence region. However, it is known that the physical behavior is qualitatively different for the two signs of λ , no matter how small $|\lambda|$ is. In this sense, the asymptotic nature of the λ series is expected. The appearance of asymptotic series usually signifies nonperturbative contributions. Well-known examples of such contributions in local QFTs include instantons [43,44] and renormalons [45], both of which give rise to Gevrey-1 type behavior for the asymptotic coefficients. Here we encounter a novel type of nonperturbative contributions whose asymptotic coefficient growth is not Gevrey-1. It is an intriguing question to find the physical interpretation of these nonperturbative contributions, which should be related to the nonlocality of $T\bar{T}$ -deformed theories.

Though our result pertains specifically to the context of $T\bar{T}$ -deformed CFTs, we believe that the origin of the nonperturbative contribution, particularly the mechanism under which new saddle points arise, can be quite universal. In fact, it is very analogous to how “islands” [46,47] arise as additional saddle points in the gravitational path-integral describing an evaporating black hole—by balancing the leading order in G_N area term with a subleading yet rapidly varying entropy term. Their discovery has played a crucial role in explaining the later part of page curve [48] that characterizes the black hole information paradox. In the future, it is fruitful to keep an eye for similar effects when studying other systems with a semiclassical limit.

From the holography perspective, although our result is not directly applicable to holographic CFTs, it suggests the possibility that as we move from $\lambda > 0$ to $\lambda < 0$, new nonperturbative contributions may emerge from the Stokes phenomena. It is thus an interesting question to look for such non-perturbative contributions and identify their bulk dual, both in the double-trace and single-trace $T\bar{T}$ deformation. Recently, an interesting non-perturbative completion has been proposed [19,20] in the context of single-trace $T\bar{T}$ -deformation. The main idea is to take into account the so-called negative winding modes. The non-perturbative correction that we found has certain similarity with the contributions from the $w = -1$ negative winding mode, but they are different. Detailed discussions can be found in the companion paper [25].

An interesting observation is that the coefficients $Z_n(\tau, \bar{\tau})$ exhibit a nice and simple structure (7), it is tempting to ask whether the “almost holomorphic characters” χ_ℓ have physical meanings, both in field theory and in

holography [49,50]. Their mathematical properties and the relations to $T\bar{T}$ -deformed characters defined in [51] is yet another interesting question.

The efficient method for the perturbative calculation which is developed in this Letter and the companion paper [25] has a wide applicability to related questions. An immediate next question is performing a similar analysis for the $J\bar{T}$ - and even more general $T\bar{T} + J\bar{T} + T\bar{J}$ deformations where the flow equation and integral representations of the deformed partition functions are known [23,52]. Related, it would also be highly interesting to consider the torus one-point function [53] and see if there are similar nonperturbative contributions.

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