Thermal Operations from Informational Equilibrium

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Thermal operations are quantum channels that play a central role in deriving thermodynamic limitations in quantum systems. However, they were originally defined by implementation procedures rather than by fundamental principles. Alternative models of thermal processes have been proposed, but they obscure the resources required for implementation. Here, we identify the universal principle that separates these models. We show that thermal operations are uniquely characterized by a purely quantum information-theoretic property, which we call informational equilibrium. This formulation does not assume a particular Hamiltonian or even a notion of temperature: a channel is a thermal operation precisely when it leaves its environment invariant whenever the system is prepared in equilibrium. In this way, the usual energetic description of equilibrium (i.e., Gibbs states) emerges as a representation of an underlying informational constraint. Extending this framework, we note that enforcing environment invariance for all system inputs leads to catalytic channels, which idealize perfectly reusable heat baths. This informational equilibrium perspective culminates in a hierarchy for doubly-stochastic quantum channels, with strict separations between the classes. Our refined hierarchy sharpens the failure of the Birkhoff–von Neumann theorem in the quantum regime, and highlights the richer structural landscape of quantum channels compared to classical stochastic processes that cannot be captured by state-convertibility relations.

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Introduction—Thermodynamics is one of the most successful frameworks in physics. However, its extension to quantum systems admits several inequivalent notions, whose precise differences are not always transparent. This is best illustrated by two competing models for thermal processes. *Thermal operations* [1–3] are defined based on global energy-preserving interaction with a thermal bath, while Gibbs-preserving maps [4–6] are processes that leave the Gibbs state invariant. Although the two classes are known to be distinct [5,7,8], identifying the physical principle that divides them has been a major problem in quantum thermodynamics.

In this Letter, we identify this missing principle as a purely information-theoretic formulation of equilibrium and equilibration: the environment must remain invariant under the equilibrating dynamics whenever the system starts in equilibrium. Information-theoretic approaches have proven exceptionally powerful for elucidating physical concepts, by providing operational interpretations of entropy and entanglement. Furthermore, this approach yields a universal principle by not assuming energetic structures such as a Hamiltonian or temperature at the outset. This result also yields a direct operational consequence: any Gibbs-preserving map that is not a thermal operation necessarily drives the environment out of equilibrium, regardless of how the process is implemented.

The concept of environmental equilibrium is closely related to catalysis [9]. These auxiliary systems play a fundamental role in activating thermodynamic transitions and achieving reversibility for quantum thermodynamics [10–14]. Our information-theoretic formulation identifies catalytic channels as those that preserve environmental equilibrium, placing them in a natural thermodynamic setting as idealized heat baths that remain unchanged after interaction. In particular, we show that thermal operations already capture all robust catalytic advantages [15]: any further catalytic enhancement necessarily relies on fragile catalysts. In other words, thermal operations act as a boundary distinguishing between robust and nonrobust catalytic advantage.

Finally, we show that catalytic channels produce an operationally meaningful strict hierarchy, given in terms of

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the behavior of the environment system in implementation, within the set of doubly stochastic quantum channels. This hierarchy is foundational to our understanding of quantum noise and the limits of environmentally-assisted error correction [16,17]. Since doubly stochastic quantum channels are a special case of Gibbs-preserving maps in the infinite temperature limit, our results suggest that analogous hierarchies should arise for general Gibbs-preserving maps at arbitrary Hamiltonians and temperatures, where connections between thermal operations and recovery maps have also been explored [18,19].

Equilibration and thermal operations—Conventionally, the equilibrium state of a (quantum) thermodynamical system is defined as the state with the lowest free energy, also known as the Gibbs state $\gamma = e^{-\beta H}/Z$, where β and H are the inverse temperature and the Hamiltonian of the system with $Z = \mathrm{Tr}[e^{-\beta H}]$ being the partition function.

However, there have been multiple notions of processes driving states to the Gibbs state. One of them is *thermal operations*, quantum channels that can be implemented through an energy-preserving unitary interaction with a thermal bath so that they can be expressed in the form of

$$T(\rho_A) = \mathrm{Tr}_B\left(U\rho_A \otimes \frac{e^{-\beta H_B}}{Z_B} U^\dagger\right) \tag{1}$$

with unitary U satisfying $[U, H_A + H_B] = 0$ [1,20] for local Hamiltonians $H_{A/B}$. This class provides a coherent framework for resource-theoretic studies of thermodynamics [21,22]. On the other hand, there are *Gibbs-preserving maps*, which form the largest class of thermal processes, defined as any quantum channel S that fixes the Gibbs state, i.e., $S(\gamma) = \gamma$. While the former enjoy straightforward operational meaning due to their constructive definition, they are harder to theoretically manipulate as they lack a simple and clear characterization. In contrast, the latter have a very simple definition so that they provide immediate upper bounds for achievability of many thermal processes, although their implementability is not always clear.

It is known that thermal operations, Gibbs-preserving maps, and many other classes in between such as enhanced thermal operations [23,24] are distinct sets. Most notably, a Gibbs-preserving map can generate quantum coherence, whereas none of thermal operations can [5]. However, pinpointing the difference in their physical characterizations that is independent of a specific choice of energetics such as Hamiltonian or temperature has been a major problem in quantum thermodynamics. Therefore, in this section, we choose the information theoretic approach to this problem in which we deal with quantum states and their transformations without further assuming additional energetic structures. If we can successfully recover the familiar thermodynamics structures in this setting, it would imply that such structures can be better understood from the informational viewpoint.

We begin with establishing the notion of informational equilibrium. While the notion of equilibrium is often associated with thermodynamics, it is possible to define equilibrium purely information theoretically without presupposing thermodynamical concepts. In a closed quantum system, dynamics (for a fixed time) is described by a unitary operator U. A state (density matrix) ω is said to be stationary with respect to U if $U\omega U^\dagger = \omega$. When the system consists of two subsystems A and B, we can say that two local states ω_A and ω_B are in (informational) equilibrium relative to U if

$$\mathrm{Tr}_B(\sigma_{AB}) = \omega_A, \qquad \mathrm{Tr}_A(\sigma_{AB}) = \omega_B, \tag{2}$$

where $\sigma_{AB} = U\omega_A \otimes \omega_B U^\dagger$ is the time-evolved state.

Starting from this very simple definition of informational equilibrium, we recover familiar properties of thermal equilibrium one by one. First, in the following Lemma 1, we show that two initially uncorrelated systems can be in equilibrium only when their composite system is stationary with respect to that interaction.

Lemma 1 (Informational zeroth law)—Suppose ω_A and ω_B are in equilibrium relative to U. Then

$$U\omega_A \otimes \omega_B U^\dagger = \omega_A \otimes \omega_B. \tag{3}$$

This lemma has an implication that informational equilibrium is transitive, just like thermodynamic equilibrium is transitive by the zeroth law of thermodynamics. More specifically, if ω_A and ω_B are in equilibrium relative to U_{AB} and ω_B and ω_C relative to U_{BC}, the combined state $\omega_A \otimes \omega_B \otimes \omega_C$ are in equilibrium relative to any U which is multiplicatively generated by $(U_{AB} \otimes 1_C)^t$ and $(1_A \otimes U_{BC})^s$ for $t, s \in \mathbb{R}$. See End Matter for further discussion on the thermodynamic interpretation of Lemma 1. Furthermore, Lemma 7 in the End Matter generalizes Lemma 1 for a generic unitary operator U that preserves the marginal states of all subsystems.

Now we give an informational definition of equilibration. Suppose that ω_A and ω_B are in equilibrium under U, but system A is prepared in a state ρ_A that may differ from ω_A. The effective dynamics on A is described by

$$T(\rho_A) = \mathrm{Tr}_B(U\rho_A \otimes \omega_B U^\dagger). \tag{4}$$

Since (ω_A, ω_B) are in equilibrium, it follows that ω_A is a fixed-point of T, i.e., $T(\omega_A) = \omega_A$. This implies that T can only drive system A closer to its equilibrium state ω_A. Hence, we can interpret T as an open-system equilibration dynamics. We formally define such equilibration dynamics as follows. Throughout this Letter, we only consider finite-dimensional Hilbert spaces.

Definition 2 (Equilibrating dilation)—Consider a quantum channel T on A, and a fixed-point ω_A of T. We say that T has an *equilibrating dilation* with respect to ω_A if there exists a dilation (U, ω_B) of T (i.e., $T(\rho_A)$ is given as Eq. (4) for any ρ_A) such that

$$\mathrm{Tr}_A(U\omega_A \otimes \omega_B U^\dagger) = \omega_B. \tag{5}$$

The dilation is called *nondegenerate* if ω_B has a nondegenerate spectrum. A channel with an equilibrating dilation is called an equilibrating channel. The dilation is called *full rank* if ω_B is full rank.

Note that this definition does not presuppose any energetic relation between systems and only concerns informational equilibration, i.e., stationarity of local quantum states, between two systems. Not every quantum channel has an equilibrating dilation, and only when the channel has an equilibrating dilation (U, ω_B) do the dynamics admit an equilibrium between ω_A and ω_B. Equilibrating channels enjoy a set of useful properties analogous to those of thermal operations as we study in Proposition 6 in End Matter, which yields one of our main results.

Theorem 3—The class of all equilibrating channels that admit a full-rank equilibrating dilation coincides with the class of all thermal operations.

Note that the full-rank assumption is very mild as they are generic (in the sense they form a measure-1 set) in the set of quantum states. This result aligns with the intuitive expectation that if a system starts in equilibrium, it not only remains stationary but also in equilibrium with the environment that remains in its own equilibrium. Conversely, Theorem 3 has a significant implication for implementability of thermal processes, as this means that if T is a quantum channel on A that is supposed to model a thermal process on A but is not a thermal operation, then *for any dilation of the channel T, the environment must change its state, even when A is already in a thermal equilibrium state.* In other words, for every dilation (U, ω_B) of the channel T we must have

$$\mathrm{Tr}_A\left(U\frac{e^{-\beta H_A}}{Z_A} \otimes \omega_B U^\dagger\right) \neq \omega_B. \tag{6}$$

Importantly, Eq. (6) implies that any such dilation *must* contain nonequilibrium resources. This interpretation fits well with the recent findings that quantum channels with a thermal fixed-point—but which are not thermal operations—may require an infinite amount of coherence for their implementation [25]. Our results also apply to enhanced thermal operations [23,24], i.e., Gibbs-preserving maps that are covariant under time translation. Thanks to the time-translation covariance, they admit a dilation consisting of an energy-preserving U satisfying $[U, H_A + H_B] = 0$, and an environment state ω_B such that $[\omega_B, H_B] = 0$ [26]. Hence, unlike general quantum channels with a thermal fixed point, no coherence is needed for the implementation. Yet, it is known that some enhanced thermal operations are not thermal operations [27]. In this case, the LHS of Eq. (6) can be interpreted as describing a thermal operation $\rho_B \mapsto \mathrm{Tr}_A[U(e^{-\beta H_A}/Z_A) \otimes \rho_B U^\dagger]$ acting on B. However, Eq. (6) indicates that ω_B cannot be a thermal state of B at inverse temperature β. It must therefore contain nonequilibrium free energy, which may be depleted after the operation.

Catalytic channels—Equilibration with a heat bath is often regarded as cost-free because the state of the bath is usually assumed to remain invariant throughout the equilibration and thus be recyclable. However, this invariance is often only approximate, justified by the vast size of the bath, and ignoring the resulting accumulated change over repeated use may lead to embezzlement [28] and thereby trivialize resource constraints. Nevertheless, it has recently been shown that some aspects of thermalization, such as randomization [29], are indeed possible without affecting the bath. It is therefore natural to ask which kinds of equilibrating processes remain possible under this stricter notion of invariance. Since such a bath facilitates the process while remaining unchanged, just like a catalyst in quantum resource theories [9,13,30,31], we characterize it as a *catalyst* and introduce the following definition:

Definition 4 (Catalytic dilation)—A quantum channel T on A has a *catalytic dilation* (U, ω_B) if

$$\mathrm{Tr}_B(U\rho_A \otimes \omega_B U^\dagger) = T(\rho_A), \tag{7}$$

$$\mathrm{Tr}_A(U\rho_A \otimes \omega_B U^\dagger) = \omega_B, \tag{8}$$

for all states ρ_A on A. The dilation is called *nondegenerate* if ω_B has a nondegenerate spectrum. A channel with a catalytic dilation is called a *catalytic quantum channel*.

We have compiled useful properties and alternative characterization of catalytic channels in Sec. I of Supplemental Material [32]. The concept of catalytic channels has been recently extended to the general advantage of having catalytic access to the randomness of the auxiliary system B [29,31,33,34], positioning the set of catalytic channels as an interesting class of its own. For instance, it is shown in [33] that if A is a finite-dimensional quantum system and ω_B has finite entropy, then the catalytic channel T must be *doubly stochastic*, i.e., $T(\mathbf{1}) = \mathbf{1}$. This implies that T can only *increase* the entropy of A, which contrasts with the dynamics expected when a system is in contact with a finite temperature heat bath, where an decrease in free energy does not necessarily accompany an increase in entropy. In the context of thermodynamic analogy, the catalytic channel T only mimics the dynamics resulting from an infinite temperature heat bath.

A simple subclass of catalytic channels is given by *mixed-unitary channels*. They have the form $T(\rho) = \sum_i p_i U_i \rho U_i^\dagger$ with some unitary operators U_i and a probability distribution $(p_i)_i$, and admit a catalytic dilation

(U, ω_B) with

$$U = \sum_i U_i \otimes |i\rangle\langle i|, \qquad \omega_B = \sum_i p_i |i\rangle\langle i|. \tag{9}$$

One can understand mixed-unitary channels as catalytic channels that can be implemented only with a classical randomness source as a catalyst, i.e., a random variable with a probability distribution p_i.

The study of catalytic channels has recently been motivated as the only way to achieve catalytic advantage which is robust in the presence of errors [15]. Typically, we say that ρ_A can be catalytically converted to σ_A by a unitary U if there exists a state ω_B such that

$$\mathrm{Tr}_B(U\rho_A \otimes \omega_B U^\dagger) = \sigma_A, \qquad \mathrm{Tr}_A(U\rho_A \otimes \omega_B U^\dagger) = \omega_B. \tag{10}$$

If ρ_A and σ_A are not unitarily equivalent, such a conversion is possible with a suitable catalyst ω_B if and only if the entropy of σ_A is larger than that of ρ_A [29,35,36]. In general, ω_B is fine-tuned with respect to ρ_A for it to be preserved exactly; however, small perturbations in the initial state ρ_A will lead to small perturbations in the final state on B. Such errors may accumulate in the catalyst upon reuse, eventually degrading it and rendering it useless. It is therefore desirable to have *robust catalysis*, where ω_B must be preserved exactly, at least for small perturbations of ρ_A. However, as observed in [15], if ρ_A has full rank, any tolerance towards a non-zero state preparation error $\epsilon > 0$ immediately implies that ω_B must be preserved exactly for *all* input states due to linearity.

One may wonder how to characterize catalytic channels. In [31] it was shown that these channels are induced by *catalytic unitaries*, which are characterized by the property that their partial transpose $U^{\top_A}$ is still unitary. Catalytic unitaries are in correspondence with *dual-unitary tensors*, a class of operators that has recently attracted significant attention for their role in constructing integrable quantum circuits [37–39]. The correspondence is shown in Sec. II of Supplemental Material [32]. Characterizing catalytic unitaries would therefore also provide a characterization of dual-unitary tensors. We leave this problem for future work.

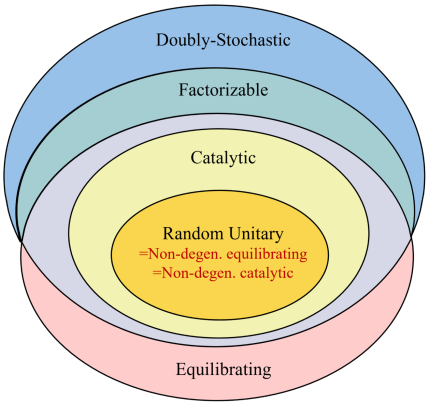

FIG. 1. A hierarchy of doubly stochastic channels, where all inclusions are strict. The strict inclusion of factorizable channels in doubly stochastic channels is shown in [43,45] and relies on the claim that Connes' embedding problem has a negative resolution [46].

Hierarchy of doubly-stochastic channels—The class of catalytic channels prompts a natural question: how do such channels fit into the broader class of doubly-stochastic quantum channels? This question is of profound importance in understanding the difference between quantum and classical theories because, by Birkhoff's theorem [40], a stochastic map in classical probability theory is doubly-stochastic if and only if it is a mixed permutation. In other words, all entropy-increasing classical processes can be implemented by simply applying a randomly chosen deterministic process. Hence, there is no nontrivial structure within doubly-stochastic maps.

On the other hand, if we identify unitaries as the natural generalization of permutations in quantum theory, we do not have a quantum analogue of Birkhoff's theorem for doubly-stochastic quantum channels and indeed some nontrivial subclasses of them have been known [41]. However, our understanding of their hierarchy is far from complete. Studying this hierarchy is important not only because it is a special case of more general Gibbs-preserving maps at infinite temperature or with a trivial Hamiltonian, but also because it may be possible to embed the structure for the nontrivial Hamiltonian case into that of the trivial one through the Gibbs-embedding map [11], thereby enabling the translation of our understanding from the latter to the former.

We consider four subclasses of doubly stochastic (DS) quantum channels, classified according to how the environment behaves in their implementation, ordered such that each class contains the preceding ones: (1) mixed unitary channels (MU), (2) catalytic channels (CAT), (3) doubly stochastic equilibrating channels (EQ $\cap$ DS), and (4) factorizable channels (F).

We assume that the catalytic and equilibrating dilating system is finite-dimensional. We have already introduced the first three classes of channels, and they can be understood as quantum channels that can be implemented with, respectively, classical catalysis, quantum catalysis, and equilibration with an already equilibrated system. Among them, the smallest set (mixed unitary channels) coincides with the other two sets when the environment state of the dilation is nondegenerate. (See End Matter for Proof.) This strengthens the intuition that randomness plays a crucial role in the catalysis of quantum randomness [29,31,34].

Lemma 5—For a doubly stochastic quantum channel T, the following are equivalent: (1) T is a mixed unitary channel, (2) T admits a nondegenerate equilibrating dilation, and (3) T admits a nondegenerate catalytic dilation.

We now discuss factorizable channels. We call a quantum channel *exactly factorizable* if it is of the form

$$T(\rho_A) = \mathrm{Tr}_B\left(U\rho_A \otimes \frac{\mathbf{1}_B}{d_B} U^\dagger\right), \tag{11}$$

and *strongly factorizable* if it is of the form $T(\rho_A) = \mathrm{Tr}_B(U\rho_A \otimes \omega_B U^\dagger)$ with $[U, \mathbf{1}_A \otimes \omega_B] = 0$. Strongly factorizable quantum channels correspond to convex mixtures of exactly factorizable channels and can be approximated arbitrarily well by exactly factorizable ones [41]. It follows immediately from Proposition 6 in End Matter that strongly factorizable quantum channels precisely correspond to doubly stochastic quantum channels that admit an equilibrating dilation:

$$\mathrm{EQ} \cap \mathrm{DS} = \text{strongly factorizable}. \tag{12}$$

From the perspective of the resource theory of thermodynamics [1,11] and informational nonequilibrium [42], exactly factorizable channels can also be seen as *noisy operations*, corresponding to thermal operations where the system Hamiltonian is fully degenerate. Then strongly factorizable maps correspond to a random choice of noisy operation.

Both exactly factorizable and strongly factorizable maps are examples of the more general class of *factorizable* quantum channels [43], where ω_B is a tracial state on an arbitrary finite von Neumann algebra $\mathcal{M}$ and $U \in \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{M}$. It is known that the set of factorizable maps on an n-dimensional Hilbert space coincides with the set of strongly factorizable maps for all n if and only if the *Connes embedding problem* [44] has an affirmative answer [43,45]. The recent MIP* = RE result [46] claims that the Connes embedding problem has a negative answer. If true, then there exists a finite dimension n and a factorizable map on n-dimensional quantum systems that cannot be approximated by exactly factorizable maps [47]. We thus find

$$\mathrm{EQ} \cap \mathrm{DS} \subsetneq \mathrm{F}. \tag{13}$$

Moreover, it has been shown in Ref. [43] that not all doubly stochastic channels are factorizable, giving F $\subsetneq$ DS, which is quite interesting from the physical viewpoint, as they are quantum channels that can only increase the entropy of an input state, however, they require *negentropy*; i.e., they can be implemented only with a nonmaximally mixed environment state.

Next, the strict inclusions MU $\subsetneq$ CAT $\subsetneq$ EQ were previously open problems [31,33] and are now proven in this Letter. The proofs for strict inclusions MU $\subsetneq$ CAT $\subsetneq$ EQ $\cap$ DS are shown in Sec. III of Supplemental Material [32]. We show CAT $\subsetneq$ EQ $\cap$ DS by first showing that catalytic channels that admit a maximally mixed catalytic dilation (hence are exactly factorizable) can be extremal among the doubly stochastic channels only if they are unitary. But the existence of nonunitary, exactly factorizable channels that are extremal among the doubly stochastic maps is known [48]. For MU $\subsetneq$ CAT, we make use of *Schur multiplier* channels, which are defined relative to some fixed basis and act as $T(\rho) = \rho \circ X$, where X is a positive semidefinite matrix with unit diagonal entries and $\circ$ is the Schur product, acting by componentwise multiplication. We show that if X has real entries, then T admits a maximally mixed catalytic dilation. However, there are examples of Schur multipliers with real X which are not mixed unitary channels [43].

In summary, we show that these classes of doubly stochastic quantum channels form a strict hierarchy (Fig. 1),

$$\mathrm{MU} \subsetneq \mathrm{CAT} \subsetneq \mathrm{EQ} \cap \mathrm{DS} \subsetneq \mathrm{F} \subsetneq \mathrm{DS}, \tag{14}$$

wherein all inclusions are strict. This demonstrates that the introduction of catalytic channels enriches this hierarchy of doubly stochastic quantum channels according to requirements on the behavior of the environment system in implementation. As this hierarchy collapses in classical thermodynamics, we can conclude that it deepens our understanding of quantum-exclusive features in quantum thermodynamics.

Conclusions—We start from a simple notion of information theoretic equilibrium, and demonstrate that local Hamiltonians naturally arise, along with the well-studied class of thermal operations. These operations are precisely characterized by their admission of equilibrating dilations, ensuring the absence of nonequilibrium resources. Our findings establish a clear distinction between thermal operations and its supersets of interest, such as enhanced thermal operations or Gibbs-preserving maps. We also use the notion of multipartite equilibrium to show that thermal operations cannot have a robust version of catalytic advantage (Lemma 8, End Matter). This resolves an open question from [15], and is particularly notable because it privileges thermal operations over their subsets of interest. While thermal operations fully capture robust catalytic advantages through the use of passive heat baths/thermal environments, their subsets are more restrictive and thus fail to encompass all such advantages. Specifically, thermal processes such as elementary thermal operations [49] and Markovian thermal operations [50] still require additional use of robust catalysts, in order to expand their achievable set of operations [30,51]. Together, these results underscore the significance of thermal operations, highlighting them as

an operationally orthodox yet sufficiently general framework that naturally encompasses thermalization processes.

Drawing inspiration from the notion of heat baths, we introduced catalytic channels, a special class of doubly stochastic quantum channels. Within this broader class, we examined the relationships between mixed, catalytic, equilibrating, and factorizable quantum channels, demonstrating that they are all distinct. Our results contribute to the broader question of what uniquely distinguishes mixed unitary channels from general doubly stochastic quantum channels, highlighting the rich mathematical structure of doubly stochastic channels, which is absent in the classical regime.

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Data availability—There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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End Matter

On the informational zeroth law—The notion of informational equilibrium defined by Eq. (2) fixes only the local marginals ω_A and ω_B, and therefore does not by itself single out a unique equilibrium state of the composite system AB. *A priori*, any bipartite state σ_{AB} with marginals ω_A and ω_B is compatible with Eq. (2).

The most natural candidate is the product state $\omega_A \otimes \omega_B$, since two systems that have independently relaxed to their local equilibrium states should be uncorrelated at the moment they are brought into contact. More importantly, Lemma 1 shows that this choice is not merely intuitive. If A and B remain in informational equilibrium relative to U_{AB}, then

$$U_{AB}(\omega_A \otimes \omega_B)U_{AB}^{\dagger} = \omega_A \otimes \omega_B,$$

so the joint equilibrium state is in fact fixed by the interaction, and no correlations can be generated between A and B.

Thus informational equilibrium singles out $\omega_A \otimes \omega_B$ as the canonical equilibrium state of the composite system AB, removing the ambiguity left by the marginal condition alone. In this sense, Lemma 1 plays the role of an informational zeroth law: equilibrium between subsystems uniquely determines the state of the joint system. Later, in Lemma 7, we show that this consistency extends to several systems brought into contact simultaneously.

Proof of Lemma 1—The von Neumann entropy $H(\omega) := -\mathrm{Tr}(\omega \log(\omega))$ is unitarily invariant, additive over tensor-products and fulfills $H(\omega_{AB}) = H(\omega_A) + H(\omega_B) - I(A:B)_{\omega}$, where I denotes the mutual information. From unitary invariance, additivity, and the equilibrium condition, we have $H(\omega_A) + H(\omega_B) = H(U\omega_A \otimes \omega_B U^{\dagger}) = H(\omega_A) + H(\omega_B) - I(A:B)_{U\omega_A \otimes \omega_B U^{\dagger}}$. It is well-known that $I(A:B)_{\omega} = 0$ if and only if $\omega_{AB} = \omega_A \otimes \omega_B$. This concludes the proof.

Properties of equilibrating channels—We study some useful properties of equilibrating channels that are analogous to those of thermal operations.

Proposition 6—Let T admit an equilibrating dilation (U, ω_B) with respect to ω_A. Then (1) $[U, \omega_A \otimes \omega_B] = 0$,

(2) for every $t \in \mathbb{R}$ and every state ρ_A on A we have $T(\omega_A^{it}\rho_A\omega_A^{-it}) = \omega_A^{it}T(\rho_A)\omega_A^{-it}$, (3) if ω_A has full rank, ω_B can be chosen to have full rank.

The third statement of Proposition 6 facilitates the thermodynamic interpretation of equilibrating channels. When a system is in thermal equilibrium with a heat bath, its equilibrium state is the Gibbs state $e^{-\beta H}/Z$ corresponding to the system Hamiltonian H and the bath temperature $\beta^{-1} = k_B T$. The Gibbs state always has full rank, and thus conversely any full-rank state ω is a Gibbs state corresponding to some Hamiltonian $\beta H + \log(Z)\mathbf{1} = -\log(\omega)$. Fixing the ambient temperature β^{-1} and the constant offsets $\log(Z_A)$ and $\log(Z_B)$ given in terms of the partition functions Z_A and Z_B for both A and B, their Hamiltonians naturally emerge from the equilibrium states as $H_A \coloneqq -\beta^{-1}[\log(\omega_A) + \log(Z_A)]$ and $H_B \coloneqq -\beta^{-1}[\log(\omega_B) + \log(Z_B)]$. If we arbitrarily fix the inverse temperature for the state ω_A on system A, this fixes the inverse temperature for all equilibrating dilations (U, ω_X) of equilibrating channels with ω_A as a fixed point.

The second statement corroborates this thermodynamic interpretation. Taking ω_A to be the Gibbs state, we have $\omega_A^{it} = e^{-i\beta t H_A}/Z_A^{it}$. Then the channel T is covariant under the time-translation symmetry generated by the Hamiltonian H_A [26,52]. Although H_A is not conserved during the evolution, since system A is open to B, if one considers the closed dynamics of AB, the first statement of the theorem implies that the total Hamiltonian $H_A + H_B$ is conserved.

Proof (Proposition 6)—Item 1 is a restatement of Lemma 1. Item 2 follows from Item 1, since

$$\begin{aligned} T(\omega_A^{it}\rho_A\omega_A^{-it}) &= \mathrm{Tr}_B(U(\omega_A \otimes \omega_B)^{it}(\rho_A \otimes \omega_B)(\omega_A \otimes \omega_B)^{-it}U^\dagger) \\ &= \mathrm{Tr}_B((\omega_A \otimes \omega_B)^{it}U(\rho_A \otimes \omega_B)U^\dagger(\omega_A \otimes \omega_B)^{-it}) \\ &= \omega_A^{it}T(\rho_A)\omega_A^{-it}. \end{aligned} \tag{C1}$$

Since $[U, \omega_A \otimes \omega_B] = 0$, U acts unitarily on the supporting subspace of $\omega_A \otimes \omega_B$. Hence Item 3 follows by restricting the Hilbert space of B to the support of ω_B. ∎

Proof of Lemma 5—The proof for $3 \to 2$ follows from the definition of the catalytic dilation. $1 \to 3$ can be proved using Eq. (9), because we can always split up probabilities into distinct summands and distribute them over more dimensions to make the catalytic dilation nondegenerate. It thus suffices to show $2 \to 1$. Suppose that a doubly stochastic channel T admits a nondegenerate equilibrating dilation (U, ω_B). From Proposition 6 we find $[U, \mathbf{1} \otimes \omega_B] = 0$, and since ω_B is nondegenerate it follows that $[U, \mathbf{1} \otimes |i\rangle\langle i|] = 0$ for all i, where $|i\rangle$ denote the eigenvectors of ω_B to the (all distinct) eigenvalues p_i. Hence $U = \sum_i U_i \otimes |i\rangle\langle i|$ and $T(\rho) = \sum_i p_i U_i \rho U_i^\dagger$, which concludes the proof.

Equilibrium for multiple subsystems—So far, we discussed equilibrium and its variations only in bipartite settings. In this section, we extend this to multi-partite scenarios. Let us start from the tripartite setting

$$\mathrm{Tr}_{\setminus X}(U\omega_A \otimes \omega_B \otimes \omega_C U^\dagger) = \omega_X, \tag{E1}$$

where $\mathrm{Tr}_{\setminus X}$ stands for the partial trace over all subsystems except for X for each $X = A, B, C$.

Lemma 7 (Zeroth law for multi-partite cases)—If $(\omega_A, \omega_B, \omega_C)$ are in equilibrium under U as in Eq. (E1),

$$U\omega_A \otimes \omega_B \otimes \omega_C U^\dagger = \omega_A \otimes \omega_B \otimes \omega_C. \tag{E2}$$

Note that this lemma does not directly follow from Lemma 1, since it is not clear whether this dilation is equilibrating with respect to the partition $AB|C$.

Proof—First define $\omega_{ABC} = U\omega_A \otimes \omega_B \otimes \omega_C U^\dagger$, and $\omega_{AB} = \mathrm{Tr}_C(\omega_{ABC})$. Then, using the entropy argument and the fact that $\mathrm{Tr}_{AB}(\omega_{ABC}) = \omega_C$,

$$\begin{aligned} H(\omega_A) + H(\omega_B) + H(\omega_C) &= H(\omega_{AB}) + H(\omega_C) \\ &\quad - I(AB:C)_{\omega_{ABC}}. \end{aligned} \tag{E3}$$

Further writing $H(\omega_{AB}) = H(\omega_A) + H(\omega_B) - I(A:B)_{\omega_{AB}}$, we obtain $I(A:B)_{\omega_{AB}} = 0$ and $I(AB:C)_{\omega_{ABC}} = 0$. The former implies $\omega_{AB} = \omega_A \otimes \omega_B$, which combined with the latter proves the theorem. ∎

It is clear that this Lemma can immediately be generalized to any multipartite settings.

No robust catalytic advantage in thermal operations—Now, we apply Lemma 7 to settings that extend beyond the catalytic dilations defined with unitaries, as in Definition 4, to incorporate more general quantum channels or processes for the dilation. A natural extension is the *catalytic channel* of the form

$$T(\rho_A) = \mathrm{Tr}_C[\mathcal{E}(\rho_A \otimes \omega_C)], \tag{F1}$$

with the constraint that for any input state ρ_A, the C marginal state $\mathrm{Tr}_A(\mathcal{E}(\rho_A \otimes \omega_C)) = \omega_C$. Similarly to the earlier case of unitary dilation, this condition is equivalent to robust catalysis, where the catalyst remains invariant under small but arbitrary errors in ρ_S [15]. If $\mathcal{E}$

in the dilation can be chosen arbitrarily, a trivial choice $\mathcal{E} = T \otimes \mathrm{id}_C$ with the identity channel id_C can be made. However, within the framework of resource theories, $\mathcal{E}$ must be chosen from a fixed set of free operations $\mathcal{F}$ defined by physical considerations. This restriction then limits the set of catalytic channels. For instance, catalytic channels defined as in Definition 4 are restricted to be a subset of doubly stochastic channels because $\mathcal{E}$ is assumed to be unitary.

In [15], a wide range of resource theories are classified into those that admit catalytic channels outside the set of free operations and those that do not. Yet, the technique used in this classification is limited to sets of free operations that are completely resource nongenerating (CRNG) [53], leaving the robust catalytic advantage for more general free operations as a largely open problem. In this Letter, we show that whenever the set of free operations is defined as those with an equilibrating dilation (a condition that does not imply CRNG), the corresponding catalytic channels also admit equilibrating dilations. In particular, this implies that for thermal operations, robust catalysis does not provide any advantage.

Lemma 8—All robust catalytic thermal operations can be implemented simply with thermal operations without a catalyst.

Proof—Suppose $\mathcal{B}$ is a thermal operation on AC with dilation

$$\mathcal{B}(\varrho_{AC}) = \mathrm{Tr}_B(U\varrho_{AC} \otimes \omega_B U^\dagger), \tag{F2}$$

and the energy-preserving unitary $[U, \omega_A \otimes \omega_C \otimes \omega_B] = 0$ for Gibbs states $\omega_A, \omega_C, \omega_B$. Hence, $\mathcal{B}$ is Gibbs preserving; i.e., $\mathcal{B}(\omega_{AC}) = \omega_{AC}$, where we denote $\omega_{AC} = \omega_A \otimes \omega_C$. Then, a robust catalytic thermal operation is a channel $T(\rho_A) = \mathrm{Tr}_C[\mathcal{B}(\rho_A \otimes \tau_C)]$ such that

$$\mathrm{Tr}_{AB}(U\rho_A \otimes \tau_C \otimes \omega_B U^\dagger) = \tau_C, \tag{F3}$$

with some catalyst state τ_C and for all system state ρ_A.

We first show that the channel T is Gibbs preserving, i.e.,

$$\mathrm{Tr}_{CB}(U\omega_A \otimes \tau_C \otimes \omega_B U^\dagger) = \omega_A. \tag{F4}$$

The monotonicity of the quantum relative entropy gives $D(\varrho_{AC}\|\omega_{AC}) \geq D(\mathcal{B}(\varrho_{AC})\|\omega_{AC})$ for any state ϱ_{AC} and any thermal operation $\mathcal{B}$, where $D(\tau\|\eta) \coloneqq \mathrm{Tr}[\tau(\log\tau - \log\eta)]$ is the quantum relative entropy. Putting $\varrho_{AC} = \omega_A \otimes \tau_C$, we get $D(\omega_A \otimes \tau_C\|\omega_{AC}) = D(\tau_C\|\omega_C)$ for the LHS from $\omega_{AC} = \omega_A \otimes \omega_C$. For the RHS, $D(\mathcal{B}(\omega_A \otimes \tau_C)\|\omega_{AC}) \geq D(T(\omega_A)\|\omega_A) + D(\tau_C\|\omega_C)$, from super-additivity of the quantum relative entropy and Eq. (F3). Since $D(T(\omega_A)\|\omega_A) \geq 0$ with equality if and only if $T(\omega_A) = \omega_A$, the channel T must be Gibbs preserving. Finally, we derive

$$\mathrm{Tr}_{AC}(U\omega_A \otimes \tau_C \otimes \omega_B U^\dagger) = \omega_B, \tag{F5}$$

again using the same argument for Eq. (F4) with $\mathcal{B}$ replaced by $\mathcal{B}'(\varrho_{CB}) = \mathrm{Tr}_A(U\omega_A \otimes \varrho_{CB}U^\dagger)$ and ϱ_{AC} replaced by $\varrho_{CB} = \tau_C \otimes \omega_B$.

Equations (F3)–(F5), when combined, demonstrate that $(\omega_A, \tau_C, \omega_B)$ are in equilibrium under U. Lemma 7 then implies $[U, \omega_A \otimes \tau_C \otimes \omega_B] = 0$. Restricting to the supporting subspace of τ_C and interpreting τ_C as a thermal state of some Hamiltonian, the channel T has the usual dilation of a thermal operation

$$T(\rho_A) = \mathrm{Tr}_{CB}(U\rho_A \otimes \omega_{CB}U^\dagger), \tag{F6}$$

with the Gibbs state $\omega_{CB} = \tau_C \otimes \omega_B$. ∎