

Quantum Cramér-Rao Precision Limit of Noisy Continuous Sensing

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Quantum sensors hold considerable promise for precision measurement, yet their capabilities are inherently constrained by environmental noise. A fundamental task in quantum sensing is determining the precision limit of noisy sensor devices. For continuously monitored quantum sensors, characterizing the optimal precision in the presence of environments other than the measurement channel is an outstanding open theoretical challenge, due to the infinite-dimensional nature of the sensor output field and the complex temporal correlation of the photons therein. Here, we establish a numerically efficient method to determine the quantum Cramér-Rao bound for continuously monitored quantum sensors subject to general environmental noise—Markovian or non-Markovian, and showcase its application with paradigmatic models of continuously monitored quantum sensors. Applicable to both constant-parameter and waveform estimation, our method provides a rigorous and practical framework for assessing and enhancing the sensor performance in realistic settings, with broad applications across experimental quantum physics.

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The performance of quantum sensors is fundamentally limited by experimental noise and imperfections; in extreme cases, even minimal noise can negate the anticipated quantum-enhanced precision [1]. A central challenge in quantum sensing is thus to characterize precision limit of sensors operating under noise. Research typically addresses this by establishing precision bounds applicable to various sensing scenarios [2–5], though these bounds may not be guaranteed to be attainable. An alternative, often more practical, approach is to develop methods to evaluate the optimally *attainable* precision for specific (classes of) sensor setups and noise models [6–9]. This approach establishes stringent criteria for evaluating whether a sensor design meets the precision requirements of a target application, and guides experimental development toward optimal performance by suggesting improved design and parameter choices. A widely adopted precision limit in this context is the quantum Cramér-Rao bound (QCRB) [10].

For single-parameter estimation, it specifies the optimally attainable mean squared error (MSE), while for

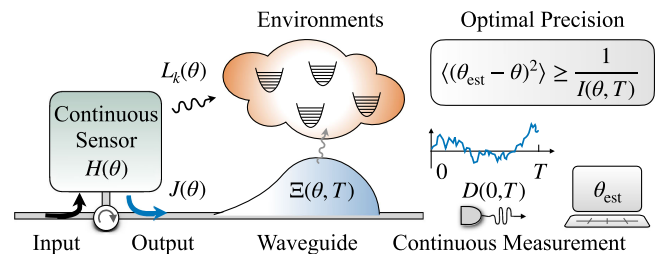


FIG. 1. Noisy continuous sensors and their precision limit. The sensor is an open quantum-optical device whose dynamics depend on the unknown parameter θ . The sensor is driven by an input quantum field (e.g., waveguide photons), with its output field subject to continuous measurement spanning a time window $[0, T]$, yielding a record $D(0, T)$ from which θ can be estimated. The optimal precision of unbiased estimation is quantified by the QFI $I(\theta, T)$ of the output-field quantum state $\Xi(\theta, T)$. The sensor and waveguide may couple with surrounding environments, resulting in decoherence that fundamentally constrains $I(\theta, T)$. A generic noisy continuous sensor is characterized by its Hamiltonian $H(\theta)$ and jump operator $J(\theta)$ [$L_k(\theta)$] responsible for the sensor-waveguide (sensor-environment) coupling. We establish a unified framework for evaluating the QFI for arbitrarily driven continuous sensors subjected to general (including non-Markovian) environmental noise, which further accommodates waveform estimation scenarios where the constant parameter θ is replaced by a function $\vartheta(t)$, $t \in [0, T]$.

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multiparameter estimation, e.g., waveform estimation [11–13], it provides a rigorous lower bound on the MSE. For unbiased estimation, the QCRB is expressed via the inverse of the quantum Fisher information (QFI) [14].

A broad class of important sensor technology relies on continuous measurement [15–20], wherein an open quantum sensor is driven by an input quantum field (e.g., a laser) and continuously monitored through detection of the output field, as illustrated in Fig. 1. Prominent examples range from gravitational-wave detectors [21,22], via atomic gas magnetometers [23,24], to driven-dissipative interacting many-body sensors [25–29]. These setups are inherently noisy, as both the open sensor and the quantum field may decohere via coupling with surrounding environments—e.g., in the form of dephasing, emission into inaccessible modes, or field attenuation. As only the designated output field is detected, the QCRB of continuous sensors can be rigorously quantified by the QFI of the monitored output field *in the presence* of environmental noise. However, the photons of the output field populate infinitely many photonic modes and possess intricate correlation both among themselves and with the external environments, posing an outstanding challenge for QCRB evaluation. Although loose precision bound may be obtained from judicious purification [30], evaluating the QFI of the output field plus all environments [31] or additionally including the sensor [11,32,33], they do not provide stringent calibration of the sensor performance under noise and hence offer limited utility for practical applications.

In this Letter, we establish a numerically efficient method for evaluating the QCRB of generic continuously monitored quantum sensors subject to arbitrary (including non-Markovian) environmental noise. Our method applies to arbitrary sensor dynamics, including non-Gaussian and time-dependent cases, with any input quantum field. Furthermore, it naturally extends to evaluating the QCRB for noisy waveform estimation [11–13].

The key insight underlying our method is that, although the output field formally contains infinitely many photonic frequency modes, its quantum state can be efficiently represented as continuous matrix product operators (cMPOs) [34] in the infinitesimal time-bin basis. The structure of this cMPO is determined entirely by the open dynamics of the sensor. This correspondence, remarkably, allows for the efficient evaluation of the output-field QFI by formulating and solving dynamical equations solely in terms of (replicas of) the open sensor variables, eliminating the need for explicit numerical discretization and manipulation of the output-field cMPO—a process that typically requires low excitation density and may suffer from numerical instability [35]. As such, our approach offers a similar level of clarity as methods [31,32] developed for the noiseless scenario. We term our approach *generalized replica master equations* (GRMEs) and develop numerically efficient method for solving them. Beyond quantum

sensing, the GRMEs established here extend to evaluating general quantum informational properties that depend nonlinearly on the output-field density operator, which we address in a follow-up article [36].

Quantum optical model—We model the sensor as a quantum-optical open system coupled with a unidirectional one-dimensional photonic field (hereafter referred to as the waveguide) as its input-output channel, with associated annihilation (creation) operators $f(\omega)$ [$f^\dagger(\omega)$] for photons at frequency ω . Following standard quantum-optical treatment [16–20,37,38], we consider photons in a bandwidth $\mathcal{B}$ around the mean optical frequency $\bar{\omega}$, and introduce the associated quantum noise operator ($\hbar = 1$ hereafter)

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{B}} d\omega f(\omega) e^{-i(\omega - \bar{\omega})t}$$

satisfying the white noise commutation relation $[f(t), f^\dagger(t')] = \delta(t - t')$. Decoherence sources can be modeled conveniently via factorization into a series of unidirectional one-dimensional bosonic environments interacting with the sensor [39]. We temporarily assume the environments are Markovian, and later extend our analysis to the non-Markovian case. We can therefore introduce quantum noise operators $b_k(t), b_k^\dagger(t)$ for each environment $k = 1, 2, \dots, K$ with the white noise condition $[b_k(t), b_j^\dagger(t')] = \delta_{kj} \delta(t - t')$. Aiming to sense a general parameter θ , the global evolution of the sensor, waveguide, and all environments in the interaction picture is hence described by a Stratonovich quantum stochastic Schrödinger equation [16–20], $i(d/dt)|\Psi\rangle = H_{\text{Glob}}|\Psi\rangle$, with

$$H_{\text{Glob}} = H(\theta) + i \left[J(\theta) f^\dagger(t) + \sum_k L_k(\theta) b_k^\dagger(t) - \text{H.c.} \right]. \quad (1)$$

Here H is the sensor Hamiltonian, J and L_k are the sensor jump operator (with jump rate absorbed into their definitions) that couple to the waveguide and the k th environment, respectively, and H.c. denotes the Hermitian conjugate. The extension to sensing a waveform $\vartheta(t)$ is straightforward, see End Matter.

The quantum evolution according to Eq. (1) formally corresponds to the canonical scheme for generating continuous matrix-product states (cMPSs) [40,41]. To illustrate this, let us assume for simplicity factorizable initial state with both the waveguide and environments in vacuum, $|\Psi(0)\rangle = |\psi_S(0)\rangle \otimes |\text{vac}\rangle_{\text{W,E}}$ (hereafter the subscripts S, W, E represent the sensor, waveguide, and environments, respectively). This allows us to transform Eq. (1) from the Stratonovich into the Itô form [16–20] and to formally write down its solution as a multifield cMPS [34,40,41]

$$|\Psi(\theta, T)\rangle = \mathcal{T} \exp \left\{ \int_0^T dt [Q(\theta) + J(\theta) f^\dagger(t) + \sum_k L_k(\theta) b_k^\dagger(t)] \right\} |\psi_S(0)\rangle \otimes |\text{vac}\rangle_{\text{W,E}}$$

with $\mathcal{T}$ denoting time ordering and $Q := -iH - (J^\dagger J + \sum_k L_k^\dagger L_k)/2$. Tracing away the waveguide and environments provides the reduced density operator of the sensor, $\rho(\theta, T) := \text{tr}_{\text{W,E}}[|\Psi(\theta, T)\rangle\langle\Psi(\theta, T)|]$, whose evolution follows a Lindblad master equation (LME)

$$\dot{\rho} = J(\theta)\rho J^\dagger(\theta) + \mathcal{L}_0(\theta)\rho, \quad (2a)$$

$$\mathcal{L}_0(\theta)\rho := -i[H(\theta), \rho] - \frac{1}{2}\{J^\dagger(\theta)J(\theta), \rho\} + \sum_k \mathcal{D}[L_k(\theta)]\rho, \quad (2b)$$

wherein $\{\cdot, \cdot\}$ is the anticommutator and $\mathcal{D}[L]\rho := L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}$ denotes the Lindblad superoperator. The LME (2a) is written in a physically intuitive form facilitating later use, with $J\rho J^\dagger$ representing emission events (quantum jumps) into the waveguide and $\mathcal{L}_0\rho$ the evolution conditioned on the absence of such jumps.

Continuous sensing relies on measuring the output photons in the waveguide, the quantum state thereof via tracing away the sensor and all environments from the global state, $\Xi(\theta, T) := \text{tr}_{\text{S,E}}[|\Psi(\theta, T)\rangle\langle\Psi(\theta, T)|]$, leading formally to a cMPO

$$\Xi(\theta, T) = \text{tr}_S \left\{ \mathcal{T} e^{\int_0^T dt [\mathcal{L}_0(\theta) + \mathcal{J}(\theta)\mathcal{G}(t) + \mathcal{K}(\theta)\mathcal{F}(t)]} \rho(0) \right\} \Xi(0). \quad (3)$$

Here $\mathcal{J}, \mathcal{K}$ are (potentially θ -dependent) superoperators of the sensor, $\mathcal{J}(\rho) := J\rho$, $\mathcal{K}(\rho) := \rho J^\dagger$; $\mathcal{G}(t), \mathcal{F}(t)$ are superoperators of the waveguide, $\mathcal{G}(t)(\Xi) := f^\dagger(t)\Xi$, $\mathcal{F}(t)(\Xi) := \Xi f(t)$. While Eq. (3) was derived assuming a specific initial global state $|\Psi(0)\rangle = |\psi_S(0)\rangle \otimes |\text{vac}\rangle_{\text{W,E}}$, it remains valid for an arbitrary sensor initial state $\rho(0)$, provided that the waveguide input is a vacuum (or coherent) state, $\Xi(0) = |\text{vac}\rangle_{\text{W}}\langle\text{vac}|$, and the environments are Markovian and Gaussian. Typical environments include vacuum environments, where $\mathcal{L}_0$ follows Eq. (2b), and thermal environments, where $\mathcal{L}_0$ is modified by replacing $\mathcal{D}[L_k]$ with $(n_k + 1)\mathcal{D}[L_k] + n_k\mathcal{D}[L_k^\dagger]$ in Eq. (2b), with n_k the thermal occupation.

Quantum Fisher information and Bargmann invariants of the waveguide—Below we develop an efficient method to evaluate the QFI, $I(\theta, T)$, of the waveguide density operator Eq. (3). For unbiased estimation of θ , the QFI quantifies the minimal MSE [42], $\min[(\theta_{\text{est}} - \theta)^2] = 1/[NI(\theta, T)]$, optimized over $N \gg 1$ repetitions of arbitrary (including adaptive) measurement of the sensor output photons for the time window $[0, T]$. The QFI can be

formally expressed as $I(\theta, T) = \sum_{\lambda+\lambda'>0} [2/(\lambda+\lambda')] |\langle\lambda|\partial_\theta\Xi(\theta, T)|\lambda'\rangle|^2$ [10,14], where $\{\lambda, |\lambda\rangle\}$ forms the eigen-system of the waveguide density operator Eq. (3), $\Xi(\theta, T) = \sum_\lambda \lambda |\lambda\rangle\langle\lambda|$. Such eigendecomposition, however, is infeasible to compute due to the continuous, infinite-dimensional nature of Eq. (3).

Instead, our method is based on relating the QFI to nonlinear powers of the waveguide density operator known as the Bargmann invariants [43],

$$B(\Theta, T) := \text{tr}[\Xi(\theta_1, T)\Xi(\theta_2, T)\dots\Xi(\theta_{m+2}, T)], \quad (4)$$

with $\Theta := \{\theta_1, \theta_2, \dots, \theta_{m+2}\}$ and $m \geq 0$. We show in [44] that $I(\theta, T)$ is the limit of a series $\{I_n(\theta, T) | n \in \mathbb{N}_0\}$, which increases monotonically and converges exponentially as $I(\theta, T) - I_n(\theta, T) = \mathcal{O}(\xi^n)$, with $\xi \equiv 1 - \min[\lambda]/\sqrt{\sum_\lambda \lambda^2}$. The elements of the series can be conveniently expressed in terms of the Bargmann invariants Eq. (4) as

$$I_n(\theta, T) = \sum_{m=0}^n (-1)^m \binom{n+1}{m+1} \frac{f_m(\theta, T)}{\Lambda^{m+1}(\theta, T)} \quad (5a)$$

$$f_m(\theta, T) = \sum_{l=0}^m D_l^m \partial_{\mu,\nu}^2 \text{tr}[\Xi(\mu, T)^{l+1} \Xi(\nu, T)^{m-l+1}]|_{\mu=\nu=\theta}, \quad (5b)$$

where $\Lambda(\theta, T) = 2\{\text{tr}[\Xi^2(\theta, T)]\}^{1/2}$, and $D_l^m = 2\binom{m}{l} - \binom{m}{l+1} - \binom{m}{l-1}$ with the convention $\binom{m}{l} = 0$ for $l < 0$ or $l > m$. Remarkably, although being nonlinear functionals of the waveguide density operator, the Bargmann invariants Eq. (4) can be efficiently evaluated via extension of theoretical techniques of open quantum systems, as we establish below. Efficient evaluation of Eq. (4) hence unlocks, via finite difference approximation of the derivatives in Eqs. (5a) and (5b), the access to the QFI.

Generalized replica master equation—As a key result of the present Letter, the waveguide Bargmann invariants Eq. (4) can be evaluated systematically from the *reduced dynamics* of the sensor alone, via

$$B(\Theta, T) = \text{tr} \varrho(\Theta, T),$$

where $\varrho(\Theta, T)$ is an operator acting on $m+2$ replicas of the sensor. It satisfies the initial condition $\varrho(\Theta, 0) = \otimes_{\alpha=1}^{m+2} \rho^{(\alpha)}(0)$, with $\rho(0)$ the sensor initial density operator, and evolves according to [44]

$$\begin{aligned} \dot{\varrho}(\Theta, t) = & \sum_{\alpha=1}^{m+2} \mathcal{L}_0^{(\alpha)}(\theta_\alpha) \varrho(\Theta, t) \\ & + \sum_{\alpha=1}^{m+2} J^{(\alpha+1)}(\theta_{\alpha+1}) \varrho(\Theta, t) J^{(\alpha)\dagger}(\theta_\alpha), \end{aligned} \quad (6)$$

with operators with superscript (α) acting on $\rho^{(\alpha)}$, and periodic boundary condition imposed on the replica index, $(m+3) \equiv (1)$. Equation (6) is not a LME and does not describe a completely positive trace-preserving map; hereafter we call it a $(m+2)$ GRME. Its first line describes independent evolution for each replica according to Eq. (2b), i.e., in the absence of emission into the waveguide. The second line of Eq. (6) describes collective emission events (quantum jumps) into the waveguide between adjacent replicas, which build up inter-replica correlations. Consequently, $\varrho(\Theta, t)$ becomes nonfactorizable with respect to the replicas for $t > 0$.

Direct numerical propagation of Eq. (6), although implementable for a small replica number, quickly becomes infeasible with increasing m —the storage cost being $\mathcal{O}(D^{2m+4})$ (with D the sensor Hilbert space dimension), and the propagation cost (i.e., the number of multiplications) $\mathcal{O}(TD^{4m+8})$. Instead, we develop a numerically efficient, scalable approach to integrate Eq. (6) based on the time-evolving block decimation (TEBD) algorithm. We Trotterize the infinitesimal-time propagator of Eq. (6) into $m+2$ gates acting on adjacent replicas α and $\alpha+1$,

$$\begin{aligned} \varrho(\Theta, t+dt) &= \prod_{\alpha=1}^{m+2} V_{\alpha, \alpha+1}(dt) \varrho(\Theta, t) + \mathcal{O}(dt^2), \\ V_{\alpha, \alpha+1}(dt) \varrho &= \exp \left\{ J^{(\alpha+1)}(\theta_{\alpha+1}) \varrho J^{(\alpha)\dagger}(\theta_{\alpha}) dt \right. \\ &\quad \left. + \frac{dt}{2} [\mathcal{L}_0^{(\alpha)}(\theta_{\alpha}) + \mathcal{L}_0^{(\alpha+1)}(\theta_{\alpha+1})] \varrho \right\}. \end{aligned} \quad (7)$$

These gates can be applied efficiently to the replica chain based on matrix-product representation of $\varrho(\Theta, t)$ across the chain [44], leading to a significant reduction of the storage cost to $\mathcal{O}[(m+2)\chi^2 D^2]$, and the computational cost to $\mathcal{O}[(m+2)T\chi^3 D^6]$, with χ the bond dimension of the matrix-product representation. Efficient propagation is achieved whenever χ can be kept small.

Crucially, the inter-replica correlation in Eqs. (6) and (7) is established solely via the collective quantum jumps $\propto J^{(\alpha+1)} \varrho J^{(\alpha)\dagger}$ which, remarkably, also suppress entanglement growth by quenching the excitation of the replicas involved. As detailed in [44], we observe that this dissipative nature leads to an *area law* of the bipartite operator entanglement entropy across the replica chain for all propagation time T , in sharp contrast to the volume law typically established by unitary entangling gates [52,53]. Consequently, TEBD can be performed for the GRME (6) with a small χ over sufficiently large m and T , enabling the efficiency and scalability of our method [54].

Illustration—As a first example, we apply our method to a typical lossy quantum optical setup: a driven sensor coupled to two photonic channels, f and b, with photon detection limited to channel f. This models, for instance, a

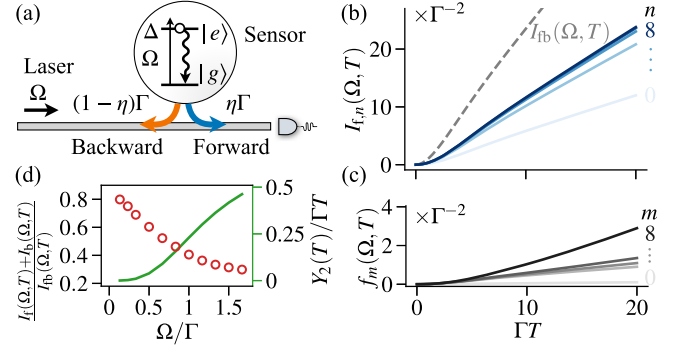


FIG. 2. Illustration of the method. (a) The sensor is an emitter driven at Rabi-frequency Ω and detuning Δ , and interfaced bidirectionally with a waveguide with a forward/backward emission rate $\eta\Gamma/(1-\eta)\Gamma$. (b) The approximate QFI of forward-emitted photons for Rabi-frequency estimation, $I_{f,n}(\Omega, T)$, for $n = 0, 2, 4, 6, 8$ (light to dark), showing excellent convergence at $n = 8$. Dashed curve shows $I_{fb}(\Omega, T)$, the joint QFI of forward and backward emitted photons, evaluated using the method of [31]. Parameters: $\Delta = \Omega = \Gamma/3$, $\eta = 2/3$. (c) The Bargmann derivatives [see Eq. (5b)] for forward-emitted photons at $m = 0, 2, 4, 6, 8$ (light to dark), evaluated at the same parameters as (b). (d) The forward and backward emitted photons become increasingly entangled as Ω increases, as reflected by the unit-time Renyi mutual information $Y_2(T)/T$. Consequently, the QFI retrievable via independent measurements of forward and backward emitted photons, $I_f(\Omega, T) + I_b(\Omega, T)$, becomes increasingly small compared to the joint QFI $I_{fb}(\Omega, T)$. Parameters: $\Delta = \Gamma/3$, $\eta = 2/3$.

nonchiral emitter-waveguide interface, where f(b) represents forward (backward) propagating modes, or atom-laser scattering, with f(b) corresponding to laser (free-space) modes. Consider a driven two-level sensor (TLS) with excited (ground) state $|e(g)\rangle$, see Fig. 2(a), whose Hamiltonian in the rotating frame reads $H = -\Delta\sigma_{ee} + (\Omega\sigma_{eg} + \text{H.c.})/2$, with $\Delta(\Omega)$ the laser detuning (Rabi frequency) and $\sigma_{ij} := |i\rangle\langle j|$. The sensor couples to the channel f and b via the jump operator $J_f = \sqrt{\eta}\Gamma\sigma_{ge}$ and $J_b = \sqrt{(1-\eta)\Gamma}\sigma_{ge}$ respectively, with $\eta \leq 1$ the photon collection efficiency and Γ the total emission rate.

Consider sensing the Rabi frequency Ω through forward photon detection. Figure 2(c) shows the corresponding $f_m(\Omega, T)$ computed from Eq. (5b) via integrating the GRME (6), assuming $\rho(0) = |g\rangle\langle g|$. From this, the approximate QFI $I_{f,n}(\Omega, T)$ is obtained via Eq. (5a), which exhibits rapid convergence with increasing n , achieving accurate estimate of $I_f(\Omega, T)$ at $n \simeq 8$, cf. Fig. 2(b). The QFI displays an initial transient behavior for $T \lesssim 1/\Gamma$, and transitions to linear growth with T for $T \gg 1/\Gamma$, i.e., as the sensor steady state is reached.

The QFI of the backward emitted photons, $I_b(\Omega, T)$, can be computed similarly. The sum of $I_f(\Omega, T)$ and $I_b(\Omega, T)$ represents the optimal precision obtainable from

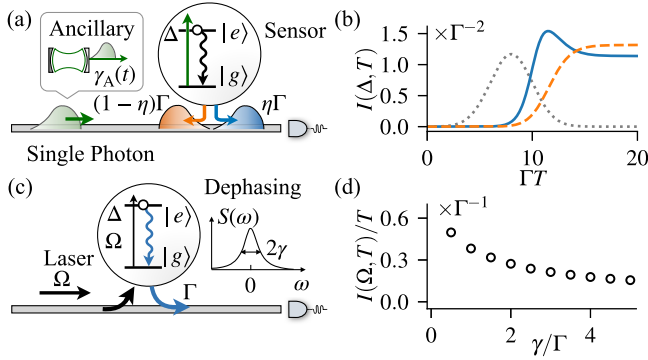


FIG. 3. Applications of the extended method. (a) A bidirectional sensor-waveguide interface with a continuous-mode single-photon input. The input field can be generated via an ancillary cavity mode (see text). (b) The temporal shape of the input photon (dotted, grey) $u(t) = \exp[-(t - t_0)^2/2\tau^2]/\sqrt{\tau\pi}^{1/4}$, and the QFI of the forward (solid, blue) and backward (dashed, orange) propagating fields. Parameters: $t_0 = 8/\Gamma$, $\tau = 1/\Gamma$, $\Delta = \Gamma/4$, $\eta = 1/2$, the sensor is initialized in $|g\rangle$. (c) A unidirectional sensor-waveguide interface, with the sensor subjected to non-Markovian dephasing. The environment has a Lorentzian noise spectrum $S(\omega) = \gamma^2/(\omega^2 + \gamma^2)$ with the spectral width γ adjustable. (d) The QFI of the emission field as a function of γ . Parameters: $\Delta = 0$, $\Omega = \Gamma$, the sensor is initialized in $|g\rangle$.

independent measurements of both channels. We can however regard f and b as a single channel and compute the corresponding QFI $I_{fb}(\Omega, T)$, which quantifies the optimal precision obtainable from any *joint* measurement of both. In Fig. 2(d) we show their ratio, $[I_f(\Omega, T) + I_b(\Omega, T)]/I_{fb}(\Omega, T)$, for $T \gg 1/\Gamma$, which monotonously decreases with increasing Ω , indicating growing entanglement between the two fields. Such entanglement can be quantified by the interfield Renyi mutual information, $Y_\ell(T) := S_{\ell,f}(T) + S_{\ell,b}(T) - S_{\ell,fb}(T)$, with $S_{\ell,i}(T) = \log_\ell[\text{tr}(\Xi_i^\ell)]/(1 - \ell)$ the ℓ th order Renyi entropy of field $i \in \{f, b\}$, which is readily computable via the Bargmann invariants Eq. (4). For $T \gg 1/\Gamma$, $Y_{\ell=2}(T)/T$ is shown in Fig. 2(d).

Extension—Our method naturally applies to sensors under time-varying driving, by allowing $\mathcal{L}_0(\theta, t)$, $J(\theta, t)$ in Eq. (6) to vary with time. Furthermore, although the GRME (6) is formulated for sensors driven by a coherent input field, our method is readily extendable to accommodate arbitrary input quantum field—by introducing an ancillary open system upstream of the sensor, see Fig. 3(a), and designing the input field of the sensor through controlling the dynamical parameters, i.e., the Hamiltonian $H_A(t)$ and jump operator $J_A(t)$, of the ancilla [55]. Correspondingly, the QFI of the sensor output field and the associated Bargmann invariants can be evaluated with Eq. (6) for the cascaded ancilla-sensor setup, as constructed via replacements $\mathcal{L}_0(\theta)q \rightarrow \mathcal{L}_0(\theta)q - i[H_A(t) + H_{\text{casc}}(t), q]$ and $J(\theta) \rightarrow J(\theta) + J_A(t)$, with $H_{\text{casc}}(t) = (i/2)[J(\theta)J_A^\dagger(t) - J_A(t)J^\dagger(\theta)]$.

As an illustration, we consider the bidirectional TLS-waveguide interface as before, with the waveguide input consisting of a continuous-mode single-photon wave packet $|\mathbf{1}_u\rangle = \int dt u(t) f^\dagger(t) |\text{vac}\rangle_W$, as relevant to lossy quantum light spectroscopy experiments [56,57]. The input photon can be generated with an upstream ancillary cavity mode ($a, a^\dagger$) initialized in the first Fock state, $\rho_A(0) = |1\rangle_A\langle 1|$, with $H_A(t) = 0$ and $J_A(t) = \sqrt{\gamma_A(t)}a$, wherein $\gamma_A(t)$ is related to the photonic temporal profile as $\gamma_A(t) = u^2(t)/[1 - \int_0^t d\tau u^2(\tau)]$ [55]. Evolving the cascaded GRME provides us with the QFI of both forward and backward propagating output fields, as shown in Fig. 3(b).

Our framework further extends to continuous sensors subject to non-Markovian noise by making use of the pseudomode representation [58–60] of structured environments. The representation replaces the original structured environment by a small number of damped bosonic modes interacting with the sensor, whose parameters are chosen to reproduce the expectation values and correlation functions of the original environment. We show in [44] that the QCRB under non-Markovian noise can be evaluated by propagating the GRME (6) for the composite system consisting of the open sensor and the pseudomodes. As an illustrative application of this theoretical extension, Figs. 3(c) and 3(d) present the QFI of the output field of a unidirectional TLS-waveguide interface, with the sensor dephased by an environment with a Lorentzian noise spectrum [44]. Finally, although we have focused on estimating a single constant parameter, our framework readily extends to waveform estimation; see End Matter.

Conclusions—We have established a framework for efficient evaluation of the QCRB of continuously monitored quantum sensors subjected to general noise. Our framework extends further to computing general properties of quantum-optical input-output channels that are nonlinear in the field density operator [36]. Future directions include extension to noisy hypothesis testing, and designing optimal measurement to saturate the QCRB. A promising approach for the latter is the recently established universal quantum noise cancellation strategy [31,61,62], for which our present work provides a rigorous benchmark for assessing the optimality under realistic noise and imperfections.

Part of the numerical simulations were performed using the QUTIP library [63].

Note added—While completing the manuscript, we became aware of a related work [64] on variational calculation of the precision limit of lossy quantum light spectroscopy. Both manuscripts have been synchronized to appear on arXiv on the same date.

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Data availability—The data that support the findings of this article are openly available [65]. Example codes are available from the authors upon reasonable request.

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End Matter

Lossy QCRB: Linear vs nonlinear sensors—Consider the scenario in Fig. 2(a), where a sensor is driven by a coherent input, and emits photons into the accessible channel (labeled as f) and the inaccessible channel

(labeled as b) at a rate $\eta\Gamma$ and $(1 - \eta)\Gamma$, respectively. An intriguing and practically relevant question is, how does the QFI of the accessible photons, $I_f(\theta, T)$, depend on the efficiency η ? As a crude guess, one might estimate

$I_f(\theta, T)$ as $\eta I_{fb}(\theta, T)$, where $I_{fb}(\theta, T)$ is the joint QFI of the accessible and inaccessible photons. Leveraging our numerical framework, here we elucidate this question by quantitative computation of model systems, providing useful insights for real-world sensor platforms.

Specifically, focusing on the estimation of the driving strength Ω (see definition below), we obtain the following results: (i) $\eta I_{fb}(\theta, T)$ is indeed a tight bound, i.e., $I_f(\theta, T) = \eta I_{fb}(\theta, T)$, for lossy *linear* amplifiers (as excellent models, e.g., of optomechanical force sensors [66] and ensemble magnetometers [23]); (ii) For nonlinear continuous sensors, such as the lossy TLS studied in the main text (for which Ω becomes the Rabi frequency), $\eta I_{fb}(\theta, T)$ is not an accurate measure of the retrievable QFI. Interestingly enough, for the lossy TLS we observe that it serves as a loose *upper bound* to the QFI, $I_f(\theta, T) \leq \eta I_{fb}(\theta, T)$.

A linear amplifier is a continuous variable system with a pair of conjugate quadratures $X = (a^\dagger + a)/\sqrt{2}$ and $P = i(a^\dagger - a)/\sqrt{2}$ satisfying $[a, a^\dagger] = 1$. These can, e.g., be the position and momentum operators of a mechanical oscillator, or be the collective internal spins along two orthogonal directions of an atomic ensemble [23]. The unknown force Ω couples with X , resulting in the sensor Hamiltonian

$$H = \omega a^\dagger a - \Omega(a + a^\dagger), \quad (\text{A1})$$

with ω the oscillation frequency.

The sensor further couples to the detectable and undetectable photonic channels via the jump operator $\sqrt{\eta}\Gamma X$ and $\sqrt{(1-\eta)}\Gamma X$, respectively. For an optomechanical force sensor [66], such a coupling is mediated by a driven damped cavity mode $(c, c^\dagger)$, which interacts with the oscillator via the linearized optomechanical coupling $V = g(c^\dagger + c)X$. The cavity damping rate is typically much larger than g , allowing us to adiabatically eliminate the cavity mode hence to arrive at the aforementioned

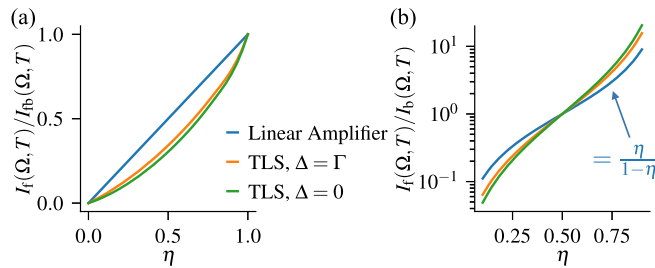


FIG. 4. Comparison between the QCRB for a linear amplifier and a TLS, assuming finite photon collection efficiency $\eta < 1$. (a) The QFI carried by the accessible photons, $I_f(\Omega, T)$, normalized by the total QFI of the emitted photons $I_{fb}(\Omega, T)$, for estimating the parameter Ω with each sensors. In the numerical calculation we choose for the amplifier $\omega = \Omega = \Gamma/3$, and for the TLS $\Omega = \Gamma/3$ and Δ adjustable. (b) The ratio between the retrievable and lost information, $I_f(\Omega, T)/I_b(\Omega, T)$, as a function of η , for both the linear amplifier and the TLS. Parameters are the same as panel (a).

quadrature jump operators. For an atomic gas magnetometer, such coupling is enabled by the Faraday rotation [23,24].

Using our numerical method, we evaluate the retrievable information for sensing the driving strength Ω , $I_f(\Omega, T)$, for various detection efficiency η , for both the lossy linear amplifier [cf. Eq. (A1)] and the lossy TLS (cf. the main text). As demonstrated in Fig. 4(a), $I_f(\Omega, T)$ precisely matches $\eta I_{fb}(\Omega, T)$ for lossy linear amplifiers, confirming its tightness; in contrast, we observe that $\eta I_{fb}(\Omega, T)$ serves only as a loose upper bound for the QFI of the lossy TLS—it becomes tight only for $\eta = 0$ or 1. While Fig. 4 is shown for specific parameters, we have checked numerically that the observed behavior persists for other parameters as well.

We can similarly verify, for linear amplifiers, that the QFI carried by the inaccessible photons is $I_b(\Omega, T) = (1 - \eta)I_{fb}(\Omega, T)$. In Fig. 4(b), we show the ratio between the retrievable and lost information, $I_f(\Omega, T)/I_b(\Omega, T)$, for both a linear amplifier and a TLS. The linear-amplifier ratio agrees exactly with $\eta/(1 - \eta)$; the TLS does not.

Our numerics lead us to conjecture that $\eta I_{fb}(\theta, T)$ is not an accurate quantifier of the precision for sensing the coherent driving strength with generic nonlinear continuous sensors [25–29], where the accessible and inaccessible photons develop intricate non-Gaussian entanglement. Whether it provides a loose upper bound for the QFI, as in the TLS case, is an interesting open question for future study.

Extension to waveform estimation—Our framework extends naturally to evaluating the QCRB for waveform estimation in the presence of noise. Therein, an open quantum sensor is subject to a time-varying waveform $\vartheta(t)$, $t \in [0, T]$, and monitored continuously. An estimator waveform $\vartheta_{\text{est}}(t)$ is then constructed from the continuously recorded signals $D(0, T)$ —a task involving joint estimation of infinitely many parameters. To make the estimation problem well defined, a prior distribution functional $P[\vartheta]$ (throughout this section, bracketed expressions of the form $f[\cdot]$ denote functionals) reflecting known constraints on the waveform distribution (e.g., its finite bandwidth) is essential [11].

Consider unbiased estimation of $\vartheta(t)$. The QCRB for this case is expressed in terms of the matrix inequality

$$\Sigma(t, t') - \mathcal{I}^{-1}(t, t') \geq 0, \quad (\text{B1})$$

where $\Sigma(t, t')$ is the weighted error matrix [11],

$$\Sigma(t, t') := \int \mathfrak{D}[\vartheta, D(0, T)] P[\vartheta] P[D(0, T) | \vartheta] \times (\vartheta_{\text{est}}(t) - \vartheta(t)) (\vartheta_{\text{est}}(t') - \vartheta(t'))$$

with $\int \mathfrak{D}[\vartheta, D(0, T)]$ a (functional) integral over all possible waveforms $\vartheta(t)$ and records $D(0, T)$, $\mathcal{I}$ the QFI matrix and $\mathcal{I}^{-1}$ its inverse [defined via

$\int_0^T ds \mathcal{I}(t, s) \mathcal{I}^{-1}(s, t') = \delta(t, t')$. In particular, Eq. (B1) implies bounds to the point MSE for unbiased estimators,

$$\langle [\vartheta_{\text{est}}(t) - \vartheta(t)]^2 \rangle \geq \mathcal{I}^{-1}(t, t). \quad (\text{B2})$$

Reference [11] shows that $\mathcal{I}$ is given by a sum of quantum and classical contributions, $\mathcal{I} = \mathcal{I}^{(Q)} + \mathcal{I}^{(C)}$, with

$$\mathcal{I}^{(C)}(t, t') = \int \mathfrak{D}[\vartheta] P[\vartheta] \frac{\delta P[\vartheta]}{\delta \vartheta(t)} \frac{\delta P[\vartheta]}{\delta \vartheta(t')} \quad (\text{B3})$$

the Fisher information matrix of the prior distribution $P[\vartheta]$, and $\delta/\delta \vartheta(t)$ a functional derivative. The other term is the weighted QFI matrix for the waveguide photons,

$$\mathcal{I}^{(Q)}(t, t') = \int \mathfrak{D}[\vartheta] P[\vartheta] \tilde{\mathcal{I}}(t, t'), \quad (\text{B4a})$$

$$\tilde{\mathcal{I}}(t, t') = \frac{1}{2} \sum_{\lambda, \lambda'} (\lambda + \lambda') \text{Re}[\langle \lambda | L_t | \lambda' \rangle \langle \lambda' | L_{t'} | \lambda \rangle], \quad (\text{B4b})$$

where Re denotes the real part, $\{\lambda, |\lambda\rangle\}$ forms the eigen-system of the waveguide density operator $\Xi[\vartheta(t)]$, and the symmetric logarithmic derivative operator L_t is defined as

$$\frac{\delta \Xi[\vartheta(t)]}{\delta \vartheta(t)} = \frac{1}{2} (L_t \Xi + \Xi L_t). \quad (\text{B5})$$

Evaluation of the QCRB Eq. (B1) hence reduces to the efficient calculation of $\tilde{\mathcal{I}}(t, t')$, based on the knowledge of the sensor dynamics and the noise model. Remarkably, this can be achieved via direct extension of our method.

Specifically, as we derive in [44], $\tilde{\mathcal{I}}(t, t')$ can be evaluated as the limit of an exponentially converging matrix series $\{\tilde{\mathcal{I}}_n | n \in \mathbb{N}_0\}$, with

$$\tilde{\mathcal{I}}_n(t, t') = \sum_{m=0}^n (-1)^m \binom{n+1}{m+1} \frac{f_m(t, t')}{\Lambda^{m+1}[\vartheta]}, \quad (\text{B6a})$$

$$f_m(t, t') = \sum_{l=0}^m D_l^m \frac{\delta^2}{\delta \mu(t) \delta \nu(t')} \text{tr}(\Xi[\mu]^{l+1} \Xi[\nu]^{m-l+1})|_{\mu=\nu=\vartheta}. \quad (\text{B6b})$$

Here, $\Lambda[\vartheta] := 2\{\text{tr} \Xi^2[\vartheta]\}^{1/2}$ and D_l^m is defined in the main text [below Eq. (5b)]. The functional derivatives in Eq. (B6b) can be implemented numerically and robustly via the finite element method [67]. All the elements in Eqs. (B6a) and (B6b) are hence accessible via the waveform-parametrized Bargmann invariants,

$$B[\tilde{\Theta}] := \text{tr}\{\Xi[\vartheta_1] \Xi[\vartheta_2] \dots \Xi[\vartheta_{m+2}]\} \equiv \text{tr}\{\varrho[\tilde{\Theta}]\}, \quad (\text{B7})$$

with $\tilde{\Theta} := \{\vartheta_1, \vartheta_2, \dots, \vartheta_{m+2}\}$ being a set of $m+2$ waveforms, and $\varrho[\tilde{\Theta}]$ evolving according to the waveform-parametrized GRME

$$\begin{aligned} \dot{\varrho}[\tilde{\Theta}] &= \sum_{\alpha=1}^{m+2} \mathcal{L}_0^{(\alpha)}[\vartheta_\alpha] \varrho[\tilde{\Theta}] \\ &+ \sum_{\alpha=1}^{m+2} J^{(\alpha+1)}[\vartheta_{\alpha+1}] \varrho[\tilde{\Theta}] J^{(\alpha)\dagger}[\vartheta_\alpha]. \end{aligned} \quad (\text{B8})$$

The above extension provides a general framework for evaluating the noisy precision limit for waveform estimation [11–13]. Toward these goals, a crucial next step is to develop a numerically robust, cost-efficient recipe that combines the Bargmann invariant evaluation with functional integration and differentiation [as required by Eqs. (B4a) and (B6b)]. We leave these prospects for future study.