High-Energy Evolution of Power-Suppressed Amplitudes

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We present a new class of evolution equations that govern the high-energy behavior of power-suppressed scattering amplitudes. The equations can be viewed as a renormalization group flow with respect to the relevant effective field theory cutoff. A distinct feature of the method is in the use of a multidimensional cutoff to separate the relevant scales in problems characterized by a complex factorization structure. By adjusting the renormalization group variables to the geometry of the effective theory modes, our method naturally extends to a broad spectrum of physical problems that include massive, massless, small, and wide-angle scattering. We present applications to the benchmark processes of electron-positron forward annihilation and light-quark-mediated Higgs boson production and decays.

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The high-energy behavior of scattering amplitudes is significantly altered by quantum effects. It is among the early applications of quantum field theory [1–5] and has become a classical problem playing a fundamental role in particle phenomenology. The asymptotic behavior of amplitudes that are not suppressed by the ratio of a characteristic infrared scale to the process energy is by now well understood [6–12]. However, the significant progress of experimental measurements and perturbative calculations in a persistent search for physics beyond the standard model has gradually shifted the focus of theoretical studies to power-suppressed contributions. Besides their phenomenological relevance, power corrections reveal many intriguing properties and pose one of the main challenges to modern effective field theory methods. The central problem of the analysis is the resummation of radiative corrections enhanced by a power of the logarithm of the large scale ratio that breaks down the finite-order perturbation theory and determines the asymptotic behavior of the amplitudes. Over the last decade, a wide spectrum of power corrections has been extensively analyzed within different frameworks (see Refs. [13–36] for the examples), but only a handful of complete closed-form all-order results for physical observables have been obtained. The general structure of the renormalization group flow turns out to be far more complex and often even the relevant effective-theory modes differ from the leading power. Hence, while the general concept of scale separation is still at the core of any analysis, the classical "hard-jet-soft" factorization [6] has to be significantly advanced. The common approach based on soft-collinear effective theory has to deal with the problem of end-point singularities, which require a nontrivial case-specific refactorization of the effective-theory building blocks [30,32,34]. At the same time, the direct resummation of the relevant Feynman diagrams becomes quite nontrivial beyond the leading logarithms [27].

In this Letter we present a solution to this problem by introducing a new class of evolution equations for power-suppressed amplitudes. These equations describe the change of the amplitudes under the variation of the effective-theory cutoff and, therefore, can be understood as a Wilsonian renormalization group flow. The main difference with respect to the existing analysis is the use of more than one cutoff dictated by the factorization structure and the geometry of the effective-theory modes, which makes the renormalization group flow multidimensional. The building blocks that form the system of evolution equations in general differ from and have a more complex multiscale structure than the standard set of hard, jet, and soft functions. The method is universally applicable to a wide range of physical processes, including massive, massless, small, and wide-angle scattering and to power corrections in mass and transverse momentum. We introduce the method by deriving the electron-to-muon forward annihilation (or backward scattering) amplitude through the next-to-leading logarithmic approximation. This is a classical problem [2] and its completely distinct spectrum of modes and factorization structure poses a severe challenge to the soft-collinear effective-theory approach [32]. Moreover, the available

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two-loop result for the electron-muon scattering amplitudes [37], obtained using infrared matching [38–40], cannot be extended to the forward region because of its peculiar infrared structure. For this reason, we complete the two-loop analysis of the annihilation process by presenting the full result for the forward amplitudes. We then extend the renormalization group analysis to the light-quark-mediated Higgs boson production and decays [17].

We start with an outline of the main idea. First, asymptotic expansion of Feynman integrals [41,42] and gauge invariance are used to reveal the factorization structure and the spectrum of dynamical modes in a given kinematical regime. This enables their effective-theory description, often avoiding the tedious construction of a complete effective action [43–48]. (Although there is no one-to-one correspondence between expansion by regions and the physical logarithmic momentum regions, the former becomes instrumental for the determination of the relevant modes at subleading power; see e.g., Refs. [14,44].) A set of cutoffs are introduced that separate the scales and shape the phase space of the effective-theory modes. We refrain from using the dimensional regularization parameter as a cutoff as it is blind to the geometry of the modes. The variation of the effective-theory amplitudes under the variation of the cutoffs is then computed yielding a system of coupled evolution equations. Solving the evolution equations and removing the auxiliary cutoffs sums up the large logarithms of scale ratios and provides the asymptotic behavior of the amplitudes. The specific form of the differential operators, which results in a closed system of the evolution equations with the corresponding boundary conditions, is determined by the geometry of the effective-theory modes. The resulting equations are, in general, of the second order, as dictated by the two-dimensional nature of the infrared singularities of the on-shell amplitudes.

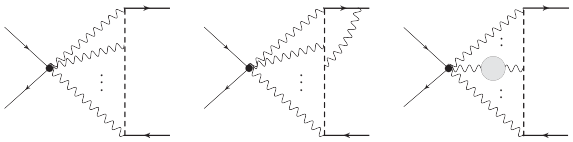
Let us now demonstrate how this program is realized for $e^+e^- \to \mu^+\mu^-$ forward annihilation or, equivalently, $e^-\mu^- \to e^-\mu^-$ backward scattering. We choose the reference frame in which, for vanishing electron mass, the electron (positron) momentum has only one light-cone component p^+ (p^-). In the high-energy limit the ratio of the fermion masses $m_{e,\mu}$ to the total energy $\sqrt{s}$ is small. The forward regime is defined for small scattering angles $\theta \lesssim m_\mu^2/s$ when the total momentum transfer scales as $(m_\mu^2 - m_e^2)/\sqrt{s}$. Thus the contribution of the forward region to the total annihilation cross section is power-suppressed by the final state phase space. In the forward regime the exponential suppression of the amplitude due to the double-logarithmic Sudakov corrections associated with the soft photon emission is absent but at the same time the amplitude develops characteristic non-Sudakov radiative corrections enhanced by the second power of the large logarithm $L = \ln(s/m_\mu^2)$ per each power of the fine structure constant α, associated with the soft fermion exchange. To account for the factorization of mass singularities, we write the amplitude of the process in the form $\mathcal{M} = \mathcal{Z}_{e\mu} A(z) \mathcal{M}_{\rm Born} + \cdots$, where α in the Born amplitude is renormalized at the physical scale $\sqrt{s}$ and the factor $\mathcal{Z}_{e\mu}$ describes the collinear renormalization of the external on-shell lines. In the leading approximation it is given by $\mathcal{Z}_{e\mu} = (s/(m_e m_\mu))^{(\alpha/\pi)\gamma_f^{(1)}}$, where $\gamma_f^{(1)} = 3/2$ is the one-loop collinear anomalous dimension. The form factor $A(z)$ is a function of the variable $z = \alpha L^2/2\pi$ and accommodates the double-logarithmic terms to all orders in α. These non-Sudakov double logarithms originate from the planar multiloop annihilation diagrams [2]. Because of the absence of momentum transfer, the electron propagator of each loop gets canceled, while the muon propagators effectively become scalar and sufficiently singular to produce the double-logarithmic corrections; see Fig. 1(a). The relevant effective theory with hard scale $\sqrt{s}$ and soft scale m_μ involves soft on-shell muons with loop momenta l_i and propagators $i\pi\delta(l_i^2 - m_\mu^2)$, and eikonal off-shell transverse photons with propagators $g_{kl}^\perp/(2l_i l_j)$. It is sufficient for the diagrammatic calculation and resummation of the double-logarithmic terms. We, however, reformulate the problem in the spirit of renormalization group flow, and introduce an auxiliary ultraviolet cutoff $\nu^+ < \sqrt{s}$ on the plus components of the light-cone momenta and an infrared cutoff $m_\mu^2/\sqrt{s} < \nu^- < \nu^+$ on the minus components. The form factor now becomes a function of the cutoffs $A(\xi, \eta)$, where we introduce the normalized logarithmic variables $\xi = \ln(\nu^+\sqrt{s}/m_\mu^2)/L, \eta = -\ln(\nu^-/\sqrt{s})/L$. The full-theory amplitude is then given by $A(z) = A(1,1)$ corresponding to the physical cutoff $\nu^+ = \sqrt{s}$, $\nu^- = m_\mu^2/\sqrt{s}$. The double-logarithmic scaling of the ℓ-loop diagrams requires the (anti)collinear $l_i^\perp \sim m_\mu$, $l_i^\pm \sim m_\mu^2/\sqrt{s}$, $l_i^\mp \sim \sqrt{s}$, or the soft $l_i^\perp \sim l_i^\pm \sim m_\mu$ scaling of the loop momenta components and the hierarchy $l_\ell^+ < \ldots l_1^+ < \nu^+, \nu^- < l_1^- < \ldots < l_\ell^-$. Hence, the second-order derivative with respect to ξ and η captures the ultraviolet and infrared singularities of the l_1 integral, and maps the ℓ-loop into the $(\ell-1)$-loop effective-theory diagram. Thus, the change of the all-order leading logarithmic form factor with the variation of the cutoffs is proportional to the form factor itself. This yields the following evolution equation:

$$\frac{\partial^2 A(\xi,\eta)}{\partial\xi\partial\eta} = z A(\xi,\eta), \tag{1}$$

with boundary conditions

$$A(\xi,0) = 1, \qquad \left.\frac{\partial A(\xi,\eta)}{\partial\eta}\right|_{\eta=\xi} = 0. \tag{2}$$

The first condition indicates that when the infrared cutoff approaches the physical hard scale $\nu^- = \sqrt{s}$, the effective-theory phase space shrinks to nothing and the amplitude



(a) (b) (c)

FIG. 1. (a) The effective all-order Feynman diagrams representing the leading logarithmic contribution. (b) An example of topology not contributing in the next-to-leading logarithmic approximation. (c) The effective all-order Feynman diagrams corresponding to the next-to-leading logarithmic contribution. The gray circle represents the one-loop single-logarithmic corrections to the annihilation amplitude.

reduces to the Born result. On the other hand, the pinch condition on the pole of the soft muon propagator $l_i^- > m_\mu^2/l_i^+$ sets a natural infrared cutoff m_μ^2/ν^+ on the l_1^- momentum component, which means that for $\nu^- \leq m_\mu^2/\nu^+$ corresponding to $\eta \geq \xi$, the amplitude does not depend on the infrared cutoff ν^-, which gives the second boundary condition. The solution of Eqs. (1) and (2) reads

$$A(\xi,\eta) = I_0\Big(2(z\eta\xi)^{1/2}\Big) - \frac{\eta}{\xi} I_2\Big(2(z\eta\xi)^{1/2}\Big), \qquad (3)$$

where $I_n(x)$ is the nth modified Bessel function. This recovers the leading logarithmic result [2]

$$A(z) = \frac{I_1(2z^{1/2})}{z^{1/2}} = 1 + \frac{z}{2} + \frac{z^2}{12} + \ldots, \qquad (4)$$

with $A(z) \sim (e^{2\sqrt{z}}/2\sqrt{\pi}z^{3/4})$ at $z \to \infty$.

The analysis readily generalizes beyond the leading-logarithmic order. First we note that in Fig. 1(a) every loop can admit the forward kinematics mandatory to produce the double-logarithmic scaling. At the same time, radiative corrections of the type exemplified in Fig. 1(b) destroy this kinematics when the extra loop momentum exceeds m_μ and, in this way, they reduce the power of the large logarithm by 2. Hence, the diagram Fig. 1(b) does not contribute to the next-to-leading running, which is instead completely determined by the diagrams of topology as in Fig. 1(c), with the gray circle representing the soft and collinear single-logarithmic terms in the one-loop forward annihilation amplitude with external momenta l_i and l_{i+1}. This simple factorization structure makes the calculation of the amplitude dependence on the cutoff at this order rather straightforward,

$$\frac{1}{z}\frac{\partial^2 A(\xi,\eta)}{\partial\xi\partial\eta} = \left[Z_\mu^2(\xi-\eta) + \frac{\alpha L}{2\pi}(\xi-\eta)(\gamma_\mu^{(1)} - 2)\right] \times A(\xi,\eta) + z\int_0^\eta d\eta' \int_{\eta'}^\xi d\xi' \delta(\xi-\eta') A(\xi',\eta') + \delta(\xi) A(0,0). \qquad (5)$$

In Eq. (5), $Z_\mu(\xi) = e^{(\alpha L/2\pi)\gamma_f^{(1)}\xi}$ is the leading collinear renormalization factor of the on-shell fermion field, and

$$\gamma_i^{(1)} = -\frac{m_i^2}{m_\mu^2 - m_e^2}\ln\left(\frac{m_\mu^2}{m_e^2}\right), \qquad i = e,\mu \qquad (6)$$

is the mass dependent part of the soft anomalous dimension, and the function

$$\delta(\xi) = \frac{\alpha(e^{\frac{L\xi}{2}} m_\mu)}{\alpha(m_\mu)} - 1 \qquad (7)$$

accommodates the leading effect of the fine structure constant running, with renormalization scale of α in the double-logarithmic variable z set to m_μ. A rather complex integrodifferential Eq. (5) can in principle be solved numerically retaining the all-order logarithmic result for the functions $Z_f(\xi)$ and $\delta(\xi,\eta)$. However, if we work in the strict next-to-leading logarithmic approximation, i.e., only keep the one-loop terms in these functions, Eq. (5) reduces to a second-order linear partial differential equation,

$$\frac{\partial^2 A(\xi,\eta)}{\partial\xi\partial\eta} = z\left[1 + \frac{\alpha L}{2\pi}(\xi-\eta) \times \left(2\gamma_f^{(1)} - \frac{\beta_0}{2} + \gamma_\mu^{(1)} - 2\right)\right] A(\xi,\eta), \qquad (8)$$

that can be solved perturbatively about the leading-logarithmic result [Eq. (3)]. The first boundary condition is now modified by matching to the one-loop amplitude

$$A(\xi,0) = 1 + \frac{\alpha L}{2\pi}\left(\gamma_e^{(1)} - 1\right), \qquad (9)$$

while the second kinematical boundary condition does not change. The final next-to-leading logarithmic result for the form factor reads

$$A(z) = \left[1 + \frac{\alpha L}{2\pi}\left(\gamma_e^{(1)} - 1\right)\right]\frac{I_1(2z^{1/2})}{z^{1/2}} + \frac{\alpha L}{2\pi}\left(2\gamma_f^{(1)} - \frac{\beta_0}{2} + \gamma_\mu^{(1)} - 2\right) k(z), \qquad (10)$$

where $\beta_0 = -\frac{4}{3}n_l$ is the QED beta function with $n_l = 2$ active flavors at the scale $\sqrt{s}$, and

$$k(z) = z\int_0^1 d\eta \int_\eta^1 d\xi (\xi-\eta) A(\xi,\eta) A(1-\eta, 1-\xi) = \frac{z}{6} + \frac{z^2}{20} + \frac{29}{5040} z^3 + \ldots, \qquad (11)$$

where $A(\xi,\eta)$ is given by Eq. (3), with the asymptotic behavior $k(z) \sim (e^{2\sqrt{z}}/4\sqrt{z})$ at $z \to \infty$. This calculation provides the $\mathcal{O}(\alpha^2)$ virtual corrections to the cross section of $e^+e^- \to \mu^+\mu^-$ annihilation in forward kinematics through the next-to-leading logarithms. Writing $d\sigma = \sum_{n=0}^{\infty}(\alpha/2\pi)^n d\sigma^{(n)}$ for the one- and two-loop terms we get

$$\frac{d\sigma^{(1)}}{d\sigma_{\rm Born}} = L^2 + \left(4 + \frac{4}{3}n_l\right)L, \tag{12}$$

$$\frac{d\sigma^{(2)}}{d\sigma_{\rm Born}} = \frac{5}{12}L^4 + \left(-\frac{1}{3}L_{\mu e} + \frac{13}{3} + \frac{14}{9}n_l\right)L^3, \tag{13}$$

where $L_{\mu e} = \ln(m_\mu^2/m_e^2)$, the coupling constant in the Born cross section is renormalized at m_μ, and we neglect the terms suppressed by powers of the electron to muon mass ratio. While Eqs. (12) and (13) are finite, the virtual corrections to the cross section become infrared divergent starting from the next-to-next-to-leading logarithmic approximation. To get the physical finite contribution to the cross section from the forward region, the real emission of soft photons with characteristic energy $\varepsilon \lesssim m_\mu^2/\sqrt{s}$ should be added, which factors out with respect to the virtual corrections discussed above. We note that for $m_e = m_\mu$ the photonic next-to-leading terms in Eqs. (12) and (13) change to $2L$ and $2L^3$, respectively, due to the mass dependence of the anomalous dimension [Eq. (6)]. Furthermore, in the equal mass case, the differential cross section for strictly forward annihilation is infrared finite beyond the logarithmic approximation since the soft emission vanishes identically along with the momentum transfer. To verify this analysis, we have analytically evaluated all two-loop terms surviving the high-energy limit for $m_e \ll m_\mu$ and $m_e = m_\mu$, building upon earlier work by some of us [49]. This allowed us to confirm the leading and next-to-leading logarithmic contribution given above. The full result for the forward amplitudes in the limit of the small electron mass is given in [50].

Let us show how the method extends to problems with quite a different dynamics. We start with the light-quark-mediated Higgs boson production and decays. A detailed analysis of the logarithmic corrections to the corresponding ggH amplitude through the next-to-leading logarithmic approximation can be found in [17,27]. The origin of the double logarithms in this case is quite different and involves a single chirality flipping soft fermion exchange accompanied by all-order Sudakov soft on-shell gluons; see Fig. 2. Nevertheless, we can apply the above formalism to derive the equations governing the renormalization group evolution of the amplitude. The hard and soft scales in the problem are given by the Higgs boson mass m_H and a light-quark mass $m_q \ll m_H$, respectively. We introduce independent infrared cutoffs $m_q^2/m_H < \nu^\pm < m_H$ on the light-cone momentum components with the corresponding

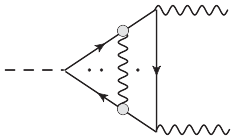


FIG. 2. The Feynman diagram with an arbitrary number of Sudakov gluon exchanges representing the leading logarithmic corrections to the light-quark-mediated $H \to gg$ amplitude. The gray circles represent the color-adjusted effective vertices.

normalized logarithmic variables $\xi = -\ln(\nu^+/m_H)/L$, $\eta = -\ln(\nu^-/m_H)/L$, where $L = \ln(m_H^2/m_q^2)$. It is convenient to define the form factor $A(z)$ normalized to 1 for the leading order one-loop mass-suppressed amplitude, $\mathcal{M} = \mathcal{Z}_g A(z)\mathcal{M}_{\rm LO} + \cdots$, where $z = \alpha_s L^2/2\pi$, α_s is the strong coupling constant $\alpha_s(\mu)$ renormalized at the scale $\mu = m_q$, and we have separated the standard Sudakov factor $\mathcal{Z}_g$ for the external on-shell gluon lines, which incorporates all the infrared divergences of the amplitude. In the effective theory, the form factor becomes a function $A(\xi,\eta)$ of the cutoff with $A(z) = A(1,1)$. By using the results of [17,27] on the factorization of the logarithmic regions, we compute its variation, which in the leading-logarithmic approximation yields the evolution equation

$$\frac{\partial^2 A(\xi,\eta)}{\partial\xi\partial\eta} = 2F(\xi,\eta,0), \tag{14}$$

where $F(\xi,\eta,\zeta)$ is the off-shell Sudakov form factor normalized to $F(\xi,\eta,\zeta)|_{z=0} = 1$. It is defined for the (anti)quark square momenta equal to $\nu^\pm m_H$, and is a function of $\zeta = -\ln(\nu^2/m_H^2)/L$ with $\nu^\pm < \nu < m_H$ being an ordinary hard momentum cutoff. The dependence of the form factor on the hard cutoff is well understood [12] and is determined by the equation

$$\frac{\partial \ln F(\xi,\eta,\zeta)}{\partial\zeta} = 4\tilde{\gamma}^{(1)}_{\rm cusp} z(\xi+\eta-2\zeta). \tag{15}$$

Here, $\tilde{\gamma}^{(1)}_{\rm cusp} = -C_F + C_A$, with $C_A = N_c$, $C_F = [(N_c^2-1)/2N_c]$, is the difference of the one-loop cusp anomalous dimension of the lightlike Wilson lines in the fundamental and adjoint representation of the SU(N_c) color group. This subtraction accounts for the color charge variation along the light-cone Wilson lines in the process of soft-quark emission, which is the physical origin of the double-logarithmic corrections to the amplitude. The evolution of the form factor is determined by the collinear, soft, and ultrasoft modes with the infrared scales set by the effective-theory cutoffs $\nu^\pm$. Thus, the form factor has a multiscale structure and its renormalization group analysis requires further scale separation and introduction of additional cutoffs [12], which disappear from the final evolution equation. Because of the scale hierarchy these auxiliary

cutoffs do not interfere with the renormalization group parameters ξ, η. The solution of Eq. (15) for $F(\zeta,\zeta,\zeta)=1$ is $F(\xi,\eta,\zeta)=e^{4\tilde{\gamma}^{(1)}_{\rm cusp} z(\xi-\zeta)(\eta-\zeta)}$. Then Eq. (14) can be integrated with the boundary conditions

$$A(\xi,0)=0, \qquad \left.\frac{\partial A(\xi,\eta)}{\partial\eta}\right|_{\eta=1-\xi}=0, \tag{16}$$

which can be derived in the same way as Eq. (2). The solution reads

$$A(\xi,\eta)=2\int_0^{\eta}\mathrm{d}\eta'\int_{1-\xi}^{1-\eta'}\mathrm{d}\xi' e^{4\tilde{\gamma}^{(1)}_{\rm cusp} z\eta'\xi'}, \tag{17}$$

and reproduces the well-known leading logarithmic result [17,20]

$$A(z)={}_2F_2\left(1,1;3/2,2;\tilde{\gamma}^{(1)}_{\rm cusp} z\right), \tag{18}$$

where ${}_2F_2(1,1;3/2,2;x)=1+(x/3)+\cdots$ is the hypergeometric function with the asymptotic behavior ${}_2F_2(1,1;3/2,2;x)\sim\sqrt{(\pi/x)}(e^x/2)$ at $x\to\infty$. As for Eq. (5), in the next-to-leading logarithmic approximation one includes the collinear renormalization of the quark fields and the renormalization group running of the strong coupling constant into the leading order equations. This results in the additional factor $Z_q(\xi)Z_q(\eta)=e^{(\alpha_s L/4\pi)\gamma_q^{(1)}(3\xi+3\eta-2)}$, $\gamma_q^{(1)}=3C_F/2$, on the right-hand side of Eq. (14), while the equation for the form factor becomes

$$\frac{\partial\ln F(\xi,\eta,\zeta)}{\partial\zeta}=4\tilde{\gamma}^{(1)}_{\rm cusp} z\int_{\zeta}^{\xi+\eta-\zeta}\mathrm{d}\zeta'\alpha_s(e^{-L\zeta'/2}m_H)+\frac{L\alpha_s}{2\pi}\gamma_q^{(1)}. \tag{19}$$

The above system of coupled equations can be solved numerically or perturbatively about the leading logarithmic solution Eq. (17), after expanding the collinear factor and the running coupling to the first order in α_s. The expansion reproduces the analytic all-order result obtained within the diagrammatic approach [27] and the numerical three-loop result [54].

Unlike the existing approaches based on dimensional regularization the method does not require additional rapidity regulators or the refactorization, and the above analysis naturally extends to the massive form factors [14,17], where the renormalization group description is yet to be formulated at subleading power in the small mass expansion. Finally we note that the method can be generalized to the subleading power dynamics involving Glauber modes (see, e.g., Ref. [36]). In this case the renormalization group variables are to be associated with the infrared cutoff on the light-cone components and an ultraviolet cutoff on the transverse components in momentum space.

In conclusion, we have developed a new approach to the analysis of the high-energy asymptotic behavior of scattering amplitudes at subleading power. It is based on multidimensional renormalization group flow described by (a system of) partial differential equations. It provides a powerful and adaptive tool applicable to a wide range of processes and kinematical regimes with a diverse spectrum of physical modes, from soft fermions to Glauber gauge bosons, that pose a serious challenge to existing techniques. We also present the $\mathcal{O}(\alpha^2)$ radiative corrections to the amplitude of $e^+e^-\to\mu^+\mu^-$ annihilation at high energy in the forward limit, which completes the QED analysis of the process at this order.

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