

Spontaneously Broken Noninvertible Symmetries in Transverse-Field Ising Qudit Chains

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Recent developments have revealed that symmetries need not form a group, but instead can be noninvertible. Here we use analytical arguments and numerical evidence to illuminate how spontaneous symmetry breaking of a noninvertible symmetry is similar yet distinct from ordinary, invertible, symmetry breaking. We consider one-dimensional chains of group-valued qudits, whose local Hilbert space is spanned by elements of a finite group G (reducing to ordinary qubits when $G = \mathbb{Z}_2$). We construct Ising-type transverse-field Hamiltonians with $\text{Rep}(G)$ symmetry whose generators multiply according to the tensor product of irreducible representations (irreps) of the group G . For non-Abelian G , the symmetry is noninvertible. In the symmetry broken phase there is one ground state per irrep on a closed chain. The symmetry breaking can be detected by local order parameters but, unlike the invertible case, different ground states have distinct entanglement patterns. We show that for each irrep of dimension greater than one the corresponding ground state exhibits string order, entanglement spectrum degeneracies, and has gapless edge modes on an open chain—features usually associated with symmetry-protected topological order. Consequently, domain wall excitations behave as one-dimensional non-Abelian anyons with nontrivial internal Hilbert spaces and fusion rules. Our Letter identifies properties of noninvertible symmetry breaking that existing quantum hardware can probe.

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Introduction—Symmetries provide powerful organizing principles that dictate the form of physical laws, constrain dynamics, and enable the classification of phases of matter. Following Landau [1], phases of matter may be classified according to patterns of spontaneous symmetry breaking (SSB), as characterized by local order parameters which are organized into charged multiplets. The prototypical example is the spontaneous breaking of $\mathbb{Z}_2$ symmetry in the transverse field Ising model (TFIM) [2] or its classical 2D equivalent [3–6]. At the next level, unbroken symmetries can protect nontrivial entanglement in the ground state wave function of a many-body system, called a symmetry protected topological phase (SPT) [7,8], which are

characterized instead by a nonlocal string order parameter [9–14]. In recent years the notion of symmetry has expanded greatly under the broad umbrella of the “generalized symmetries paradigm” [15,16]. Whereas ordinary, invertible, symmetries are captured by their group structure, it is now understood that symmetries can instead be noninvertible (technically, described by a fusion category structure) [17–24]. The past few years have seen an explosion of activity in the study of such noninvertible symmetries, with particular emphasis on their formal mathematical underpinnings and field theory manifestations [25–33]. Recently, more attention is being paid to condensed-matter-relevant lattice systems [34–55]. However, there remains a large disconnect between these very technical developments and the broader condensed matter community, for whom invertible internal and spatial symmetries are textbook knowledge yet noninvertible symmetries remain largely mysterious.

The aim of this Letter is to bridge that divide, by studying spontaneous breaking of a noninvertible symmetry in a model with a tensor product Hilbert space which

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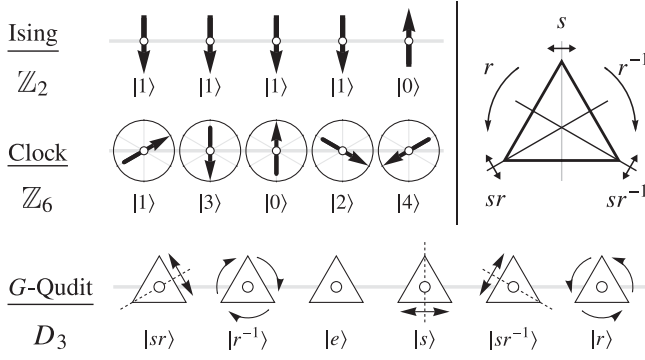


FIG. 1. We consider Ising-like chains of G -qudits, whose internal states are labeled by the elements of a finite group G . They reduce to the transverse-field Ising model when $G = \mathbb{Z}_2$ and clock models when $G = \mathbb{Z}_N$, but can harbor noninvertible $\text{Rep}(G)$ symmetries when G is non-Abelian. The smallest discrete non-Abelian group is the dihedral group D_3 , generated by a rotation r and a reflection s .

is a direct generalization of the celebrated TFIM, see Fig. 1. We show how ground states with distinct entanglement structures arise and how seemingly disparate features—conventional local order and nonlocal string order—coexist in harmony, protected by the noninvertible symmetry. Our results offer guidance for ongoing analog and digital quantum simulations with qudits and open new avenues toward non-Abelian quantum computation in one spatial dimension.

Ising models: From qubits to G -qudits—The TFIM is the workhorse of quantum many-body physics, and for our purposes is the canonical example of invertible symmetry breaking. The Hamiltonian is

$$H_{\text{TFIM}} = -J \sum_i Z_i Z_{i+1}^\dagger - h \sum_i X_i + \text{H.c.}, \quad (1)$$

where X and Z are Pauli operators, $J > 0$ is the ferromagnetic Ising coupling, and h the transverse field strength. This Hamiltonian has an invertible $\mathbb{Z}_2$ symmetry generated by the unitary operator $U = \prod_i X_i$. At large field there is a unique symmetric gapped ground state, the paramagnetic product state of the symmetric superposition $|0\rangle + |1\rangle$ on every site. At small field there are two ferromagnetic gapped ground states, $|0\rangle \equiv \otimes_i |0\rangle_i$ and $|1\rangle \equiv \otimes_i |1\rangle_i$; since $U|0\rangle = |1\rangle$ and vice versa, the $\mathbb{Z}_2$ symmetry is spontaneously broken. This symmetry breaking is detected by operators such as Z_i which are charged under the symmetry, $UZ_i = -Z_iU$, which serve as local order parameters. In the symmetry broken phase the gapped excitations are fermionic domain walls, which behave as deconfined quasiparticles in this 1D system [2].

The natural generalization of the Ising $\mathbb{Z}_2$ symmetry to an Abelian $\mathbb{Z}_N$ symmetry is a clock model [56]. We upgrade the two-state qubits with N -state qudits, with states $|n\rangle$, and generalize the Pauli operators to “clock”

operator, $Z^k|n\rangle = e^{i2\pi kn/N}|n\rangle$, and “shift” operator $X^m|n\rangle = |n+m\rangle$, with addition modulo N . They satisfy

$$Z^k X^m = e^{i2\pi km/N} X^m Z^k. \quad (2)$$

Using the same Hamiltonian, Eq. (1), the global $\mathbb{Z}_N$ symmetry is generated by $U = \prod_i X_i$. The trivial state at large field is the symmetric superposition $\sum_n |n\rangle$ on every site, and the symmetry broken states at small field are $|n\rangle \equiv \otimes_i |n\rangle_i$. The symmetry generator exchanges ground states, $U^m|n\rangle = |n+m\rangle$, and the Z_j operators again serve as local order parameters, as they are charged under U due to Eq. (2), with $\langle n|Z_j|n\rangle = e^{2\pi i n/N}$. The domain walls are parafermionic quasiparticles [56,57].

We now consider how this generalizes to an arbitrary finite, and in particular non-Abelian, group G . Following the pattern, the local Hilbert space is spanned by states $|g\rangle$ for each group element $g \in G$, which we refer to as a G -qudit [58]. The clock and shift operators generalize to [51,59–62]

$$\bar{X}^h|g\rangle = |hg\rangle, \quad \bar{X}^h|g\rangle = |gh^{-1}\rangle, \quad Z_{\alpha\beta}^\Gamma|g\rangle = \Gamma_{\alpha\beta}^g|g\rangle. \quad (3)$$

The non-Abelian shift operators are self-explanatory—they perform group multiplication, either from the left or right (which are inequivalent for non-Abelian groups). The non-Abelian clock operators are labeled by an irreducible representation (irrep) Γ , where $\Gamma_{\alpha\beta}^g$ is the d_Γ -dimensional unitary representation matrix of group element g . The non-Abelian generalization of Eq. (2) is [58]

$$Z_{\alpha\beta}^\Gamma \bar{X}^g = \Gamma_{\alpha\gamma}^g \bar{X}^g Z_{\gamma\beta}^\Gamma, \quad (4)$$

with implied summation of the repeated index γ (but not g). These reduce to the Pauli operators for $G = \mathbb{Z}_2$ and clock operators when $G = \mathbb{Z}_N$.

Using the G -qudits, the simplest G -symmetric transverse-field Hamiltonian analogous to Eq. (1) is [60,63]

$$H_G = -J \sum_i \sum_\Gamma d_\Gamma \text{Tr}[Z_i^\Gamma \cdot Z_{i+1}^\Gamma] - h \sum_i \sum_g \bar{X}_i^g + \text{H.c.}, \quad (5)$$

where in the first term the dot $(\cdot)$ indicates contraction of the neighboring indices of the Z operators, while the trace indicates contraction of their outer indices, and the bar indicates $\bar{\Gamma}^g = \Gamma^{g^{-1}}$. This Hamiltonian has a G symmetry generated by invertible operators $U_g = \prod_i \bar{X}_i^g$. It also has other symmetries irrelevant to the current discussion; its most general version is given in Appendix A. The first term is a projector which is zero unless neighboring sites are in the same group element state, thus at small field the ground states are $|g\rangle \equiv \otimes_i |g\rangle_i$, spontaneously breaking the G symmetry. The second term serves to disorder the group

elements, such that at large field the unique gapped ground state is the symmetric superposition $\sum_g |g\rangle$ on every site.

Noninvertible Rep(G) symmetry—It is well known that the TFIM and clock models (1) are self-dual, known as Kramers-Wannier duality, effectively exchanging the X and Z operators. The dual of Eq. (5) is (Appendix B)

$$H_{\tilde{G}} = -J \sum_i \sum_g \tilde{X}_i^g \tilde{X}_{i+1}^g - h \sum_i \sum_{\Gamma} d_{\Gamma} \text{Tr}[Z_i^{\Gamma}] + \text{H.c.} \quad (6)$$

Whereas Eq. (5) has a G symmetry generated by the X operators, Eq. (6) has a symmetry generated by the Z operators, namely, $R_{\Gamma} = \text{Tr}[\prod_i Z_i^{\Gamma}]$, where all neighboring indices of the Z 's are contracted. These are best expressed as matrix product operators (MPOs), by treating the indices of the Z 's as the virtual legs of the MPO,

$$Z_{\alpha\beta}^{\Gamma} \equiv \alpha \bullet \left[\begin{array}{c} | \\ \hline Z^{\Gamma} \\ \hline \end{array} \right] \bullet \beta, \quad R_{\Gamma} = \left(\begin{array}{c} | \\ \hline Z^{\Gamma} \\ \hline \end{array} \right) \cdots \left(\begin{array}{c} | \\ \hline Z^{\Gamma} \\ \hline \end{array} \right). \quad (7)$$

The symmetry operators R_{Γ} multiply according to the representation algebra [38,51,61,64],

$$R_{\Gamma_a} R_{\Gamma_b} = \sum_c N_{ab}^c R_{\Gamma_c} \Leftrightarrow \Gamma_a \otimes \Gamma_b = \bigoplus_c N_{ab}^c \Gamma_c, \quad (8)$$

where N_{ab}^c are non-negative integers which count the number of times Γ_c appears in the decomposition of $\Gamma_a \otimes \Gamma_b$ into irreps. This algebra is called $\text{Rep}(G)$, the dual of the G symmetry. For an Abelian group every irrep is one dimensional, the product of two irreps is another 1D irrep, and thus there is a unique irrep on the right-hand side of Eq. (8), so the symmetry is invertible—the R_{Γ} are unitary operators. This is Pontryagin duality—to wit, addition modulo N (performed by X 's) is equivalent to multiplication of N th roots of unity (performed by Z 's). Non-Abelian groups have irreps with $d_{\Gamma} > 1$, so some tensor products of irreps must be reducible, resulting in multiple terms on the right-hand side of Eq. (8). This makes the $\text{Rep}(G)$ symmetry noninvertible when G is non-Abelian—some of the R_{Γ} have a nontrivial kernel [65]. The simplest way to see this is to consider the action of R_{Γ} on a product state, which computes the character in irrep Γ of the ordered product of group elements

$$R_{\Gamma}[g_1, \dots, g_L] = \text{Tr}[\Gamma^{g_1 \cdots g_L}] |g_1, \dots, g_L\rangle. \quad (9)$$

If $d_{\Gamma} > 1$ then at least one group element has zero character, so the R_{Γ} annihilates some states and therefore cannot be inverted.

Noninvertible symmetry breaking—We now explore the spontaneous breaking of this noninvertible symmetry [24,34–38]. To express the low-field symmetry broken ground states we introduce the dual of the $|g\rangle$ basis [58], defined by

TABLE I. (a) The character table of the dihedral group D_3 (isomorphic to the permutation group S_3), where the characters are the traces of the irrep matrices $\text{Tr}[\Gamma^g]$. The E irrep has a zero character, which reflects the noninvertibility of $\text{Rep}(D_3)$ symmetry. (b) The $\text{Rep}(D_3)$ algebra, where A_1 is the identity, A_2 generates a $\mathbb{Z}_2$, and E is a noninvertible generator due to $E \otimes E$ being reducible.

(a)			
D_3	$[e]$	$[r]$	$[s]$
A_1	1	1	1
A_2	1	1	−1
E	2	−1	0
(b)			
$\Gamma_a \otimes \Gamma_b$	A_1	A_2	E
A_1	A_1	A_2	E
A_2	A_2	A_1	E
E	E	E	$A_1 \oplus A_2 \oplus E$

$$|\Gamma_{\alpha\beta}\rangle = \sqrt{\frac{d_{\Gamma}}{|G|}} \sum_{g \in G} \Gamma_{\alpha\beta}^g |g\rangle \equiv |\alpha\rangle \bullet \left[\begin{array}{c} | \\ \hline \Gamma \\ \hline \end{array} \right] \bullet |\beta\rangle, \quad (10)$$

where $|G|$ is the number of group elements. In the second equality we have expressed this as a matrix product state (MPS) tensor, with indices α and β labeling virtual states transforming in irrep Γ and its dual, respectively (Appendix C). In this basis the X operators are block-diagonalized, $\tilde{X}^g |\Gamma_{\alpha\beta}\rangle = \Gamma_{\alpha\beta}^{g^{-1}} |\Gamma_{\gamma\beta}\rangle$. It follows that when $h \rightarrow 0$, the $\text{Rep}(G)$ symmetry-broken ground states of Eq. (6) are fully contracted MPSs which are invariant under the action of the two-site $\tilde{X} \tilde{X}$ term [58],

$$|\Gamma\rangle = \sum_{\{\alpha_i\}} \bigotimes_i \frac{1}{\sqrt{d_{\Gamma}}} |\Gamma_{\alpha_i \alpha_{i+1}}\rangle = \left(\begin{array}{c} | \\ \hline \Gamma \\ \hline \end{array} \right) \circ \cdots \circ \left(\begin{array}{c} | \\ \hline \Gamma \\ \hline \end{array} \right). \quad (11)$$

The normalization of the MPS is naturally incorporated as a virtual bond insertion of the diagonal matrix $\Lambda_{\alpha\beta} \equiv \delta_{\alpha\beta} / \sqrt{d_{\Gamma}}$ formed by Schmidt eigenvalues for this MPS. It is indicated by the open circle. One can check that the MPS is expressed in its canonical form. The $\tilde{X}_i^{g \neq e}$ and $\tilde{X}_i^{g \neq e}$ operators serve as local order parameters of spontaneous symmetry breaking. Indeed it is easy to show that $\langle \Gamma | \tilde{X}_i^{g \neq e} | \Gamma \rangle = \text{Tr}[\Gamma^g] / d_{\Gamma}$ [58]. In contrast, deep in the paramagnetic regime ($J \rightarrow 0$) the ground state is given by the product state $|e\rangle \equiv \bigotimes_i |e\rangle_i$, where e is the group identity, and the order parameter vanishes, $\langle e | \tilde{X}_i^{g \neq e} | e \rangle = 0$.

As an example, consider the smallest non-Abelian group—the dihedral group D_3 , the symmetries of an equilateral triangle, generated by a threefold rotation r

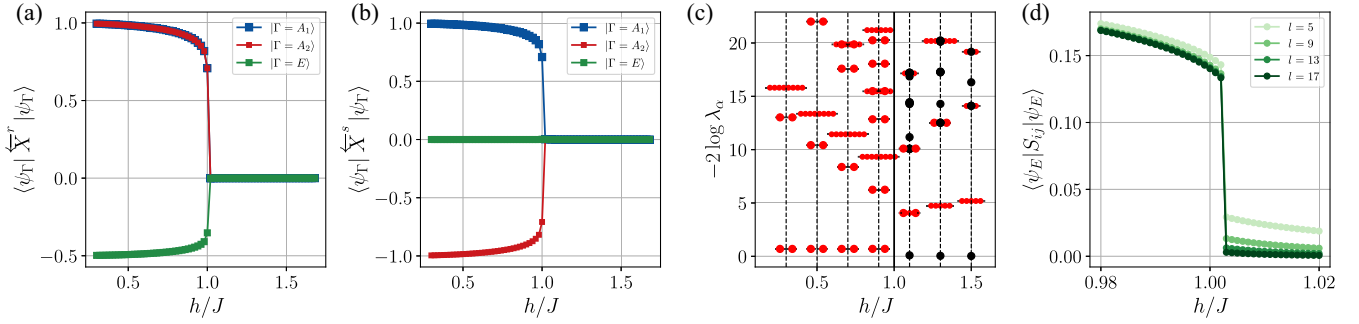


FIG. 2. Spontaneously breaking a noninvertible $\text{Rep}(G)$ symmetry mixes local and nonlocal order. Here we illustrate it for the group D_3 with the Hamiltonian Eq. (6) using iDMRG: (a),(b) Local order parameters $\langle \tilde{X}_i^r \rangle$ and $\langle \tilde{X}_i^s \rangle$ in each symmetry-broken ground state; (c) The entanglement spectrum under a bipartition of the ground state $|E\rangle$. Nondegenerate Schmidt eigenvalues are colored black, while degenerate ones are colored red; (d) The string order parameter measured in the 2D irrep $|E\rangle$ ground state for different string lengths. All states are obtained by initializing in the exact $|\Gamma\rangle$ ground state at $h = 0$, then incrementally increasing h/J and numerically converging to the ground state in the same symmetry sector [66].

and a reflection s , illustrated in Fig. 1. Its character table is given in Table I(a), with three conjugacy classes and three irreps—the 1D trivial irrep A_1 , the 1D sign irrep A_2 , and a 2D irrep E acting as rotations in the plane, permuting the corners of the triangle. Note that the 2D E irrep has a zero character on the $[s]$ conjugacy class, reflecting the non-invertibility of $\text{Rep}(D_3)$ symmetry. The $\text{Rep}(D_3)$ algebra is given in Table I(b). We can see that A_2 generates an invertible $\mathbb{Z}_2$ subsymmetry, while E generates the non-invertible part of the symmetry.

When $h \rightarrow 0$ there are three symmetry broken ground states: two product states $|A_1\rangle$, $|A_2\rangle$ and an MPS $|E\rangle$ of bond-dimension 2. To study $h > 0$ we perform infinite density matrix renormalization group (iDMRG) calculations, initializing with one of the exact $h = 0$ ground states (11), incrementally increasing h , and reconverging to the same symmetry-broken sector. Figures 2(a) and 2(b) illustrate the expectation values for the local order parameters $\tilde{X}_i^r$ and $\tilde{X}_i^s$ as a function of h for each ground state, showing a phase transition to the large- h trivial phase [66].

Ground state entanglement—So far noninvertible $\text{Rep}(G)$ symmetry breaking does not look so different from ordinary G symmetry breaking, but let us juxtapose them more closely. Symmetry broken ground states of the G -symmetric Hamiltonian Eq. (5) are labeled by a group element, and any two ground states are related by the symmetry action, $U_g|h\rangle = |gh\rangle$. This is not the case in the noninvertible case: instead we have $R_{\Gamma_a}|\Gamma_b\rangle = \sum_c N_{ab}^c |\Gamma_c\rangle$. In particular, the action of the symmetry on a ground state can create a linear superposition of macroscopically distinct ground states, a cat state, which never happens for invertible symmetries.

Notice that the bond dimension of the ground state MPS in Eq. (11) is equal to the dimension of the irrep Γ , indicating that different ground states have inequivalent entanglement structures—1D irreps are unentangled product states, while higher-dimensional ones carry nontrivial entanglement. Such entanglement structure is a hallmark of

SPT order, for which it is reflected in (i) the presence of zero modes at the boundary of an open system, (ii) exact degeneracy of the entanglement spectrum, and (iii) the existence of a nonlocal string order parameter. We will now show that all three of these SPT markers are present for $\text{Rep}(G)$ symmetry broken ground states with $d_\Gamma > 1$.

First, whereas on a closed chain the number of ground states is equal to the number of irreps, on an open chain the ends of the MPS tensors Eq. (11) are no longer contracted. For a given irrep Γ , the ground states carry free indices at the ends

$$|\Gamma_{\alpha\beta}\rangle = |\alpha\rangle \bullet \boxed{|\Gamma\rangle} \circ \cdots \circ \boxed{|\Gamma\rangle} \bullet |\beta\rangle. \quad (12)$$

The number of distinct ground states on an open chain is therefore $\sum_\Gamma d_\Gamma^2 = |G|$, which we verified numerically with open-chain DMRG calculations. For each irrep with $d_\Gamma > 1$ the symmetry broken ground state has localized gapless edge modes. Since $\tilde{X}_L^g$ at the left end of the chains and $\tilde{X}_R^g$ at the right end of the chain commute with the Hamiltonian, but do not commute with the $\text{Rep}(G)$ symmetry, they are strong zero modes [69]. By acting on an energy eigenstate, these local operators generate a nonequivalent state of the same energy but with a different internal state of the noncontracted external leg. In summary, ground states $|\Gamma_{\alpha\beta}\rangle$ span the Hilbert space of a single G -qudit. In contrast to ordinary symmetry breaking, part of it is fractionalized between the two ends of the chain.

Second, we obtain the entanglement spectrum (the spectrum of the reduced density matrix of a bipartition) using iDMRG for each ground state as h is varied. Figure 2(c) shows the Schmidt eigenvalues for the $|E\rangle$ ground state of the $\text{Rep}(D_3)$ -symmetric Hamiltonian, which exhibit exact double degeneracies of every eigenvalue throughout the symmetry broken phase, compared to singlet eigenvalues in the trivial phase. Beyond D_3 , we find that the degeneracy is robust—a $|\Gamma\rangle$ ground state has a

d_Γ -degenerate entanglement spectrum within the SSB phase (Appendix D). There we have also checked that these entanglement degeneracies are robust to generic $\text{Rep}(G)$ -preserving perturbations of the Hamiltonian which break any additional global symmetries.

The existence of entangled SSB states is an intrinsic property of the noninvertible symmetry originating from the structure of its symmetry multiplets, which mix local and nonlocal operators [30,34]. Consider first an ordinary G symmetry: local charged operators form irreducible multiplets which transform as $U_g \mathcal{O}_\Gamma^\alpha = (\Gamma_{\alpha\beta}^g \mathcal{O}_\Gamma^\beta) U_g$. In contrast, for the order parameters of $\text{Rep}(D_3)$ symmetry, we can derive the operator equality

$$R_E \vec{X}_j^r = \left[\text{Re}(\omega) \vec{X}_j^r + i \text{Im}(\omega) \left(\prod_{k < j} Z_k^{A_2} \right) \vec{X}_j^r \right] R_E, \quad (13)$$

where $\omega = \exp(i2\pi/3)$ (Appendix F). This demonstrates that the local operator $\vec{X}_j^r$ forms an irreducible multiplet together with a nonlocal string operator, a remarkable property ensured by the noninvertible $\text{Rep}(G)$ symmetry. In particular in the SSB phase this implies that if a local order parameter is nonzero then a string order parameter from the same multiplet must also be nonzero. Starting from Eq. (13), in Ref. [58] we derive relations between two-point correlation functions of local charged operators and string order parameters. Using iDMRG, we observe in Fig. 2(d) that the string order parameter $S_{ij} = \vec{X}_i^r \prod_{i < k < j} Z_k^{A_2} \vec{X}_j^r$ has a vacuum expectation value in the $|E\rangle$ ground state.

Anyonic domain walls—The mixing of local and nonlocal order and the presence of nontrivial entanglement in the noninvertible symmetry broken phase is reflected in the nature of the domain wall excitations. Letting Γ_0 denote the trivial representation, the state $|\Gamma_0\rangle$ plays a special role—any ground state can be created as $|\Gamma\rangle = R_\Gamma |\Gamma_0\rangle$ since Γ_0 acts as the identity element of the $\text{Rep}(G)$ algebra. Acting on $|\Gamma_0\rangle$ with a truncated R_{Γ_a} MPO creates a Γ_a domain inside the Γ_0 ground state, with a Γ_a ($\Gamma_{\bar{a}}$) domain wall at the left (right) end, where $\bar{a}$ denotes the dual representation (Appendix C). Since the truncated MPO contains an uncontracted virtual bond, each domain wall carries an internal d_{Γ_a} -dimensional Hilbert space. These quasiparticles fuse according to the tensor product of their corresponding irreps

$$|\psi_{ab}\rangle = \cdots - [\Gamma_a] - [\Gamma_a] - \bullet \xrightarrow{|\psi_{\bar{a}}\rangle} \bullet \xrightarrow{|\psi_b\rangle} [\Gamma_b] - [\Gamma_b] - \cdots \quad (14)$$

where $|\psi_b\rangle$ and $|\psi_{\bar{a}}\rangle \equiv \langle\psi_a|$ denote the internal states of the quasiparticles, with virtual Hilbert space dimensions d_{Γ_a} and d_{Γ_b} , respectively. We can make a unitary change of basis for the joint internal state of the two quasiparticles, $|\psi_{\bar{a}}\rangle \otimes |\psi_b\rangle$, decomposing into different irreducible fusion channels using the Clebsch-Gordan decomposition of the

tensor product $\Gamma_{\bar{a}} \otimes \Gamma_b = \bigoplus_c N_{\bar{a}b}^c \Gamma_c$ (Appendix E),

$$|\Gamma_{c_n}, \gamma\rangle = \sum_{\alpha, \beta} [\mathcal{C}_{\bar{a}b}^{c_n}]_{\alpha\beta}^\gamma |\Gamma_{\bar{a}}, \alpha\rangle \otimes |\Gamma_b, \beta\rangle, \quad (15)$$

where $n = 1 \dots N_{\bar{a}b}^c$ labels the copies of Γ_c in the direct sum, and the Greek indices label the basis states of each irrep. The Clebsch-Gordan coefficients may be viewed as a unitary matrix $\mathcal{C}_{\bar{a}b}$ with rows indexed by (c_n, γ) and columns labeled by $(\bar{a}, \beta)$. Expanding $|\psi_{\bar{a}}\rangle = \sum_{\bar{\alpha}=1}^{d_{\Gamma_{\bar{a}}}} \psi_{\bar{a}}^{\bar{\alpha}} |\Gamma_{\bar{a}}, \bar{\alpha}\rangle$ and similarly for $|\psi_b\rangle$, the two domain wall state Eq. (14) can then be decomposed as

$$|\psi_{ab}\rangle = \sum_c \sum_{n=1}^{N_{\bar{a}b}^c} \cdots - [\Gamma_a] - [\Gamma_a] - \bullet \xrightarrow{|\psi_{c_n}\rangle} \bullet \xrightarrow{|\psi_b\rangle} [\Gamma_b] - [\Gamma_b] - \cdots \quad (16)$$

where the virtual bond insertions are $d_{\Gamma_a} \times d_{\Gamma_b}$ matrices constructed from the Clebsch-Gordan coefficients which encode the internal state of a Γ_c quasiparticle,

$$\langle \bar{\alpha} | \psi_{\bar{a}} \rangle \bullet \xrightarrow{[\mathcal{C}_{\bar{a}b}^{c_n}]_{\gamma}^{\bar{\alpha}\beta}} \bullet \langle \beta | \psi_b \rangle \equiv \sum_{\gamma} [\mathcal{C}_{\bar{a}b}^{c_n}]_{\gamma}^{\bar{\alpha}\beta} \psi_{\bar{a}}^{\bar{\alpha}} \psi_b^{\beta} |\Gamma_{c_n}, \gamma\rangle, \quad (17)$$

where $\mathcal{C}^{\bar{a}b}$ denotes the inverse of $\mathcal{C}_{\bar{a}b}$. Equation (16) describes a superposition of states with a single Γ_c quasiparticle at the interface between a Γ_a and Γ_b domain, whose internal state depends on the initial states $|\psi_{\bar{a}}\rangle$ and $|\psi_b\rangle$. Thus colliding the Γ_a and Γ_b quasiparticles gives rise to different possible fusion channels, which is a defining property of non-Abelian anyons [70,71]. Notably, if $\Gamma_c \in \Gamma_{\bar{a}} \otimes \Gamma_a$, then a Γ_c quasiparticle can propagate within a Γ_a domain.

We have seen that $\text{Rep}(G)$ SSB ground states, by virtue of being labeled by the objects of a fusion category, behave differently from the familiar invertible SSB and share many properties with SPTs. Notably, a static junction between distinct SPTs protected by invertible symmetries necessarily carries zero modes. Despite not being SPTs in the ordinary sense, certain domain wall interfaces between different $\text{Rep}(G)$ SSB ground states carry zero modes, and so behave like dynamical SPT junctions.

Using Eq. (16) we can also understand the origin of the entanglement spectrum degeneracies. For an SPT protected by an invertible group symmetry, the action of a truncated symmetry operator U_g on the ground state inserts a symmetry defect V_g on the virtual leg at its ends; for Abelian symmetries, if different defects do not commute, $[V_g, V_{g'}] \neq 0$, the representation must be projective, indicating symmetry fractionalization and a protected entanglement spectrum degeneracy [72]. A similar argument can be applied to the entanglement spectrum degeneracies of $\text{Rep}(G)$ SSB states: the defects (17) are inserted at the ends of truncated symmetry operators; focusing on the defects

$C_{c_n}^{aa}$ which can appear in a single $|\Gamma_a\rangle$ ground state, the failure of these defects to commute [despite the commutation of the R_Γ generators of $\text{Rep}(G)$ symmetry] implies degeneracy of the entanglement spectrum. This happens provided the defects commute with the matrix of Schmidt eigenvalues. As shown in Appendix F, in the $\text{Rep}(D_3)$ case the $C_{A_2}^{EE}$ defect in the $|\mathbf{E}\rangle$ ground state is σ^z , while the two C_E^{EE} defects are $\sigma^\pm$, implying twofold degeneracy of the entanglement spectrum of this symmetry-broken ground state.

Outlook—Our Letter connects closely to discrete lattice gauge theories such as 2D quantum double models [62]—non-Abelian generalizations of the $\mathbb{Z}_2$ toric code—which are naturally built from G -qudits. In this context, the Ising-type terms in the Hamiltonian Eq. (6) may be viewed as 1D equivalents of the “star” operators which measure the local electric charge. Taking this perspective, the $\text{Rep}(G)$ SSB ground states are those with zero charge. The symmetry operators R_Γ are the Wilson line operators of the gauge theory—creation operators of electric field lines. The trivial irrep corresponds to the zero-field state, and each state $|\Gamma\rangle = R_\Gamma|\Gamma_0\rangle$ corresponds to a single non-Abelian electric field string running through the system. The domain walls are non-Abelian electric charges. Alternatively, our 1D model may be viewed as describing the gapped edge of a quantum double model [60], closely related to the so-called SymTFT construction [24,30,61,73]. As with quantum double models, our model can be further generalized from finite groups to Hopf algebras [74–78].

Significant effort has been put towards realizing 2D non-Abelian topological order for anyon quantum computation [70,79,80]. A natural intermediate step towards that goal is to realize 1D G -qudit models with noninvertible $\text{Rep}(G)$ symmetry. An implementation using neutral atoms trapped in optical tweezer arrays has been proposed in Ref. [54]. An alternative promising direction is to prepare the fixed-point $\text{Rep}(G)$ symmetry-broken ground states, Eq. (12), and their cat superpositions [38] using existing quantum hardware based on trapped ions of calcium atoms [81] which allows for the implementation of qudits with dimension up to $d = 7$. This can be done by constructing sequential circuits made of two-qudit entangling controlled- X gates [38,51,59]. Moreover, Ref. [82] demonstrates the ability to prepare an SPT state, manipulate its edge modes, and measure local and string order parameters on existing quantum trapped-ion hardware. These methods can be directly transferred to the preparation, manipulation, and diagnostics of the $\text{Rep}(G)$ SSB states investigated here.

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Data availability—The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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- [66] The reader may notice that this phase transition occurs precisely at $h = J$. This is a consequence of the fine-tuned nature of Eq. (6): it is dual to Eq. (5), which can be written in the form $H_G = -J \sum_i \delta(g_i, g_{i+1}) - h \sum_i |+\rangle_i \langle +|_i$, where we define the equal superposition state $|+\rangle = |G|^{-1/2} \sum_g |g\rangle$. This Hamiltonian is actually a $|G|$ -state Potts model, which is self-dual [67] with a first-order transition if $|G| \geq 4$ [68]. Generic Hamiltonians Eqs. (A1) and (A2), can exhibit continuous or discontinuous transitions depending on the parameters.
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End Matter

Appendix A: General form of Hamiltonian—In the main text we presented only a special fine-tuned version of the nearest-neighbor generalized TFIM Hamiltonians for G symmetry, Eq. (5), and $\text{Rep}(G)$ symmetry, Eq. (6). The most general G -symmetric Hamiltonian is

$$H_G = - \sum_i \sum_{\Gamma} \sum_{\alpha, \beta=1}^{d_\Gamma} J_{\alpha\beta}^\Gamma [Z_i^\Gamma \cdot Z_{i+1}^\Gamma]_{\alpha\beta} - \sum_i \sum_{g \in G} h_g \tilde{X}_i^g + \text{H.c.} \quad (\text{A1})$$

with the G symmetry generated by $U_g = \prod_i \tilde{X}_i^g$. So long as all $J_{\alpha\beta}^\Gamma > 0$, the ground state as $h_g \rightarrow 0$ completely breaks the G symmetry, with one ground state for each group element g . The most general $\text{Rep}(G)$ -symmetric Hamiltonian is

$$H_{\text{Rep}(G)} = - \sum_i \sum_{g \in G} J_g \tilde{X}_i^g \tilde{X}_{i+1}^g - \sum_i \sum_{\Gamma} \sum_{\alpha, \beta=1}^{d_\Gamma} h_{\alpha\beta}^\Gamma [Z_i^\Gamma]_{\alpha\beta} + \text{H.c.}, \quad (\text{A2})$$

this completely breaks the $\text{Rep}(G)$ symmetry when $h_{\alpha\beta}^\Gamma \rightarrow 0$ so long as all $J_g > 0$, with one ground state for each irrep Γ on a closed chain, Eq. (11).

Appendix B: Gauging duality—Here, we review how gauging relates a G -symmetric model to a dual $\text{Rep}(G)$ -symmetric model, Eqs. (A1) and (A2). For Abelian groups these are isomorphic, so it is a self-duality called Kramers-Wannier duality, while for non-Abelian groups it relates two distinct models. Here we sketch how this works: starting from the G -symmetric Eq. (A1), we fully gauge the G symmetry, meaning that we minimally couple the ZZ term to a G -valued gauge field on the links and enforce the Gauss law. The Hilbert space of a G gauge field is nothing but a G -qudit, and we denote the corresponding link operators with $\mathcal{Z}$ and $\mathcal{X}$. The minimally coupled gauged Hamiltonian is

$$H_G \rightarrow -\sum_i \sum_{\Gamma, \alpha\beta} J_{\alpha\beta}^\Gamma [\mathcal{Z}_i^\Gamma \cdot \mathcal{Z}_{i+\frac{1}{2}}^\Gamma \cdot \mathcal{Z}_{i+1}^\Gamma]_{\alpha\beta} - \sum_i \sum_g h_g \tilde{X}_i^g + \text{H.c.} \quad (\text{B1})$$

The operators $G_i^g \equiv \tilde{X}_{i-\frac{1}{2}}^g \tilde{X}_i^g \tilde{X}_{i+\frac{1}{2}}^g$ perform gauge transformations, and we enforce the Gauss constraint $G_i^g \equiv 1$. This constraint can be resolved by fixing to unitary gauge, where every original G -qudit on the sites is rotated to the state $|e\rangle$. Acting with $\tilde{X}_i^g$ changes this state to $|g^{-1}\rangle$, which is undone by a G_i^g gauge transformation, meaning that in the gauge-fixed Hilbert space $\tilde{X}_i^g$ acts as $\tilde{X}_{i-\frac{1}{2}}^g \tilde{X}_{i+\frac{1}{2}}^g$. Meanwhile, the $[\mathcal{Z}_i^\Gamma]_{\alpha\beta}$ acting on this gauge-fixed state produce Kronecker deltas $\delta_{\alpha\beta}$. Thus the gauge-fixed Hamiltonian is

$$H_{\text{Rep}(G)} = -\sum_i \sum_{\Gamma, \alpha\beta} J_{\alpha\beta}^\Gamma [\mathcal{Z}_{i+\frac{1}{2}}^\Gamma]_{\alpha\beta} - \sum_i \sum_g h_g \tilde{X}_{i-\frac{1}{2}}^g \tilde{X}_{i+\frac{1}{2}}^g + \text{H.c.} \quad (\text{B2})$$

which is Eq. (A2) after relabeling $h \leftrightarrow J$, and translating by half a site.

Appendix C: Peter-Weyl decomposition of G -qudit Hilbert space—Mathematically, the Peter-Weyl theorem applied to finite groups states that the Hilbert space of a single G -qudit decomposes as [83]

$$\mathcal{H} \cong \bigoplus_\Gamma V_\Gamma \otimes V_\Gamma^*, \quad (\text{C1})$$

where V_Γ is a d_Γ -dimensional vector space and V_Γ^* is its dual. The basis $|\Gamma_{\alpha\beta}\rangle$, Eq. (10), precisely encodes this decomposition, i.e., we may write $|\Gamma_{\alpha\beta}\rangle \equiv |\Gamma_\alpha\rangle \otimes \langle\Gamma_\beta^*|$, where $|\Gamma_\alpha\rangle$ form an orthonormal basis for V_Γ and $\langle\Gamma_\beta^*|$

are the dual basis vectors. The operators $\tilde{X}^g$ act on V_Γ and the operators $\tilde{X}^g$ act on V_Γ^* . When considering the states $|\Gamma_{\alpha\beta}\rangle$ as MPS tensors, the left index labels states $|\Gamma_\alpha\rangle$ in the virtual Hilbert space V_Γ , and the right index labels states $\langle\Gamma_\beta^*|$ in the dual space V_Γ^* . Throughout this text, we denote the dual representation by $\Gamma_a^* \equiv \Gamma_{\bar{a}}$.

Appendix D: Entanglement spectrum degeneracies for generic $\text{Rep}(G)$ symmetric Hamiltonians—In the main text we focused our analysis on the simplest case of $\text{Rep}(G)$ -symmetric Hamiltonians with couplings which are real and uniform. Here, we show how different generic choices do not affect our conclusions regarding the degeneracies of the entanglement spectrum, which is therefore to be attributed entirely to the $\text{Rep}(G)$ invariance and not to other accidental symmetries. In Fig. 3 we present the entanglement spectra for the group D_3 , addressed already in the main text, and for the alternating group A_4 . The latter is a non-Abelian group with 12 elements, three one-dimensional irreps and one three-dimensional irrep. In contrast to Fig. 2, we now consider group-element dependent couplings $J_g = J + \delta J_g$, where the deviations δJ_g are small enough not to modify the symmetry breaking pattern (e.g., they do not introduce “antiferromagnetic” order), but break the symmetry which rotates between the different group elements. δJ_g are chosen to be complex, thus breaking time reversal. In the ordered phase, we target the ground state corresponding to the high-dimensional irreps by choosing an appropriate initial state for the iDMRG simulations. We observe that up until the phase transition point the whole entanglement spectrum of the system exhibits exact d_Γ -fold degeneracies.

Appendix E: Clebsch-Gordan coefficients, fusion, and operator multiplets—The Clebsch-Gordan coefficients $C_{ab}^{c_n}$ of the group G , Eq. (15), play a key role in the fusion of defects and in determining the symmetry

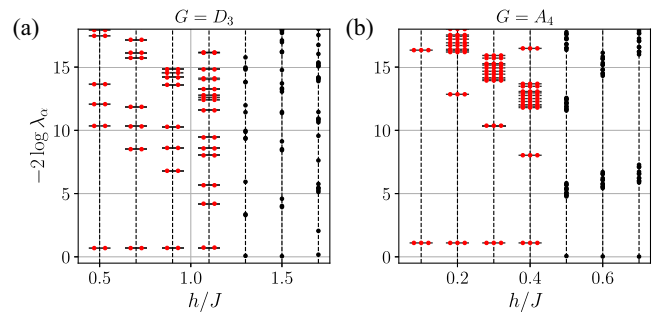


FIG. 3. Entanglement spectra for $\text{Rep}(G)$ symmetric Hamiltonians with generic complex J_g couplings as a function of the “transverse field” h for the groups D_3 (a) and A_4 (b). Both groups have exactly one representation of dimension $d_\Gamma > 1$, with $d_\Gamma = 2$ and 3 , respectively. For small values of h the symmetry broken ground states corresponding to such representations exhibit d_Γ -fold degeneracy of the entanglement spectrum.

multiplets of operators. This is because they encode the intertwiners between representations, i.e., linear maps $\phi_{ab}^{c_n}: V_{\Gamma_a} \otimes V_{\Gamma_b} \rightarrow V_{\Gamma_{c_n}}$,

$$\phi_{ab}^{c_n} = \sum_{\alpha, \beta, \gamma} [\mathcal{C}_{ab}^{c_n}]_{\alpha\beta}^\gamma |\Gamma_{c_n}, \gamma\rangle \otimes \langle \Gamma_a, \alpha| \otimes \langle \Gamma_b, \beta|, \quad (\text{E1})$$

where $|\Gamma_a, \alpha\rangle$ is a basis for V_{Γ_a} . Note that for dual irreps $\Gamma_{\bar{a}}$ acting on $V_{\Gamma_a}^*$, we have the dual basis $|\Gamma_{\bar{a}}, \bar{\alpha}\rangle \equiv \langle \Gamma_a, \alpha|$. We can write the Clebsch-Gordan as a trivalent junction,

$$[\mathcal{C}_{ab}^{c_n}]_{\alpha\beta}^\gamma \equiv \alpha \bullet \begin{array}{c} \gamma \\ \circ \\ \mathcal{C}_{ab}^{c_n} \end{array} \bullet \beta. \quad (\text{E2})$$

These are the matrices that appear on the MPS virtual legs when fusing a and b domain walls, Eq. (17). Since they block-diagonalize the tensor products of representation matrices, they do the same when multiplying Z operators,

$$Z_{\alpha\alpha'}^{\Gamma_a} Z_{\beta\beta'}^{\Gamma_b} = \sum_{\Gamma_c \in \Gamma_a \otimes \Gamma_b} \sum_{n=1}^{N_{ab}^c} \sum_{\gamma, \gamma'=1}^{d_{\Gamma_c}} [\mathcal{C}_{ab}^{c_n}]_{\alpha\beta}^\gamma Z_{\gamma\gamma'}^{\Gamma_c} [\mathcal{C}_{c_n}^{ab}]_{\gamma'\gamma}^{\alpha'\beta'}, \quad (\text{E3})$$

where $\mathcal{C}^{ab}$ is the inverse of the unitary Clebsch-Gordan matrix $\mathcal{C}_{ab}$ with rows indexed by (c_n, γ) and columns indexed by (α, β) . In particular, this tells us how to fuse truncated symmetry operators,

$$\begin{array}{c} \beta \\ \bullet \\ \begin{array}{|c|} \hline Z^b \\ \hline \end{array} \end{array} = \sum_{c \in a \otimes b} \sum_{n=1}^{N_{ab}^c} \begin{array}{c} \begin{array}{|c|} \hline Z^a \\ \hline \end{array} \bullet \begin{array}{c} \gamma \\ \circ \\ \mathcal{C}_{ab}^{c_n} \end{array} \bullet \begin{array}{|c|} \hline Z^{c_n} \\ \hline \end{array} \end{array}. \quad (\text{E4})$$

Given these we can identify the symmetry multiplets of local order parameters of $\text{Rep}(G)$ symmetries. Commuting $\vec{X}^g$ through R_Γ symmetry operators inserts a Γ^g matrix on the virtual leg of the MPO,

$$\begin{array}{c} \begin{array}{|c|} \hline Z^{\Gamma_a} \\ \hline \end{array} \begin{array}{|c|} \hline Z^{\Gamma_a} \\ \hline \end{array} \begin{array}{|c|} \hline Z^{\Gamma_a} \\ \hline \end{array} = \begin{array}{c} \begin{array}{|c|} \hline Z^{\Gamma_a} \\ \hline \end{array} \begin{array}{c} \vec{X}^g \\ \circ \\ \Gamma_a^g \end{array} \begin{array}{|c|} \hline Z^{\Gamma_a} \\ \hline \end{array} \begin{array}{|c|} \hline Z^{\Gamma_a} \\ \hline \end{array} \end{array}. \quad (\text{E5})$$

The virtual leg insertion matrix Γ_a^g can be written as linear combinations of the Clebsch-Gordan coefficient matrices $\mathcal{C}_{ab}^a$, which form a basis for linear maps $V_{\Gamma_a} \rightarrow V_{\Gamma_a}$,

$$[\Gamma_a^g]_{\alpha\alpha'} = \sum_b \sum_{\beta=1}^{d_{\Gamma_b}} [v_{ab}^g]_\beta [\mathcal{C}_{ab}^a]_{\alpha\beta}^{\alpha'}. \quad (\text{E6})$$

We can identify the right-hand side of this equation as $\vec{X}^g$ at the end of the half-infinite Z^{Γ_b} string operator whose open end is contracted with the coefficients $[v_{ab}^g]_\beta$.

Appendix F: Clebsch-Gordan coefficients for D_3 —Here we illustrate the construction for the group D_3 , using the following basis for the E irrep:

$$E^r = \begin{pmatrix} \omega & 0 \\ 0 & \omega^* \end{pmatrix} \equiv \text{Re}(\omega) \mathbb{1} + i \text{Im}(\omega) \sigma^z, \quad E^s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \sigma^x, \quad (\text{F1})$$

and the dual $\bar{E}$ matrices are the complex conjugates of these. In this basis, the matrix of Clebsch-Gordan coefficients encoding the decomposition $E \otimes \bar{E} = A_1 \oplus A_2 \oplus E$ is given by

$$\underbrace{\begin{pmatrix} |A_1, 1\rangle \\ |A_2, 1\rangle \\ |E, 1\rangle \\ |E, 2\rangle \end{pmatrix}}_{|\Gamma_{c_n}, \gamma\rangle} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{[\mathcal{C}_{E\bar{E}}^{c_n}]_{\alpha\beta}^\gamma} \underbrace{\begin{pmatrix} |E, 1\rangle \otimes |\bar{E}, 1\rangle \\ |E, 1\rangle \otimes |\bar{E}, 2\rangle \\ |E, 2\rangle \otimes |\bar{E}, 1\rangle \\ |E, 2\rangle \otimes |\bar{E}, 2\rangle \end{pmatrix}}_{|E, \alpha\rangle \otimes |\bar{E}, \beta\rangle}, \quad (\text{F2})$$

which are given by

$$\begin{aligned} [\mathcal{C}_{E\bar{E}}^{A_1}]_{\alpha\beta}^1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\delta_{\alpha\beta}}{\sqrt{2}}, & [\mathcal{C}_{E\bar{E}}^E]_{\alpha\beta}^1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sigma_{\alpha\beta}^+, \\ [\mathcal{C}_{E\bar{E}}^{A_2}]_{\alpha\beta}^1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\sigma_{\alpha\beta}^z}{\sqrt{2}}, & [\mathcal{C}_{E\bar{E}}^E]_{\alpha\beta}^2 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sigma_{\alpha\beta}^-. \end{aligned} \quad (\text{F3})$$

These form a basis for the domain wall defects, Eq. (17), which can appear in an $|E\rangle$ ground state. Note that the trivial irrep A_1 corresponds to no domain wall, correspondingly it simply gives the normalization matrix Λ from Eq. (11).

To obtain the noninvertible symmetry multiplets such as Eq. (13), the relevant matrices are

$$\begin{aligned} [\mathcal{C}_{EA_1}^E]_{\alpha 1}^\gamma &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta_{\alpha\gamma}, & [\mathcal{C}_{EA_2}^E]_{\alpha 1}^\gamma &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_{\alpha\gamma}^z, \\ [\mathcal{C}_{EE}^E]_{\alpha 1}^\gamma &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sigma_{\alpha\gamma}^+, & [\mathcal{C}_{EE}^E]_{\alpha 2}^\gamma &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sigma_{\alpha\gamma}^-, \end{aligned} \quad (\text{F4})$$

where α labels the rows and γ labels the columns. These matrices form a basis for maps $V_E \rightarrow V_E$, and so can be used to expand the E^g matrices, Eq. (E6). The simple case is

E^r , which using Eq. (F1) can be separated into an identity component and a σ^z component, the first of which is inserted by fusion with an A_1 line and the second of which is inserted by fusion with an A_2 line, thus obtaining Eq. (13). The $E^s = \sigma^x$ case is more complex. Using Eq. (E6) we see that fusing an open E string with the endpoint contracted with $|+\rangle = |1\rangle + |2\rangle$ into the R_E MPO

inserts a σ^x defect, but it also produces an A_1 and A_2 fusion channel. This implies that three operators are mixed in this multiplet—a local $\vec{X}^s$ operator, an $\vec{X}^s$ operator at the end of an E string with endpoint $|+\rangle$, and an $\vec{X}^s$ operator at a junction between an E string with endpoint $|-\rangle$ and an A_2 string. More detailed derivations can be found in Ref. [58].