

Long-Term Stable Nonlinear Evolutions of Ultracompact Black-Hole Mimickers

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We study the stability of ultracompact boson stars admitting light rings combining a perturbative analysis with 3 + 1 numerical-relativity simulations with and without symmetry assumptions. We observe excellent agreement between all perturbative and numerical results, which uniformly support the hypothesis that this family of black-hole mimickers is separated into stable and unstable branches by extremal-mass configurations. This separation includes, in particular, thin-shell boson stars with light rings located on the stable branch, which we conclude to represent long-term stable black-hole mimickers.

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Introduction—Black-hole solutions in closed analytic form have played a crucial role in many aspects of the theoretical understanding of general relativity (GR) almost right from the moment Einstein published his theory. Starting with observations of x-ray sources (e.g., Cygnus X-1 [1]) and quasars (e.g., 3C 273 [2,3]) and their eventual identification with accretion onto compact objects, black holes (BHs) acquired an equally prominent role as seemingly common citizens of our Universe. The rapidly increasing number of gravitational wave (GW) detections by LIGO and Virgo [4,5], all compatible with theoretical predictions for BHs and neutron stars in GR [6], and direct observations of BH environments by the Event Horizon Telescope [7,8], have further cemented the prominence of BHs in observational astronomy.

In both mathematical modeling as well as the popular imagination, the most characteristic feature of a BH is its event horizon, the 2 + 1 dimensional outer boundary of the set of spacetime points from which no null geodesics can reach future null infinity. In terms of their observational phenomenology, however, the decisive feature of BHs is their (unstable) light rings (LRs), circular null geodesics on which photons can orbit; for a

Schwarzschild BH of mass M , the event horizon is located at areal radius $r = 2GM/c^2$ and the LR at $r = 3GM/c^2$. The appearance of accreting BHs is strongly influenced by the presence and properties of the LR, while quasinormal modes have been shown to have a close connection with null unstable geodesics [9–14]. The observational evidence therefore points more directly at the existence of LRs, not necessarily BHs. Additionally, classical black holes come with theoretical issues, like the singularity problem [15] or the information paradox [16], motivating the study of BH mimickers. Even if one accepts Occam’s razor and parsimoniously interprets BHs as the most likely sources, there remains the intriguing possibility that a population of ultracompact objects (UCOs) is hiding inside a more conventional BH society and may provide us with clues about the dark matter candidate fields they are composed of. We also note that the GW emission from compact boson star (BS) binaries exhibits significant degeneracy with that from BH binaries [17].

Clearly, dynamical longevity is a key requirement of any viable alternative to the BH paradigm. The stability of UCOs or *BH mimickers*—both implied henceforth to denote objects with a LR but no horizon—has, consequently, gained much attention in recent years. Under a broad set of assumptions, BH mimickers have been shown to possess two LRs, one stable and one unstable, located at the minimum and maximum, respectively, of the null geodesic’s potential [18,19]; Kerr BHs, in contrast have unstable LRs only. The existence of a minimum in the effective potential leads to *stable trapping*, i.e., null geodesics originating close to the minimum may remain indefinitely bound to it. At linear order, waves propagating on a background spacetime with stable trapping exhibit

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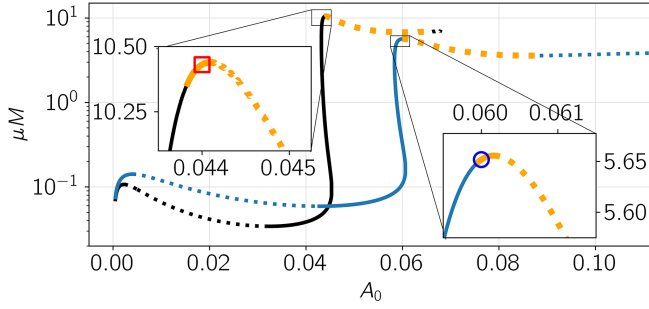


FIG. 1. The ADM mass is shown versus the central scalar-field amplitude A_0 for the one-parameter families of BS models with solitonic potential parameter $\sigma_0 = 0.06$ (black) and $\sigma_0 = 0.08$ (blue). For each family, solid (dashed) segments mark perturbatively stable (unstable) branches, and we highlight in orange color BSs exhibiting a pair of LR. The insets indicate the two perturbatively stable BH mimickers with parameters detailed in Table I that form the focus of our numerical investigation.

decay slower than polynomial or exponential, as occurs for unstable trapping [20]. Guided by results on the instability of anti-de Sitter spacetime and its similar decay rates [21–23], it has been heuristically argued that UCOs may also be nonlinearly unstable, thus ruling them out as plausible BH mimickers [24]. In the absence of rigorous analytic results, however, numerical evolutions are the only available tool for testing this hypothesis.

To date, literature results investigating the ultimate fate and stability of UCOs are inconclusive. By simulating two families of ultracompact BSs, Ref. [25] finds either migration to nonultracompact configurations or collapse to a BH, typically on timescales $\lesssim \mathcal{O}(10^3)\mu^{-1}$ in units of the scalar mass parameter and concludes that the LR trigger instability. Reference [26] explains the instability as the result of the backreaction of perturbations at the stable LR appearing to deepen the geodesic potential. In contrast, simulations of the merger and ringdown of ultracompact spinning BSs yield no evidence of this instability [27], although the timescales the simulations span in that work are considerably shorter. Further simulations in spherical symmetry of scalarized Reissner-Nordström BHs with LR also exhibited long-term stability [28]. In a recent study using a toy-model spacetime with two LR, Ref. [29] analyzes the differences between linear and nonlinear dynamics in more detail: within the region of stable trapping, the slow linear decay leads to a growth of higher-order derivatives and energy in the nonlinear evolution, thereby resulting in turbulent behavior. This phenomenon is proven in the perturbative regime [30], where the authors argue how it may not necessarily trigger an instability. The ultimate end state of these configurations, however, remains an open question.

Here, we explore the stability of a class of spherically symmetric ultracompact BSs with LR characterized by *thin-shell* profiles [31] and reaching compactness $C \sim 0.32$. We combine a set of numerical simulations in spherical

symmetry (1 + 1), axisymmetry (2 + 1), and full 3 + 1 dimensions with perturbative calculations of the radial oscillation frequency, observing excellent agreement between all approaches compatible with the standard result that stable and unstable branches of stars, including UCOs with LR, are separated by local extrema in the mass-radius diagram. In particular, we observe no signs of instability on the stable branch containing BH mimickers, in agreement with the restabilization, i.e., real frequency, of the fundamental radial oscillation mode predicted in this regime by our perturbative calculations. Henceforth, we employ Planck units $c = G = \hbar = 1$ but keep the scalar-field mass parameter μ , which controls the scale invariance of the spacetimes in question. Code units are translated into SI (or similar) by fixing μ and scaling with appropriate powers thereof: e.g., $\mu = 1.33 \times 10^{-10}$ eV corresponds to $\mu^{-1} = M_\odot = 1.48$ km.

Theory and numerical codes—We consider BSs in GR composed of one minimally coupled, complex scalar field φ , with solitonic potential, described by the action

$$S = \int \sqrt{-g} \left\{ \frac{R}{16\pi} - \frac{1}{2} [g^{\mu\nu} \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + V(\varphi)] \right\} d^4x, \quad (1)$$

$$V(\varphi) = \mu^2 |\varphi|^2 \left(1 - 2 \frac{|\varphi|^2}{\sigma_0^2} \right)^2, \quad \sigma_0 = \text{const.}$$

Varying S results in the Einstein-Klein-Gordon equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \quad \nabla^\mu \nabla_\mu \varphi = \frac{dV}{d\varphi},$$

$$T_{\mu\nu} = \frac{1}{2} \nabla_{(\mu} \bar{\varphi} \nabla_{\nu)} \varphi - \frac{g_{\mu\nu}}{2} [g^{\alpha\beta} \nabla_\alpha \bar{\varphi} \nabla_\beta \varphi + V(\varphi)]. \quad (2)$$

We compute stationary BS models in spherical symmetry, writing $\varphi(t, r) = A(r)e^{i\omega t}$ and using a GR version with quadruple precision of the two-way shooting code of Ref. [32]. The resulting single-BS spacetimes are evolved in time using four different codes in spherical symmetry, axisymmetry, or full 3D using the BSSN [33,34] and the conformal covariant Z4 (CCZ4) [35] formulations. (i) A SPHERICALBSEVOLVER achieves spherically symmetric time evolutions through dimensional reduction with SO(3) symmetry according to Ref. [36] using a uniform radial grid. (ii) A 2D code implements axisymmetry using the same modified cartoon method but for SO(2) isometry. It is based on (iii) ExoZvezda [37], a code for exotic matter built on GRChombo [38,39], which evolves 3D domains with adaptive mesh refinement provided by Chombo [40]. (iv) LEAN [41] is based on the CACTUS computational toolkit [42] and evolves 3D domains with mesh refinement provided by Carpet [43]. We will refer to these codes and their respective simulations with the acronyms Sph, Axi, E3D, and L3D. Unless specified otherwise, our simulations employ fourth-order discretization, a resolution $h = 1/(12\mu)$ with outer boundary at $256\mu^{-1}$ on a unigrid (Sph) or using five refinement levels (Axi, E3D, L3D).

TABLE I. Nickname, solitonic parameter σ_0 , central scalar-field amplitude A_0 , ADM mass M , Noether charge $\mathcal{N}$, the radius r_{99} encompassing 99% of the total mass, compactness $C := M/r_{99}$, and fundamental radial oscillation frequency χ_0^2 for the two BS models marked by square and circle in Fig. 1.

Model	σ_0	A_0	μM	$\mu^2 \mathcal{N}$	μr_{99}	C	χ_0^2/μ^2
S06A044	0.06	0.044	10.43	65.9	31.7	0.33	7.52×10^{-5}
S08A06	0.08	0.06	5.65	26.1	17.9	0.31	2.68×10^{-5}

Time evolutions—Our Letter focuses on BS families for two values of the solitonic parameter, $\sigma_0 = 0.08$ and $\sigma_0 = 0.06$; the resulting BS models are displayed in the plane spanned by the central scalar amplitude A_0 and the Arnowitt-Deser-Misner (ADM) mass M in Fig. 1. There, we mark in orange color the ultracompact BSs possessing a pair of LRs; for both σ_0 values, this set of UCOs extends to the left, beyond the global maximum of the $M(A_0)$ curve, i.e., into the radially stable branch. This regime comprises the stars of interest in our Letter, and we mark for each potential one representative specimen with properties as summarized in Table I.

We begin our assessment of these two models by plotting in Fig. 2 the deviation ΔA_0 of the central scalar-field amplitude from its initial value over time as obtained with different symmetry assumptions. For the S06A044 model, we observe periodic oscillations with a relative magnitude

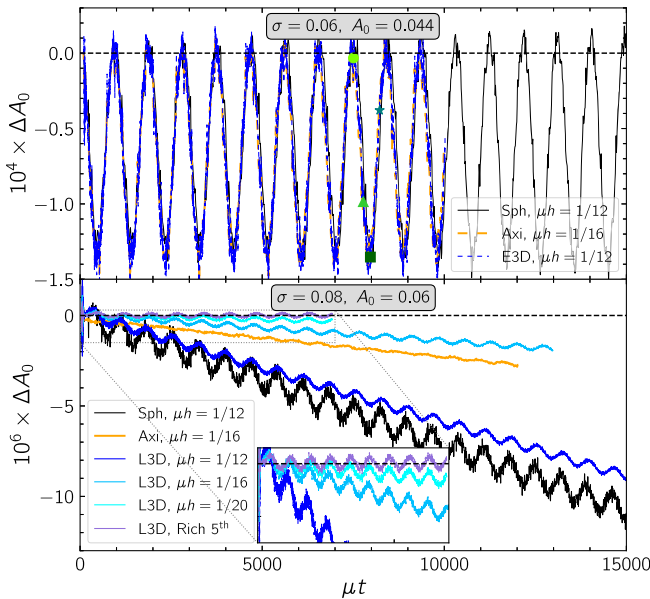


FIG. 2. Time evolution of the deviation ΔA_0 of the central scalar-field amplitude for the S06A044 (top) and S08A06 (bottom) models obtained in spherical symmetry, axisymmetry, and 3D. The linear drift visible for S08A06 converges away, as shown in the inset. The four symbols in the top panel mark times when we evaluate the LR potential; cf. Fig. 4.

$\sim 0.3\%$ and good agreement by all codes. For S08A06, the oscillations are 2 orders of magnitude smaller and exhibit a linear drift, which converges away with increasing resolution. This is demonstrated by the three blue-shaded curves obtained with L3D using different grid spacing h . Closer analysis of the differences yields fifth-order convergence, and the Richardson extrapolated function oscillates with high precision around zero.

Two features manifest in this figure require further explanation. First, we attribute the considerably larger oscillations observed for S06A044 to a larger numerical error in the initial data; for this model, round-off uncertainty in the BS frequency $\sim 10^{-32}$ can effect relative variations in the ADM mass or central lapse comparable to the level $\Delta A_0/A_0$ observed. The S08A06 model is less challenging and results in initial data with small uncertainty compared to those incurred in the time evolutions. This feature also explains the level of agreement between time evolutions employing different symmetry: the perturbations of S06A044 are initial data dominated, with comparatively smaller contributions from the time evolution. For S08A06, on the other hand, details of the time evolution dominate, resulting in more pronounced relative—albeit tiny absolute—differences. This observation is confirmed by artificially lowering the quality of the initial data or time evolution, which amplifies drift or oscillation magnitude but with no change in frequency. The 3D simulations displayed here have been computed using octant symmetry. We have performed additional versions of the 3D simulations of S08A06 using L3D (E3D) with $h = 1/12$ in full grid mode up to $t \gtrsim 3000$ ($t \gtrsim 1000$); these give identical results to the octant runs within round-off precision.

The time evolution of the maximal Kretschmann scalar $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ exhibits, up to an overall scaling, nearly identical behavior as ΔA_0 , including the oscillatory pattern, drift and fifth-order convergence toward the equilibrium value. The Noether charge is conserved with identical convergence to within $\Delta \mathcal{N}/\mathcal{N} \lesssim 10^{-3}$ and 10^{-4} for S06A044 and S08A06, respectively, predicting $\Delta \mathcal{N}/\mathcal{N} \lesssim 10^{-6}$ in the continuum limit for S08A06. The constraints exhibit a rapid drop to L_2 norms $\sim 10^{-9}$ due to the CCZ4 damping [44], but converge only between the first and fourth order [45].

Perturbative predictions—The oscillatory behavior of ΔA_0 in Fig. 2 can be understood in terms of linearized calculations. For this purpose, we compute the frequency χ_0 of the fundamental radial oscillation mode for our BS families using the approach of Refs. [46,47]; cf. Ref. [45] for details. The transitions between stable and unstable branches in Fig. 1 are characterized by zero crossings of χ_0^2 such that a real (imaginary) χ_0 implies radial (in)stability. For our BS models S06A044 and S08A06, we obtain $\chi_0^2 > 0$, as expected from their location on the rising slope of the $M(A_0)$ curve. The behavior of these BSs, as shown in Fig. 2, exhibits excellent agreement with the perturbative

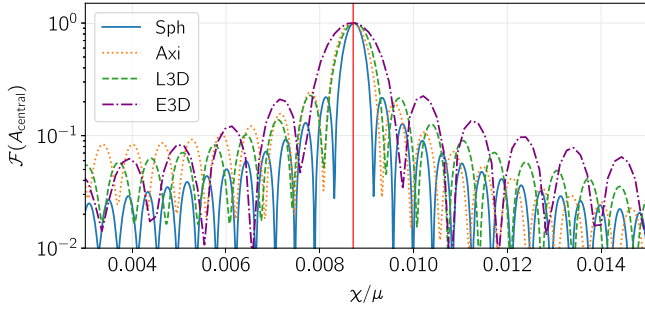


FIG. 3. Power spectrum of $\Delta A_0(t)$ for S08A060; the red line marks the fundamental radial mode frequency χ_0 . Units on the vertical axis are arbitrary.

predictions: numerical imperfections trigger perturbations around the equilibrium solution, whose time evolution is dominated by radial oscillations with the expected frequency. We quantify this agreement by plotting in Fig. 3 for S08A06 the Fourier spectrum of $\Delta A_0(t)$, i.e., the data of Fig. 2. The spectrum peaks at the χ_0 value predicted by the perturbative calculation. The corresponding analysis for S06A044 yields similar agreement [45].

Light-ring structure—Having demonstrated the long-term stable behavior of our BH mimickers S06A044 and S08A06, we now turn to the question of whether their LR structure remains preserved. In a stationary and axisymmetric spacetime, a LR is defined as a null geodesic with a tangent vector field that is always a linear combination of the Killing vectors ∂_t and ∂_ϕ , assuming coordinates (t, r, θ, ϕ) adapted to the symmetry. The conditions for a LR can then be recast as follows [cf. Eqs. (1)–(8) in Ref. [18]]:

$$\partial_\mu H_\pm = 0, \quad \text{where } H_\pm := \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}}. \quad (3)$$

Furthermore, (un)stable LRs are located at local (maxima) minima of $H_\pm$. In spherical symmetry with adapted coordinates, $H_\pm$ simplifies to a single potential function $H = \sqrt{-g_{tt}/g_{\phi\phi}}$.

Under the assumption that an evolved spacetime remains nearly spherically symmetric and stationary, the potential function can be approximated by an *adiabatic effective potential* (AEP) H_{eff} computed via an averaging procedure analogous to Ref. [25]. Specifically, we transform between standard Cartesian and spherical coordinates $x^\alpha = (t, x, y, z)$, $\hat{x}^\mu = (t, R, \theta, \phi)$. For a sphere of fixed coordinate radius R , we compute the proper area $\mathcal{A}$, the areal radius $r := \sqrt{\mathcal{A}/(4\pi)}$, and the AEP $H_{\text{eff}}(r) = \sqrt{(\langle \alpha^2 \rangle - \langle \gamma_{ij}\beta^i\beta^j \rangle)}/r$, where α , β^i , and γ_{ij} are the lapse, shift, and spatial metric in Cartesian coordinates, and $\langle \cdot \rangle$ denotes averaging over the sphere.

Repeating this procedure for a sequence of ~ 50 coordinate spheres with $\Delta R \approx 0.1$ across the relevant shell region, we obtain at each time t an estimate for the potential $H_{\text{eff}}(r)$. We validate the robustness of this method and the assumptions in Supplemental Material [45].

Our analysis of the resulting potentials for S06A044 and S08A06 indeed yields one minimum and one maximum at all t . The locations of the respective LRs exhibit minor periodic variation in line with the BS oscillations observed in the other diagnostics, but the pair of LRs remains in place with no indication of instability. We show these variations together with selected profiles of the AEP over a representative oscillation cycle in Fig. 4 for S06A044. The same behavior, merely with smaller oscillation amplitude, is found for S08A06.

Nonlinear stability—We complement our detailed analysis of the BS models S06A044 and S08A06 with a larger set of time evolutions of the $\sigma = 0.08$ BS sequence. We have used all codes for this analysis, but restricted 3D evolutions to $\mu t \lesssim 3000$ for reasons of computational cost. In this analysis, we define the instability time as the moment when $A_0(t)$ departs by 10% from its equilibrium value. If $A_0(t)$ yields no signature of departing from the oscillatory pattern seen in Fig. 2 within $3000\mu^{-1}$, the model is designated stable. The result is shown in Fig. 5 and fully agrees with the perturbative predictions: models on unstable branches succumb to exponentially growing perturbations on a timescale that diverges at the threshold to stable branches. These transition points coincide with extrema in the $M(A_0)$ curves, i.e., points where the oscillation frequency χ_0 of the fundamental mode switches between real and imaginary.

Concluding remarks—We have performed a series of fully nonlinear evolutions of the Einstein-Klein-Gordon

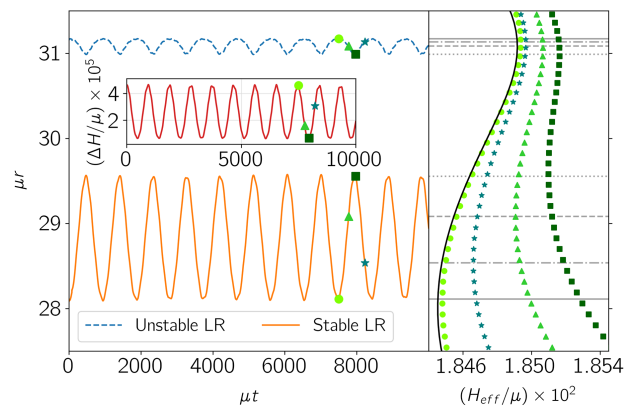


FIG. 4. Left: areal radius of the (un)stable light ring over time as extracted from the AEP H_{eff} , right: during the time evolution of S06A044 performed with E3D. The inset shows the difference between the extrema of the AEP over time. The green markers correspond to the snapshots of the AEP given in the right panel. The gray lines indicate the positions of the extrema. The solid line is the effective potential (3) at the start of the simulation.

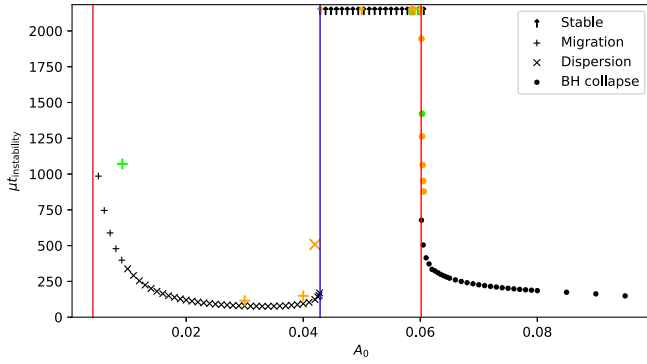


FIG. 5. Instability timescales as a function of A_0 for solitonic BSs with $\sigma_0 = 0.08$. The vertical lines mark the transition between stable and unstable branches as predicted by linear theory; cf. Fig. 1. The marker style indicates dynamical fate, while the color indicates the code used: black for Sph, lime green for Axis, orange for L3D. BS models whose time evolution show no sign of instability are represented with arrows at the top of the figure.

equations governing the dynamics of solitonic boson stars in the thin-shell regime in $3+1$ dimensions imposing spherical, axi-, octant, or no symmetry. A subset of the models under consideration are ultracompact, exhibiting a pair of light rings, one of which is stable. We have evolved these objects for $\gtrsim 1000$ light crossing times, finding no evidence of an instability due to the appearance of the stable light ring, as previously conjectured [20,24] and reported for spinning BSs [25]. We observe excellent agreement between the axisymmetric/3D evolutions and those in spherical symmetry, suggesting that the inclusion of nonspherical degrees of freedom does not significantly affect the stability of these models. Taking into account also the agreement of our numerical results with perturbative predictions for stability and fundamental radial oscillation frequencies, we interpret our findings as strong numerical evidence in favor of the stability of this class of BH mimickers.

As a cautionary note, we add that poor choices of numerical ingredients, in particular the constraint-damping parameter κ_1 of the CCZ4 formulation, can result in seemingly plausible instabilities of our thin-shell BSs. With $M\kappa_1 \gg 0.1$, for example, both models S06A044 and S08A06 collapse to a BH; the erroneous nature of these simulations reveals itself only through large and growing constraint violations and complete lack of convergence.

If the presence of the stable light ring is not sufficient to destroy these solitonic boson stars, this family of spherically symmetric models, i.e., with quantized spin zero, as well as similar UCOs, can be considered as viable black-hole mimickers again. The thin-shell solutions under consideration here can mimic the phenomenology of black

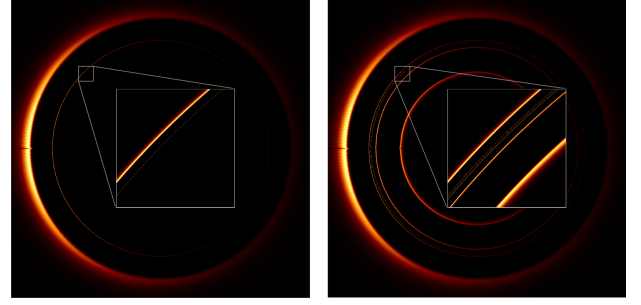


FIG. 6. Side-by-side comparison of ray-traced images of (left) a Schwarzschild BH and (right) the S06A044 model of the same ADM mass. The color scale indicates the observed brightness intensity, in arbitrary units, from a simple geometrically thin disk located in the equatorial plane. Both panels also provide a close-up of the photon ring region, exhibiting their fractal structure. Images are obtained with FOORT [48,49].

holes in several different ways, as, for example, in the GW emission from binaries [17] or the creation of BH like shadows in VLBI images [54]. Figure 6 illustrates how the S06A044 model could resemble a Schwarzschild BH with the same ADM mass in the presence of an equatorial, optically thin accretion disk. The image of the BS clearly displays additional structure toward the center of the image, which is a result of the absence of a horizon as well as the pair of light rings. The outermost direct emission band, as well as the first lensed photon ring, are nearly indistinguishable for both images.

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Data availability—The data are not publicly available. The data are available from the authors upon reasonable request.

- [1] S. Bowyer, E. T. Byram, T. A. Chubb, and H. Friedmann, *Science* **147**, 394 (1965).
- [2] C. Hazard, M. B. Mackey, and A. J. Shimmins, *Nature (London)* **197**, 1037 (1963).
- [3] M. Schmidt, *Nature (London)* **197**, 1040 (1963).
- [4] B. P. Abbott *et al.*, *Phys. Rev. Lett.* **116**, 061102 (2016).
- [5] R. Abbott *et al.* (KAGRA, Virgo, and LIGO Scientific Collaborations), *Phys. Rev. X* **13**, 041039 (2023).
- [6] R. Abbott *et al.* (LIGO Scientific, Virgo, and KAGRA Collaborations), [arXiv:2112.06861](https://arxiv.org/abs/2112.06861).
- [7] K. Akiyama *et al.* (Event Horizon Telescope Collaboration), *Astrophys. J. Lett.* **875**, L1 (2019).
- [8] K. Akiyama *et al.* (Event Horizon Telescope Collaboration), *Astrophys. J. Lett.* **930**, L12 (2022).
- [9] J. L. Synge, *Mon. Not. R. Astron. Soc.* **131**, 463 (1966).
- [10] V. Cardoso, A. S. Miranda, E. Berti, H. Witek, and V. T. Zanchin, *Phys. Rev. D* **79**, 064016 (2009).
- [11] K. Jusufi, *Phys. Rev. D* **101**, 084055 (2020).
- [12] Y. Koga, N. Asaka, M. Kimura, and K. Okabayashi, *Phys. Rev. D* **105**, 104040 (2022).
- [13] S. H. Völkel, N. Franchini, E. Barausse, and E. Berti, *Phys. Rev. D* **106**, 124036 (2022).
- [14] D. Pedrotti and M. Calzà, *Phys. Rev. D* **111**, 124056 (2025).
- [15] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
- [16] S. D. Mathur, *Classical Quantum Gravity* **26**, 224001 (2009).
- [17] T. Evstafyeva, U. Sperhake, I. M. Romero-Shaw, and M. Agathos, *Phys. Rev. Lett.* **133**, 131401 (2024).
- [18] P. V. P. Cunha, E. Berti, and C. A. R. Herdeiro, *Phys. Rev. Lett.* **119**, 251102 (2017).
- [19] P. V. P. Cunha and C. A. R. Herdeiro, *Phys. Rev. Lett.* **124**, 181101 (2020).
- [20] J. Keir, *Classical Quantum Gravity* **33**, 135009 (2016).
- [21] P. Bizoń and A. Rostworowski, *Phys. Rev. Lett.* **107**, 031102 (2011).
- [22] G. Holzegel and J. Smulevici, *Commun. Pure Appl. Math.* **66**, 1751 (2013).
- [23] G. Holzegel and J. Smulevici, *Anal. PDE* **7**, 1057 (2014).
- [24] V. Cardoso, L. C. B. Crispino, C. F. B. Macedo, H. Okawa, and P. Pani, *Phys. Rev. D* **90**, 044069 (2014).
- [25] P. V. P. Cunha, C. Herdeiro, E. Radu, and N. Sanchis-Gual, *Phys. Rev. Lett.* **130**, 061401 (2023).
- [26] P. V. Cunha, *J. Cosmol. Astropart. Phys.* **05** (2025) 083.
- [27] N. Siemonsen, *Phys. Rev. Lett.* **133**, 031401 (2024).
- [28] G. Guo, P. Wang, and Y. Zhang, [arXiv:2403.02089](https://arxiv.org/abs/2403.02089).
- [29] G. Benomio, A. Cárdenas-Avendaño, F. Pretorius, and A. Sullivan, *Phys. Rev. D* **111**, 104037 (2025).
- [30] J. Redondo-Yuste and A. Cárdenas-Avendaño, *Phys. Rev. D* **111**, 124009 (2025).
- [31] L. G. Collodel and D. D. Doneva, *Phys. Rev. D* **106**, 084057 (2022).
- [32] T. Evstafyeva, R. Rosca-Mead, U. Sperhake, and B. Bruggmann, *Phys. Rev. D* **108**, 104064 (2023).
- [33] T. W. Baumgarte and S. L. Shapiro, *Phys. Rev. D* **59**, 024007 (1998).
- [34] M. Shibata and T. Nakamura, *Phys. Rev. D* **52**, 5428 (1995).
- [35] D. Alic, C. Bona-Casas, C. Bona, L. Rezzolla, and C. Palenzuela, *Phys. Rev. D* **85**, 064040 (2012).
- [36] W. G. Cook, P. Figueras, M. Kunesch, U. Sperhake, and S. Tunyasuvunakool, *Int. J. Mod. Phys. D* **25**, 1641013 (2016).
- [37] ExoZvezda code, GitHub repository (2024), <https://github.com/GRTLCollaboration/ExoZvezda>.
- [38] M. Radia, U. Sperhake, A. Drew, K. Clough, P. Figueras, E. A. Lim, J. L. Ripley, J. C. Aurrekoetxea, T. França, and T. Helfer, *Classical Quantum Gravity* **39**, 135006 (2022).
- [39] T. Andrade *et al.*, *J. Open Source Software* **6**, 3703 (2021).
- [40] M. Adams *et al.*, Chombo software package for AMR applications—Design document, Technical Report LBNL-6616E, Lawrence Berkeley National Laboratory, 2019.
- [41] U. Sperhake, *Phys. Rev. D* **76**, 104015 (2007).
- [42] G. Allen, T. Goodale, J. Massó, and E. Seidel, in *Proceedings of Eighth IEEE International Symposium on High Performance Distributed Computing, HPDC-8, Redondo Beach, 1999* (IEEE Press, Piscataway, NJ, 1999).
- [43] E. Schnetter, S. H. Hawley, and I. Hawke, *Classical Quantum Gravity* **21**, 1465 (2004).
- [44] D. Alic, W. Kastaun, and L. Rezzolla, *Phys. Rev. D* **88**, 064049 (2013).
- [45] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/lk48-7r2f>, which includes Refs. [25,46–53], for details of the perturbative calculations.
- [46] M. Gleiser and R. Watkins, *Nucl. Phys.* **B319**, 733 (1989).
- [47] B. Kain, *Phys. Rev. D* **103**, 123003 (2021).
- [48] D. R. Mayerson, F. Bacchini, and S. Staelens, FOORT: Flexible Object-Oriented Ray Tracing (2024), <https://github.com/drmayerson/FOORT> and version number 1.0.0.
- [49] D. R. Mayerson, F. Bacchini, and S. J. Staelens (to be published).
- [50] B.-X. Ge, E. A. Lim, U. Sperhake, T. Evstafyeva, D. Cors, E. de Jong, R. Croft, and T. Helfer, [arXiv:2410.23839](https://arxiv.org/abs/2410.23839).
- [51] S. E. Gralla, A. Lupsasca, and D. P. Marrone, *Phys. Rev. D* **102**, 124004 (2020).
- [52] H.-Y. Pu, K. Akiyama, and K. Asada, *Astrophys. J.* **831**, 4 (2016).
- [53] A. Cárdenas-Avendaño, A. Lupsasca, and H. Zhu, *Phys. Rev. D* **107**, 043030 (2023).
- [54] J. L. Rosa, C. F. B. Macedo, and D. Rubiera-Garcia, *Phys. Rev. D* **108**, 044021 (2023).