

Exact Coherent Structures as Building Blocks of Turbulence on Spatially Extended Domains

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Exact unstable solutions of the Navier-Stokes equations are thought to underpin the dynamics of turbulence but are usually computed in minimal computational domains. Here, we extend this dynamical systems approach to spatially extended turbulent flows featuring multiple interacting “substructures” and show how new simple invariant solutions can be constructed by spatial tiling of exact solutions from small-box calculations. Candidate solutions are found via gradient-based optimization of a scalar loss function which targets autorecurrence in spatially localized regions of the flow. We first identify 25 relative periodic orbits (RPOs) which are combinations of high-dissipation, small-box solutions with laminar patches. We then show that vertically stacked combinations of pairs of distinct small-box RPOs can form robust guesses for dynamically relevant two-tori in the larger domain. Finally, we show how our optimization procedure can identify “turbulent” trajectories which locally shadow a small-box RPO for multiple periods in a subdomain. These small-box combinations are possible as the flow spends prolonged periods in a regime where it can be effectively considered to be a pair of weakly coupled small-box systems. The new large-scale states are the first examples of solutions featuring multiple interacting substructures and support the long-standing hypothesis that turbulence can be built up as a spatiotemporal tapestry of smaller domain solutions.

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The dynamical systems view of turbulence considers a realization of a particular flow as a trajectory in a high-dimensional state space, pinballing between exact, unstable solutions of the Navier Stokes equations [1]. This picture has been realized experimentally in both two-dimensional [2] and three-dimensional [3] flows. Early applications of these ideas explained many aspects of the subcritical transition to turbulence in parallel shear flows [4,5], resulting in intensive focus on the identification and computation of exact solutions in fully developed turbulence [6–8]. The exact solutions, which are typically relative periodic orbits (RPOs) due to the presence of continuous symmetries in many of the canonical fluid systems, each contain a closed cycle of physical processes helping to sustain the fully turbulent state [9]. Computing RPOs is notoriously challenging, particularly in more strongly turbulent systems, and has restricted much of the work to date to weak turbulence in “minimal” computational domains (e.g., see the review by Ref. [10]). While

recent approaches based on optimization have produced enough solutions to reconstruct turbulent statistics at modest Reynolds numbers [11], Re , there are to our knowledge no examples of exact solutions featuring multiple spatially localized substructures typically associated with turbulence in larger domains.

There have been notable examples of spatially localized exact solutions obtained in extended domains for various parallel shear flows. This includes the identification of spanwise-localized roll-streak solutions (and associated homoclinic snaking) obtained by Ref. [12] and streamwise-localized RPOs computed at transitional Re in a pipe [13]. Solutions which localize in the *cross-shear* direction have also been found in planar configurations [14,15] and appear to continue indefinitely up in Re . A slightly different perspective was adopted by Ref. [16], who sought new large-box exact solutions (in a reduced system of equations) which are spanwise-spatially modulated small-box states. However, these “substructures” are all dominated by a single cross-shear length scale. More recent efforts to examine dynamics involving multiple substructures have considered the construction of quasiperiodic states within a reduced (quasilinear) model [17] or by attempting to identify (spatially) local shadowing of small-box solutions in a large convection cell [18].

In this Letter we construct spatially extended solutions of the Navier-Stokes equations from combinations of

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small-box exact solutions, inspired by the ideas introduced by Cvitanović and coauthors [19,20] describing spatiotemporal chaos as a space-time tapestry composed from smaller solutions and to date only demonstrated convincingly in the much simpler Kuramoto-Sivashinsky equation [21,22]. Our starting point is a large library of RPOs assembled in two-dimensional Kolmogorov flow on a $2\pi \times 2\pi$ domain [23] which we seek to combine in a larger $2\pi \times 4\pi$ box. Computation of the new states is performed with a combination of gradient-based optimization and classical Newton-Krylov methods, which we adapt to target near-recurrence in localized patches of the full domain. The new solutions include (i) exact RPOs, which are spatially tiled combinations of a small-box RPO with the laminar solution; (ii) unstable two-tori, which are constructed from pairs of nontrivial small-box RPOs; and (iii) turbulent trajectories, which locally shadow a small-box solution in a subregion of the flow domain.

We consider two-dimensional Kolmogorov flow driven by a monochromatic force in the horizontal, $\mathbf{f}^* := \hat{\mathbf{x}}\chi^* \sin(2\pi ny^*/L^*)$ (an asterisk indicates a dimensional variable). The forcing amplitude, χ^* , and fundamental wave number associated with the box width, $k^* := 2\pi/L^*$, are used to nondimensionalize the system. The out-of-plane vorticity evolves according to

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega + \frac{1}{Re} \Delta \omega - n \cos(ny), \quad (1)$$

where the velocity and vorticity are related by the definition $\omega := (\nabla \times \mathbf{u}) \cdot \hat{\mathbf{z}}$ and the Reynolds number based on our nondimensionalization is $Re := (1/\nu) \sqrt{\chi^*/k^{*3}}$. We fix the number of forcing waves per reference-box-height of 2π at $n = 4$ but, contrary to previous studies [8,11], consider “tall” boxes of size $(L_x, L_y) = (2\pi, 4\pi)$ (so that there are eight forcing waves in the box) for most of this work, with some additional results shown at $(L_x, L_y) = (4\pi, 4\pi)$ (note that nondimensionalization is based on the smaller $2\pi \times 4\pi$ box). The Reynolds number is held fixed at $Re = 40$ throughout this Letter. Equation (1) is equivariant under continuous translations in x , under discrete shift-reflects over half-wavelengths of the forcing in y , and under rotation through π [8].

We seek relative periodic and quasiperiodic solutions in the $4\pi \times 2\pi$ box which are constructed from RPOs found in the “standard” $2\pi \times 2\pi$ geometry. Our expectation that this is possible is due to the fact that the tall box can often be considered to be a pair of weakly interacting small boxes. To quantify the interaction, consider a partitioning of the vorticity in the vertical coordinate, $\omega(x, y) = \omega(x, y)\mathbf{1}_{[0, 2\pi)}(y) + \omega(x, y)\mathbf{1}_{[2\pi, 4\pi)}(y) := \omega_1 + \omega_2$, with matching conditions across the interfaces $[\omega] := \lim_{y \rightarrow 2\pi^+} \omega - \lim_{y \rightarrow 2\pi^-} \omega = 0$ and $[\partial_y \omega] = 0$. The vertically partitioned vorticity fields, ω_1 and ω_2 , each evolve according to

$$\partial_t \omega_i + (\mathbf{u}_1 + \mathbf{u}_2) \cdot \nabla \omega_i = \frac{1}{Re} \Delta \omega_i - n \cos(ny), \quad (2)$$

on their respective domains. Here the velocity has been decomposed into components induced by the vorticity on each half domain, with $\mathbf{u}_i := (-\hat{\mathbf{x}}\partial_y + \hat{\mathbf{y}}\partial_x)\Delta^{-1}\omega_i$.

We then quantify coupling between the vorticity fields by the action of the induced velocity from one field onto another, defining a *coupling coefficient* $i \rightarrow j$:

$$c_{ij} := \|\mathbf{u}_i(\omega_i) \cdot \nabla \omega_j\|, \quad (3)$$

where $\|\cdot\| := [(8\pi^2)^{-1} \iint |\cdot|^2 d^2 \mathbf{x}]^{1/2}$ and the definition (3) allows for examination of “self-interaction” when $i = j$. It is helpful to consider a normalized version of (3): $C_{ij} := c_{ij}/(c_{ij} + c_{jj})$. These normalized coefficients provide a measure of the degree to which one region influences the advection in another via the induced velocity.

The cross-coupling terms, C_{12} and C_{21} , were computed for snapshots on a long turbulent trajectory (minimized over discrete $\pi/2$ translations of the full solution in y) and a joint probability density function (pdf) of these variables is reported in Fig. 1(a). The peak in the pdf occurs when $C_{12} \approx C_{21} \approx 0.35$, indicating that the turbulence spends most of its time in a weakly coupled state where velocity generated by the vorticity local to the $2\pi \times 2\pi$ window under consideration dominates the total advective term, with excursions to more strongly or weakly coupled regimes. Some examples of these excursions are visualized in panels (b)–(d) of Fig. 1, where we report contours of $\mathbf{u}_j(\omega_j) \cdot \nabla \omega_i$, with the driving vorticity fields overlaid. The most relevant example to the exact solutions discovered herein is shown in panel (d), in which the advection in regions 1 and 2 due to induced velocity from regions 2 and 1, respectively. This effect is associated with particularly intense vorticity fields with many closely packed vortices providing a shielding effect for the other half of the domain.

Motivated by the weak vertical coupling, we now seek exact coherent states in the tall domain which are recurrent in a spatially localized region of the full box. We accomplish this via gradient-based optimization of a functional featuring a *mask* $\mathcal{M}$ which can zero the velocity and vorticity outside of the region of interest. The functional to be minimized takes the form

$$\mathcal{L}(\mathbf{u}, T, \alpha) = \frac{1}{2} \|\mathcal{M}[T^\alpha \mathbf{f}^T(\mathbf{u}) - \mathbf{u}]\|^2 + \text{constraints}. \quad (4)$$

Here $T^\alpha: \mathbf{u}(x, y) \rightarrow \mathbf{u}(x + \alpha, y)$ is a continuous shift operator, and $\mathbf{f}^T$ is the flow map with T being the target time. The mask $\mathcal{M}(\mathbf{u}) \equiv \mathbb{L}[M(y)\mathbf{u}]$, where $M(y)$ is the vertical windowing function and $\mathbb{L}\mathbf{u} = \mathbf{u} - \nabla \Delta^{-1}(\nabla \cdot \mathbf{u})$ is the Leray projection, allows us to target temporal recurrence in spatial subregions of the full domain via careful choice of

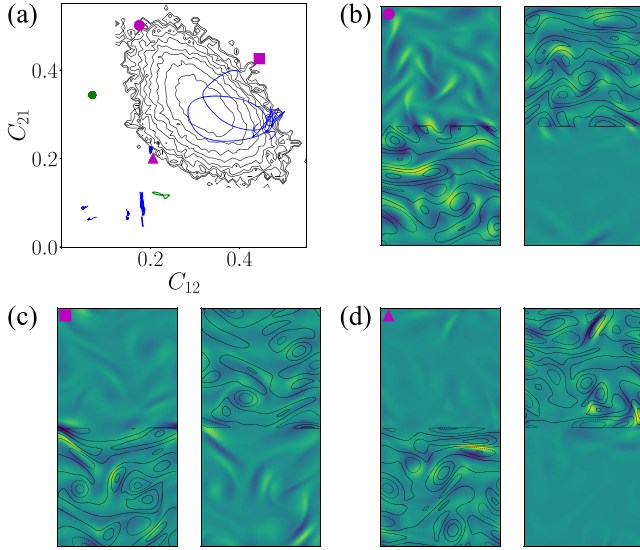


FIG. 1. Cross-stream coupling in the tall-box Kolmogorov flow. (a) Joint pdf of the normalized cross-coupling terms $C_{12} = c_{12}/(c_{12} + c_{22})$ and $C_{21} = c_{21}/(c_{21} + c_{11})$ [see Eq. (3) for the definition of c_{ij}] computed from a turbulent $2\pi \times 4\pi$ trajectory of total length $T = 10^5$ (contour levels evenly spaced logarithmically from 0.13 to 61.3). Also shown are the coupling coefficients for the RPO-laminar tiles (green curves and points) and the RPO-RPO tiles (blue curves). (b)–(d) Examples of three vorticity fields with varying coupling strengths, corresponding to the magenta symbols in (a). Background filled contours are the explicit coupling terms $\mathbf{u}_1 \cdot \nabla \omega$ (left) and $\mathbf{u}_2 \cdot \nabla \omega$ (right) with contours in ranges (a) $[-21.7, 21.7]$, (b) $[-30.0, 30.0]$, and (c) $[-32.8, 32.8]$. Line contours show the vorticity, evenly spaced $-11.4 < \omega < 11.4$, in each half domain which is used to compute the induced velocities $\{\mathbf{u}_i\}$.

$M(y)$, which we discuss further below [see Supplemental Material (SM) [24] for full details].

The forward fields satisfy the Navier-Stokes equations which is enforced in Eq. (4) via constraints $\int_0^T \langle \mathbf{v}^\dagger \cdot [\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - Re^{-1} \Delta \mathbf{u} - \hat{\mathbf{x}} \sin(ny)] \rangle dt$ and $\int_0^T \langle \pi^\dagger \nabla \cdot \mathbf{u} \rangle dt$, where $\langle \bullet \rangle := (8\pi^2)^{-1} \iint \bullet d^2 \mathbf{x}$ and $\mathbf{v}^\dagger, \pi^\dagger$ are adjoint velocity and pressure fields, respectively.

We solve the optimization problems described below using variational techniques which relate the gradients of the objective functional to the adjoint field $\mathbf{v}^\dagger(t=0)$ (see SM [24]). The optimization is complemented with classical Newton-Krylov techniques for convergence of exact RPOs and as a minimization approach for non- T -periodic states (see SM [24]). Forward and backward time integration is performed with the Dedalus solver [28] with a Fourier decomposition in both x and y ($N_x \times N_y = 128 \times 256$ on the $2\pi \times 4\pi$ domain) while time integration is performed with a third-order operator-splitting Runge-Kutta scheme. Gradients are passed to an AdaGrad optimizer [29] (see SM [24] for details of specific hyperparameters).

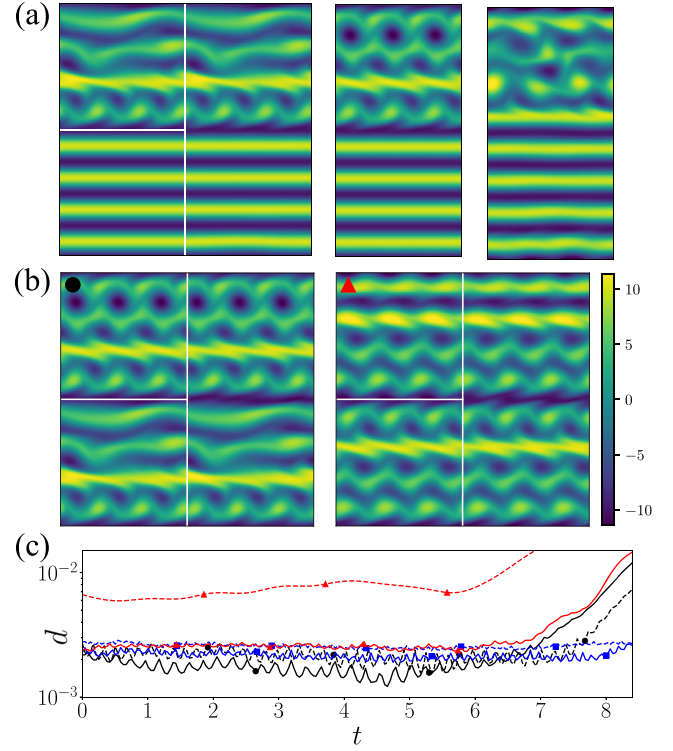


FIG. 2. Some example exact coherent states in the tall $2\pi \times 4\pi$ box. (a) Snapshots of out-of-plane vorticity for RPOs formed from a combination of a small-box ($2\pi \times 2\pi$) RPO with the laminar solution. The first panel includes the “starting” pair of solutions alongside the converged state. (b) Snapshots of out-of-plane vorticity for trajectories shadowing two-tori in the $2\pi \times 4\pi$ box. Left panels show snapshots from the pair of RPOs used to initialize the optimization, and right panel is a snapshot from the final state. (c) Masked near-recurrence for the solutions in (b), with d defined in Eq. (6). Colors match the symbols on the panels in (b); dashed lines are the top solution, and solid lines are the bottom. Symbols in (c) identify integer multiples of the fundamental period within the masked subdomains (blue solution shown in SM [24]).

We first attempt to tile (i.e., vertically stack alongside another state) RPOs from the extensive library assembled in Ref. [23] with the laminar solution to form new RPOs. To do this, the masking function $\mathcal{M}$ is set to the identity function. We also found it necessary to modify Eq. (4) to a multipoint shooting form which splits the full loop $t \in [0, T]$ into subintervals (see SM [24]). Following optimization, candidate RPOs were passed to a Newton solver and converged to a relative error $< 10^{-10}$.

Three examples of the 25 successful RPO-laminar convergences are reported in Fig. 2(a). For all converged states, the candidate small-box RPOs tend to be very high dissipation, with many consisting of counterpropagating rows or “jets” of vorticity which shield the laminar patches from distortion (though note the weak distortion of the laminar stripes in the third panel). The converged RPO-laminar tiles

are also shown in green in Fig. 1, with the majority sitting away from the turbulent attractor in an extremely weakly coupled regime.

We next attempt to combine pairs of (nonlaminar) RPOs to form two-tori by vertical stacking, with the expectation being that weak coupling allows for structures which are approximately periodic on half of the domain. The loss function (4) is therefore adjusted to seek near recurrence in two distinct regions:

$$\begin{aligned} \mathcal{L}(\mathbf{u}, T_1, \alpha_1, T_2, \alpha_2) = & \frac{1}{2} \|\mathcal{M}_1[\mathcal{T}^{\alpha_1} \mathbf{f}^{T_1}(\mathbf{u}) - \mathbf{u}]\|^2 \\ & + \frac{1}{2} \|\mathcal{M}_2[\mathcal{T}^{\alpha_2} \mathbf{f}^{T_2}(\mathbf{u}) - \mathbf{u}]\|^2 \\ & + \text{constraints.} \end{aligned} \quad (5)$$

The (differentiable) window function $M(y)$ used to define the masks $\mathcal{M}_j$ is formed by a double vertical convolution of a pair of square waves (see SM [24]). We incrementally increase the target time over which (5) is minimized: once $\mathcal{L}$ is reduced such that the relative error $\varepsilon_i := \|\mathcal{M}_i[\mathcal{T}^{\alpha_i} \mathbf{f}^{T_i}(\mathbf{u}) - \mathbf{u}]\| / \|\mathcal{M}_i \mathbf{u}\| \leq 0.01$ in each region, we then restart the process with $T_i \rightarrow 2T_i$. All results presented here were converged below this tolerance to at least two fundamental periods in both halves of the domain.

We apply this procedure for all combinations of higher-dissipation RPOs that successfully tiled with the laminar state. From this search we obtain six trajectories which converge to a relative error within the masked regions of less than 0.01 for both regions over at least two periods. Some examples are included in Fig. 2(b), with Fig. 2(c) showing the corresponding evolution of the minimum relative distance from the original $2\pi \times 2\pi$ states vertically stacked on the 4π domain,

$$d_i(t) = \min_{\alpha, t'} \|\mathcal{M}_i[\mathbf{u}_{4\pi}(t) - \mathcal{T}^\alpha \mathbf{u}_{2\pi,i}(t')]\| / \|\mathbf{u}_{2\pi,i}(t')\|, \quad (6)$$

where $\mathbf{u}_{4\pi}$ and $\mathbf{u}_{2\pi,i}$ are the tiled $2\pi \times 4\pi$ solution and the $2\pi \times 2\pi$ RPOs ($i \in \{1, 2\}$) used to generate it in the corresponding half domain, respectively, over several periods. The distinct fundamental frequencies, $2\pi/T_1$ and $2\pi/T_2$, and low values of d_i extending over several Lyapunov times ($t_\lambda \sim 2$ here) are strongly suggestive of shadowing of underlying two-tori.

Similar to the laminar tiles discussed above, the solutions reported in Fig. 2 are all extremely weakly coupled and with vortices contained in distinct vertical bands. However, five “dynamically relevant” candidate tori were also obtained; dynamical relevance being determined by the overlap of low-dimensional projections of the tori with the appropriate (joint) pdf of the turbulence (see both the coupling coefficients in Fig. 1 and the production and dissipation values in Fig. 3). These solutions are overlaid on the turbulent attractor visualized in Fig. 1. In contrast to

the states reported in Fig. 2, which are difficult to distinguish from the starting small-box RPOs, these dynamically relevant tori are altered in the optimization process. Example visualizations are included in SM [24], where we also report full time evolutions to confirm the masked autorecurrence in each half of the spatial domain.

The various RPOs and tori described above are all visualized in a production-dissipation diagram in Fig. 3 [the production $I := \langle \mathbf{u} \cdot \sin(ny) \rangle$]. The “dynamically relevant” two tori which combine distinct high-dissipation RPOs overlap the turbulent attractor (see blue curves in the lower panel). In addition, we also visualize in red a number of turbulent trajectories which *locally* shadow a small-box RPO. These events were found via minimization of Eq. (4) with the mask applied to a single region with the dynamics outside essentially unconstrained.

This approach is very effective and produces $d \leq 0.01$ over > 2 full periods of the underlying RPO for many RPOs in our library, producing 32 (16 unique RPOs) such solutions from the 21 RPOs used to construct two-tori. An example trajectory is reported in Fig. 4(a), where near recurrence in the top half of the domain can be clearly seen (compare snapshots vertically at equivalent times modulo the period T). In panel (b) we also show contours of the masked autorecurrence function,

$$R(t, \tau) = \min_{\alpha} \|\mathcal{M}(\mathcal{T}^\alpha \mathbf{u}(t) - \mathbf{u}(t - \tau))\| / \|\mathcal{M} \mathbf{u}\|, \quad (7)$$

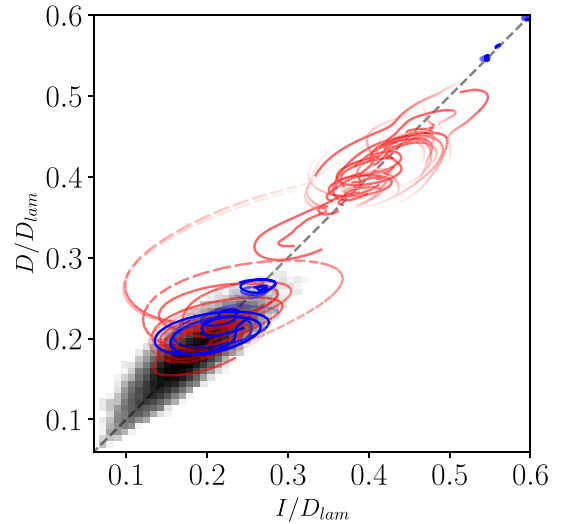


FIG. 3. Production and dissipation (I and D , respectively) in the $2\pi \times 4\pi$ domain normalized by the laminar dissipation D_{lam} . Gray histogram is a pdf of a long (4000 advective time units) turbulent orbit, which runs logarithmically from 0.4 to 275.8. Also shown are the candidate two-tori (blue) and some example “turbulent” trajectories which shadow an RPO in a masked region of the domain (red). The dashed and faded components of these curves span the first half of the optimization period nT for tiles with turbulence and $\min(n_1 T_1, n_2 T_2)$ for RPO-RPO tiles.

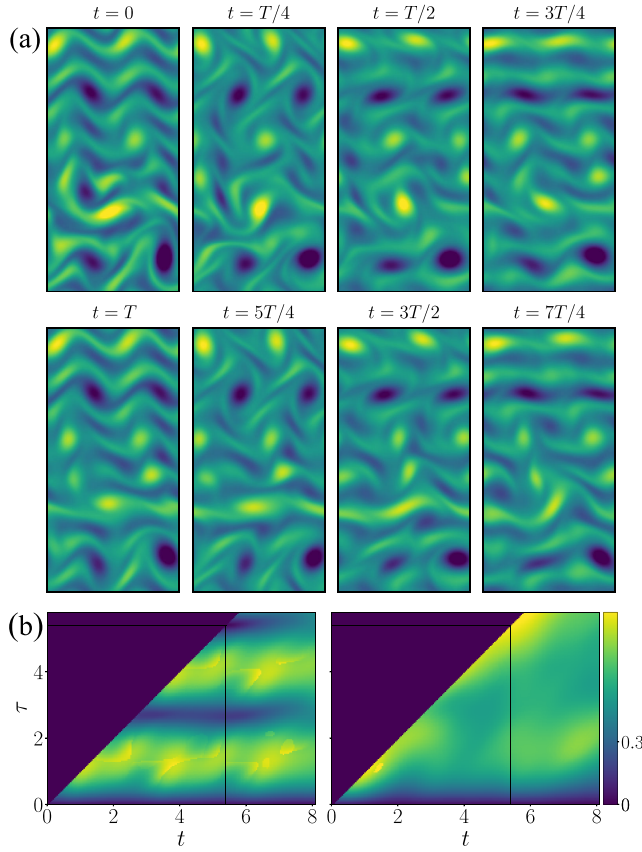


FIG. 4. Example trajectory with RPO shadowing in the top half of the domain. (a) Out-of-plane vorticity above two periods of the RPO used to construct the solution (contours run $-11.4 \leq \omega \leq 11.4$). (b) Masked autorecurrence $R(t, \tau)$ computed using Eq. (7) in the top (left panel) and bottom (right panel) halves of the domain.

which confirms the shadowing in the top half while the lower half of the domain shows no temporal autocorrelation beyond $\tau \approx 1$.

The ability of many of the small-box RPOs to coexist alongside aperiodic dynamics extends to larger domains: so far we have found 4 instances of spatially localized shadowing in a larger $4\pi \times 4\pi$ box, beginning with two copies of an RPO in the top half of the domain and minimizing Eq. (4) in the same manner as for the $2\pi \times 4\pi$ domain. One such example is reported in Fig. 5, with the autorecurrence plots in panel (b) emphasizing the time-periodic dynamics (or lack thereof) in the (bottom half) top half of the domain. Further examples are included in SM [24].

In this Letter we have shown that RPOs converged in a “small-box” calculation can be combined to form new solutions of the Navier-Stokes equation on a larger spatial domain. This spatial tiling has produced exact RPOs, two-tori involving weakly interacting substructures, and more general “turbulent” orbits which shadow exact solutions on patches of the domain. We showed this phenomenon to be

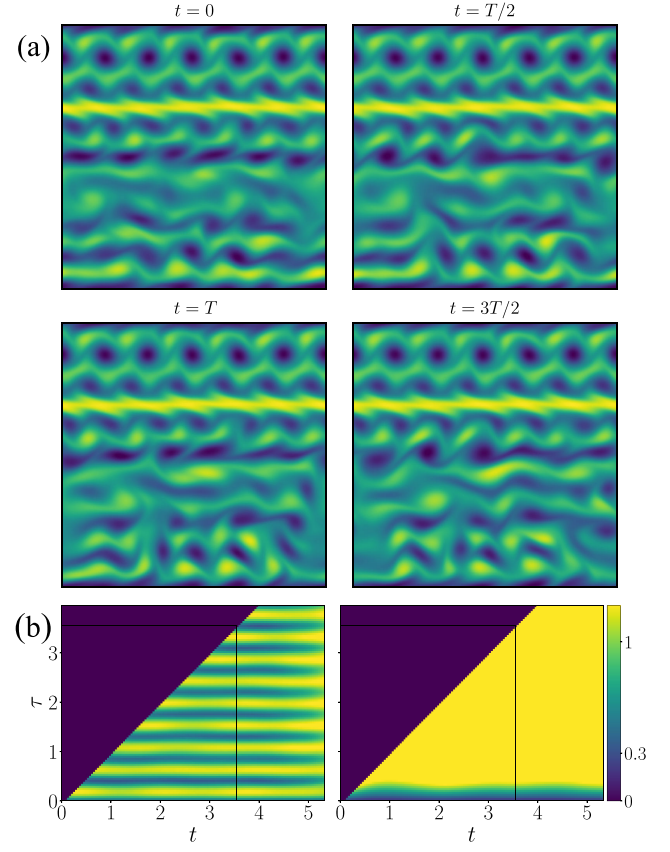


FIG. 5. Example trajectory exhibiting RPO shadowing in a $4\pi \times 2\pi$ subdomain of a larger $4\pi \times 4\pi$ box. (a) Out-of-plane vorticity above two periods of the RPO used to construct the solution (contours run $-11.4 \leq \omega \leq 11.4$). (b) Masked autorecurrence $R(t, \tau)$ as per (7) in the top (left panel) and bottom (right panel) halves of the domain.

robust: (1) long-lived shadowing of small-box solutions is possible for a variety of chaotic dynamics in the other half of the domain and (2) it is also observed as the domain size is increased. This spatially localized shadowing of substructures is in line with Cvitanović’s [19,20] view of turbulence as a “spatiotemporal tapestry” of connected states, and the results presented set the stage for a systematic hunt for local autorecurrence in large-scale flows. The algorithms employed here are general and easily adapted to variable solution geometry (e.g., varying aspect ratio) and ultimately to three-dimensional problems, where a long-standing challenge has been the identification of RPOs with distinct substructures.

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Data availability—The data that support the findings of this article are openly available at Ref. [30].

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