

Universal Precision Limits in General Open Quantum Systems

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The intuition that the precision of observables is constrained by thermodynamic costs has recently been formalized through thermodynamic and kinetic uncertainty relations. While such trade-offs have been extensively studied in Markovian systems, corresponding constraints in the non-Markovian regime remain largely unexplored. In this Letter, we derive universal bounds on the precision of generic observables in open quantum systems that interact with their environments at arbitrary coupling strengths and are subjected to two-point measurements. By introducing an asymmetry term that quantifies the disparity between forward and backward processes, we show that the relative fluctuation of any time-antisymmetric current is constrained not only by entropy production but also by this asymmetry. For general observables, we further prove that their relative fluctuation is always bounded from below by a generalized activity term that characterizes environmental changes. These results establish a comprehensive framework for understanding the fundamental limits of precision in a broad class of open quantum systems, beyond the traditional Markovian setting.

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Introduction—Understanding the fundamental limits of precision is essential for the development of quantum machines such as sensors, clocks, and heat engines. It is intuitive that high precision cannot be achieved for free and some cost must inevitably be paid. In recent years, this intuition has been rigorously formalized through a class of uncertainty relations, initially developed for classical Markov jump processes [1]. The thermodynamic uncertainty relation (TUR) states that achieving high precision of currents—quantified as the squared mean divided by the variance—requires a corresponding increase in dissipation [2–7]. Complementarily, the kinetic uncertainty relation (KUR) indicates that improving the precision of counting observables, defined in the same way, necessitates increased jump activity [8–10]. These relations not only reveal the costs associated with enhancing precision in small, fluctuating systems, but also carry significant implications for nonequilibrium physics [11–15].

In the quantum regime, uncovering trade-off relations between precision and thermodynamic costs becomes significantly more nontrivial due to uniquely quantum features such as coherence and entanglement. It has been

shown that classical uncertainty relations—namely, the TUR and KUR—can be violated in quantum systems [16–32]. For Markovian quantum dynamics, several extensions of these relations have been proposed, revealing how quantum coherence can enhance the precision of observables [33–46]. These studies demonstrate that quantum effects can relax classical bounds, enabling high precision even at low thermodynamic costs. In the context of fermionic transport, precision bounds for particle currents beyond the Markovian regime have also been explored, highlighting the crucial role of coherent dynamics in enhancing precision [47–49]. As a result, coherence and entanglement have been widely recognized as essential ingredients in modifying precision trade-offs. Nevertheless, the ultimate limits that quantify how thermodynamic costs and quantum features *jointly* constrain the precision of trajectory observables in the strong-coupling, non-Markovian regime remain largely unexplored. This gap motivates the development of precision bounds that apply to arbitrary system-environment couplings and extend beyond the limitations of Markovian approximations [50–52].

In this Letter, we address this gap by deriving universal precision bounds for generic observables in open quantum systems. We consider a general setup in which a system interacts with its environment at an arbitrary coupling strength, and two-point measurements are performed on the total system, yielding stochastic outcomes from which trajectory observables are defined (see Fig. 1). To capture dynamical factors relevant to precision, we introduce a novel quantity, termed the *forward-backward asymmetry*,

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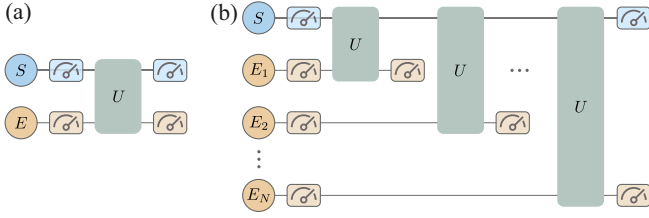


FIG. 1. Schematic illustration of general open quantum systems interacting with uncorrelated environments and subjected to two-point measurements. (a) The system and environment evolve under a single unitary transformation, with measurements performed at the initial and final times. (b) The system repeatedly interacts with fresh, uncorrelated environments, where each environment is projectively measured before and after the interaction.

which quantifies the disparity between forward and backward processes. Within a thermodynamically consistent framework, we prove that the relative fluctuation of any time-antisymmetric current is jointly constrained by entropy production and the forward-backward asymmetry [Eq. (8)]. This finding reveals the mechanism of precision enhancement in the quantum regime: currents can attain higher precision due to significant asymmetry, even at low dissipation. For general observables, we further show that their relative fluctuation is always bounded from below by a generalized activity term that characterizes environmental changes [Eq. (10)]. Notably, this result substantially tightens and extends previous findings to more general setups [50,53]. Taken together, these bounds generalize both the TUR and KUR to the strong-coupling quantum regime; remarkably, they are saturable and broadly applicable, and establish fundamental limits on the precision of observables in open quantum systems. We stress that these results are distinct from metrological bounds [54–56], which constrain the fluctuations of *trajectory-independent* estimators via the quantum Fisher information, a quantity rooted in the geometry of quantum state distinguishability [57].

Setup—We consider a finite-dimensional system S , which sequentially interacts with fresh, uncorrelated environments E over a total duration of T [58]. During each interaction, the composite system undergoes unitary evolution governed by the total Hamiltonian,

$$H = H_S + H_E + H_I, \quad (1)$$

where H_S , H_E , and H_I denote the system, environment, and interaction Hamiltonians, respectively. The coupling between the system and the environment can be arbitrarily strong. Two-point measurements are performed on both the system and the environment. The system is projectively measured only at the initial time $t = 0$ and the final time $t = T$, using an orthonormal basis $\{|n\rangle\}$, with outcomes n and m , respectively. The postmeasurement system state is $\rho_S = \sum_n p_n |n\rangle\langle n|$, where $\{p_n\}$ is the probability distribution over measurement outcomes. During each time interval

$[(i-1)\tau, i\tau]$, for $i = 1, 2, \dots, N$, the system is coupled to a new, uncorrelated environment initialized in a generic state $\rho_E = \sum_\mu p_\mu |\mu\rangle\langle\mu|$. The environment is measured before and after each interaction in the basis $\{|\mu\rangle\}$, yielding outcomes (ν_i, μ_i) . Note that the initial projective measurement on the environment does not disturb its state. A stochastic trajectory of measurement outcomes is denoted by $\gamma = \{n, (\nu_1, \mu_1), \dots, (\nu_N, \mu_N), m\}$. For $N = 1$, scenario Fig. 1(b) reduces to scenario Fig. 1(a). As shown later, this setup establishes a thermodynamically consistent framework that captures non-Markovian dynamics of finite-dimensional systems with arbitrarily engineered system-environment couplings; it also provides operational access to observables and recovers Markovian dynamics in the limit $\tau \rightarrow 0$ [59]. The reduced dynamics of the system for each interaction is described by a completely positive trace-preserving map,

$$\mathcal{E}(\circ) := \sum_{\mu,\nu} M_{\mu\nu}(\circ) M_{\mu\nu}^\dagger, \quad (2)$$

where the Kraus operators are defined as $M_{\mu\nu} = \sqrt{p_\nu} \langle\mu|U|\nu\rangle$, with U denoting the unitary operator generated by the total Hamiltonian H in Eq. (1). These operators satisfy the completeness condition $\sum_{\mu,\nu} M_{\mu\nu}^\dagger M_{\mu\nu} = \mathbb{1}_S$. The probability of observing the stochastic trajectory γ is given by

$$\mathbb{P}(\gamma) = p_n |\langle m | M_{\mu_N \nu_N} \dots M_{\mu_1 \nu_1} | n \rangle|^2. \quad (3)$$

We are interested in trajectory-dependent observables $\phi(\gamma)$, which include any time-integrated counting observables. Our goal is to elucidate the relationship between the relative fluctuation $\text{Var}[\phi]/\langle\phi\rangle^2$ and thermodynamic costs. Here, $\text{Var}[\phi] := \langle\phi^2\rangle - \langle\phi\rangle^2$ is the observable variance, and $\langle\circ\rangle$ denotes the ensemble average over all stochastic trajectories.

Next, we briefly describe the thermodynamics of open quantum systems and introduce several key quantities. To ensure thermodynamical consistency, we assume that the initial state of each environment is a thermal Gibbs state at inverse temperature β (i.e., $\rho_E = e^{-\beta H_E} / \text{tr} e^{-\beta H_E}$) [60]. Within the framework of quantum thermodynamics, entropy production—quantifying the degree of thermodynamic irreversibility—is defined as the sum of the von Neumann entropy change ΔS in the system and the heat Q dissipated into the environment [61]:

$$\Sigma = \Delta S + \beta Q. \quad (4)$$

For simplicity, we assume that the system is initially in a stationary state, i.e., $\rho_S = \mathcal{E}(\rho_S)$, as the generalization to arbitrary initial states is straightforward. Under this assumption, the system entropy change ΔS vanishes, and the entropy production reduces to heat dissipation and takes

the explicit form $\Sigma = ND(q'_{SE} \| q_S \otimes q_E)$ [62]. Here, $q'_{SE} = U(q_S \otimes q_E)U^\dagger$ represents the ensemble state of the composite system immediately after each interaction, and $D(\cdot \| \cdot)$ denotes the quantum relative entropy.

The thermodynamics of open quantum systems can also be formulated at the level of individual trajectories. To proceed, we define the time-reversed (backward) process using an antiunitary time-reversal operator $\Theta = \Theta_S \otimes \Theta_E$, which satisfies $\Theta i = -i\Theta$ and $\Theta\Theta^\dagger = \Theta^\dagger\Theta = \mathbb{1}$ [70]. The initial states of the system and environment in the backward process are $\Theta_S q_S \Theta_S^\dagger$ and $\Theta_E q_E \Theta_E^\dagger$, respectively, and projective measurements are performed in the time-reversed bases $\{\Theta_S |n\rangle\}$ and $\{\Theta_E |\mu\rangle\}$. The unitary operator $\tilde{U}$ governing the evolution in the backward process is generated by the time-reversed Hamiltonian $\Theta H \Theta^\dagger$. The probability of observing the time-reversed trajectory $\tilde{\gamma} = \{m, (\mu_N, \nu_N), \dots, (\mu_1, \nu_1), n\}$ in the backward process is given by

$$\tilde{\mathbb{P}}(\tilde{\gamma}) = p_m |\langle n | \Theta_S^\dagger \tilde{M}_{\nu_1 \mu_1} \dots \tilde{M}_{\nu_N \mu_N} \Theta_S | m \rangle|^2, \quad (5)$$

where the Kraus operators in the backward process are defined as $\tilde{M}_{\nu\mu} = \sqrt{p_\mu} \langle \nu | \Theta_E^\dagger \tilde{U} \Theta_E | \mu \rangle$. The entropy production Σ can then be expressed as the average logarithmic ratio between forward and backward trajectory probabilities:

$$\Sigma = \left\langle \ln \frac{\mathbb{P}(\gamma)}{\tilde{\mathbb{P}}(\tilde{\gamma})} \right\rangle. \quad (6)$$

This representation highlights Σ as a measure of time-reversal symmetry breaking due to thermodynamic dissipation. In addition to Σ , another key quantity relevant to time-reversal symmetry is the relative entropy between the forward and backward probabilities of the *same* trajectory, defined as

$$\Sigma_* := \left\langle \ln \frac{\mathbb{P}(\gamma)}{\tilde{\mathbb{P}}(\gamma)} \right\rangle \geq 0. \quad (7)$$

The quantity Σ_* is strictly positive whenever the forward and backward path probabilities, $\mathbb{P}(\gamma)$ and $\tilde{\mathbb{P}}(\gamma)$, differ, and it vanishes only when the two processes are identical (i.e., $\tilde{\mathbb{P}} \equiv \mathbb{P}$). In the presence of magnetic fields (i.e., odd variables under time reversal), the asymmetry $\mathbb{P} \neq \tilde{\mathbb{P}}$ generally holds, leading to a nonzero Σ_* (e.g., see Sec. S5 of Ref. [62]). Even without magnetic fields, Markovian dynamics that preserve quantum coherence in the system's state can also give rise to a positive asymmetry, as discussed later. More generally, we can prove that the ability of the dynamics to generate quantum entanglement between the system and its environment is a necessary condition for Σ_* to be nonzero [62]. Additionally, we show that Σ_* vanishes

in relevant cases, including incoherent Markovian dynamics and a class of thermal operations [62]. Thus, Σ_* quantifies the asymmetry between forward and backward processes, arising from dynamical features such as quantum coherence, entanglement, or external magnetic fields. Owing to both its definition and its physical interpretation, we refer to Σ_* as the forward-backward asymmetry. Together, Σ and Σ_* provide complementary characterizations of time-reversal symmetry breaking in open quantum systems: the former captures thermodynamic irreversibility, while the latter reflects intrinsic dynamical asymmetries.

Main results—With the key quantities defined above, we are now ready to present our main results; the proof is deferred to Appendix A. We begin by considering current-type observables that satisfy the time-antisymmetry condition $\phi(\tilde{\gamma}) = -\phi(\gamma)$. These include, but are not limited to, time-integrated currents commonly studied in conventional TUR formulations [3]. As our first main result, we prove that the relative fluctuation of any time-antisymmetric current is bounded from below by a function of both the entropy production Σ and the forward-backward asymmetry Σ_* :

$$\frac{\text{Var}[\phi]}{\langle \phi \rangle^2} \geq f(\Sigma + \Sigma_*). \quad (8)$$

Here, $f(x) = 4[\Phi(x/2)/x]^2 - 1 \in (0, +\infty)$ is a monotonically decreasing function, and Φ denotes the inverse function of $x \tanh(x)$. The relation (8) can be interpreted as a generalized quantum TUR. It implies that achieving high precision in current-type observables necessarily requires either high dissipation or significant forward-backward asymmetry. In other words, both thermodynamic irreversibility and dynamical asymmetry contribute to constraining fluctuations in open quantum systems. A generalization of this result to arbitrary initial states is provided in Ref. [62], where a boundary term arising from initial conditions is incorporated into the bound.

Several remarks on the result (8) are in order. First, the lower-bound function behaves as $f(x) \approx 2/x$ for $x \ll 1$, and decays exponentially to zero as $x \rightarrow \infty$. This reflects the fact that the bound is applicable not only to time-extensive but also to time-intensive currents. Second, the result highlights that the asymmetry Σ_* plays a role equally important to that of the entropy production Σ in constraining current precision. As demonstrated numerically later, the relative fluctuation of currents cannot, in general, be bounded solely by entropy production. We note that contributions to the asymmetry, such as magnetic fields and initial conditions, have classical counterparts, whereas genuinely quantum features like coherence and entanglement do not [cf. Eq. (S102) in Ref. [62]]. Third, in the case where $\Sigma_* = 0$ (i.e., the forward and backward processes coincide), the bound (8) reduces to a TUR previously derived from the detailed fluctuation theorem [71–73]. This

fluctuation-theorem TUR is known to be tight and saturable in certain cases, indicating that our generalized result inherits this desirable property. It is worth noting that another TUR, derived from the fluctuation theorem, also holds in our setup [74]. However, this relation involves the statistics of currents in the *backward* process, thus rendering it unsuitable for characterizing the ultimate precision limits of currents in the *forward* dynamics. Finally, Markovian dynamics can be recovered from our setup in the limit of short-time interactions (i.e., $\tau \rightarrow 0$) [75]. For Markovian processes governed by a Hamiltonian H and jump operators $\{L_k\}_k$ that satisfy the local detailed balance condition [76], the forward-backward asymmetry Σ_* can be lower bounded in the short-time limit as [62]

$$\Sigma_* \geq \frac{8}{9} \frac{|\langle [H, \mathbf{L}] \rangle_S|^2}{\langle 2H^2 + \mathbf{L}^2/2 \rangle_S} T^2, \quad (9)$$

where $\mathbf{L} := \sum_k L_k^\dagger L_k$ and $\langle \circ \rangle_S := \text{tr}(\circ \rho_S)$. This inequality confirms that $\Sigma_* > 0$ whenever H and $\mathbf{L}$ do not commute, a clear signature of quantum coherent dynamics. A general lower bound of Σ_* for finite times is presented in Ref. [62].

Next, we turn to the case of generic observables, without imposing any time-antisymmetry condition. Moreover, both the system and the environment may be initialized in arbitrary states, which need not be stationary or thermal Gibbs states, and projective measurements can be taken in any basis. Let $\mathcal{I}$ denote the subset of stochastic trajectories in which no change is detected in the environment, $\mathcal{I} = \{\gamma | \mu_i = \nu_i \ \forall i\}$. We impose a minimal condition on observables that $\phi(\gamma) = 0$ for any $\gamma \in \mathcal{I}$. This covers a wide range of observables, such as energy currents, particle transport, and the total number of quantum jumps detected in the environment. For this general setup, we prove the following lower bound on the relative fluctuation of any such observable:

$$\frac{\text{Var}[\phi]}{\langle \phi \rangle^2} \geq \frac{1}{\mathcal{P}^{-1} - 1}. \quad (10)$$

Here, $\mathcal{P}$ denotes the probability of detecting no environmental change, defined as $\mathcal{P} := \sum_{\gamma \in \mathcal{I}} \mathbb{P}(\gamma) < 1$. This constitutes our second main result, which holds under highly general conditions: for arbitrary initial states and system-environment interactions. Since $\mathcal{P}$ quantifies the inactivity of the environment, its inverse $\mathcal{P}^{-1}$ can be interpreted as an activity term. The bound (10) thus implies that achieving high precision requires correspondingly high activity. In the short-time limit of Markovian dynamics, the denominator $\mathcal{P}^{-1} - 1$ reduces exactly to the dynamical activity [62]. Therefore, the inequality (10) can be regarded as a generalized quantum KUR.

We provide several remarks on the result (10). First, the inequality is tight and can be saturated. Specifically, the equality holds for the observable ϕ that simply detects any

environmental change, i.e., $\phi(\gamma) = 1$ for all $\gamma \notin \mathcal{I}$ and zero otherwise. Second, although we focus on the case where $\mathcal{I}$ represents no-jump trajectories, the result remains valid for any subset $\mathcal{I}$ of trajectories, as long as the condition $\phi(\gamma) = 0$ for any $\gamma \in \mathcal{I}$ is fulfilled. This generality significantly broadens the applicability of the bound across a wide range of physical settings. Third, the inequality (10) generalizes and sharpens a pivotal result previously reported in Ref. [50]. In particular, when the environment is initialized in the pure state $|0\rangle\langle 0|$, it was shown that the precision of observables satisfies $\text{Var}[\phi]/\langle \phi \rangle^2 \geq 1/(\mathcal{A} - 1)$, where $\mathcal{A} := \text{tr}(V_0^\dagger V_0)^{-1} \rho_S$ is known as the survival activity [50] and $V_0 = M_{00}^N$. Using the inequality $|\text{tr}(AB)|^2 \leq \text{tr}(A^\dagger A) \text{tr}(B^\dagger B)$ and noting that $\mathcal{P} = \text{tr}(V_0^\dagger V_0) \rho_S$, we obtain

$$1 \leq \text{tr}\{(V_0^\dagger V_0)^{-1} \rho_S\} \text{tr}\{(V_0^\dagger V_0) \rho_S\} = \mathcal{A} \mathcal{P}, \quad (11)$$

which immediately yields $\mathcal{P}^{-1} \leq \mathcal{A}$. Therefore, the bound (10) is strictly tighter than the previous finding, and more importantly, it naturally extends to arbitrary initial states of the environment. Finally, in the case of Markovian dynamics with arbitrary Hamiltonian and jump operators, the inactivity $\mathcal{P}$ can be explicitly expressed as $\mathcal{P} = \text{tr}(e^{-iH_{\text{eff}} T} \rho_S e^{iH_{\text{eff}}^\dagger T})$, where $H_{\text{eff}} = H - (i/2) \sum_k L_k^\dagger L_k$ is the effective non-Hermitian Hamiltonian that describes no-jump evolution. In this setting, Ref. [53] derived an alternative precision bound using the Loschmidt echo approach: $\text{Var}[\phi]/\langle \phi \rangle^2 \geq 1/(\eta^{-1} - 1)$, where $\eta = |\text{tr}(e^{-iH_{\text{eff}} T} \rho_S)|^2$ is the Loschmidt echo. Applying the Cauchy-Schwarz inequality gives

$$\eta \leq \text{tr}\left(e^{-iH_{\text{eff}} T} \rho_S e^{iH_{\text{eff}}^\dagger T}\right) \text{tr}(\rho_S) = \mathcal{P}, \quad (12)$$

which implies that the result (10) is always sharper than the Loschmidt echo-based bound in this setting.

Example—We illustrate our results [Eqs. (8) and (10)] in a qubit system interacting with a finite-dimensional environment. The total Hamiltonian is given by

$$H = \frac{1}{2}(\omega_z \sigma_z + \omega_x \sigma_x) \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E + \lambda V_S \otimes V_E, \quad (13)$$

where $\sigma_{x,z}$ are the Pauli matrices, and λ denotes the coupling strength between the qubit and the environment. The environment may represent a heat bath or an ancillary qudit. Two-point measurements are performed on the total system only at the initial and final times (i.e., $N = 1$). In the strong-coupling regime (i.e., large λ), the system and the environment become significantly entangled, leading to the potential for precision enhancement and violation of conventional TURs.

To demonstrate the bounds, we fix the energy eigenstates of the environmental Hamiltonian H_E , while randomly

sampling its eigenvalues, as well as the matrix elements of the Hermitian operators V_S and V_E . Observables are defined as $\phi(\gamma) = c_{\mu\nu}$, where $\{\nu, \mu\}$ are the measurement outcomes on the environment at the initial and final times, and the coefficients $c_{\mu\nu}$ are also randomly chosen. For each instance of the parameter set, we compute the relative fluctuation of the observable, the entropy production Σ , the forward-backward asymmetry Σ_* , and the inactivity $\mathcal{P}$. The numerical results are summarized in Fig. 2.

For current-type observables that satisfy $c_{\mu\nu} = -c_{\nu\mu}$, Fig. 2(a) shows that their relative fluctuation is always bounded from below by $f(\Sigma + \Sigma_*)$. In contrast, significant violations occur when the forward-backward asymmetry Σ_* is ignored, demonstrating that fluctuations cannot, in general, be constrained solely by entropy production. For generic observables, Fig. 2(b) confirms that their relative fluctuation is always bounded from below by the inverse of the generalized activity term, $1/(\mathcal{P}^{-1} - 1)$, in agreement with Eq. (10). Both bounds are shown to be tight and saturable across the sampled parameter space.

We next investigate the relationship between the forward-backward asymmetry Σ_* and the quantum entanglement generated between the system and the environment. Since the composite system is initially in a pure product state after the projective measurements and subsequently evolves unitarily, quantum entanglement can be quantified via the entanglement entropy. Specifically, we define the average entanglement entropy as

$$\bar{S}_{\text{EE}} := \sum_{n,\nu} p_n p_\nu S_{\text{EE}}(q'_{n,\nu}), \quad (14)$$

where $q'_{n,\nu} = \text{tr}_E(U|n, \nu\rangle\langle n, \nu|U^\dagger)$ is the conditional system state at the final time and $S_{\text{EE}}(\rho) = -\text{tr}(\rho \ln \rho)$ is the von Neumann entropy. We vary the coupling strength λ while keeping other parameters fixed, and plot Σ_* , $\bar{S}_{\text{EE}}$, and the

quality factor $\mathcal{Q} := (\text{Var}[\phi]/\langle\phi\rangle^2)/f(\Sigma)$ as functions of λ in Fig. 2(c). Note that $\mathcal{Q} < 1$ indicates the violation of the conventional TUR [71–73] based solely on entropy production. As shown, both the forward-backward asymmetry Σ_* and the entanglement entropy $\bar{S}_{\text{EE}}$ monotonically increase with λ , while the quality factor $\mathcal{Q}$ significantly drops below 1. This supports that quantum entanglement contributes to the emergence of dynamical asymmetry and enables precision enhancement beyond the classical TUR.

Conclusion and outlook—In this Letter, we established universal precision bounds for both time-antisymmetric (current-type) and generic observables in general open quantum systems subjected to two-point measurement protocols. Our analysis reveals that, beyond the well-known role of dissipation, the forward-backward asymmetry serves as a fundamental factor limiting the precision of currents. We further anticipate that this notion of asymmetry will play an essential role in nonequilibrium thermodynamics more broadly, as evidenced in Ref. [62], where it is shown to govern a generalized TUR for underdamped Langevin systems. Meanwhile, for general observables, we showed that the generalized activity, which captures the underlying kinetic structure of quantum trajectories, imposes a tight constraint on their precision. These results offer a comprehensive and experimentally relevant framework for understanding the fundamental limits of precision in open quantum systems.

We conclude by highlighting several open directions for future research. A first important direction is to extend our framework to more realistic settings, including continuous-variable systems [77] and infinitely large environments [78], which are fundamental to quantum optics, condensed matter, and many-body physics. Establishing precision bounds in such regimes would clarify the universality of thermodynamic constraints on fluctuations. Second, it remains open whether structurally refined precision-cost

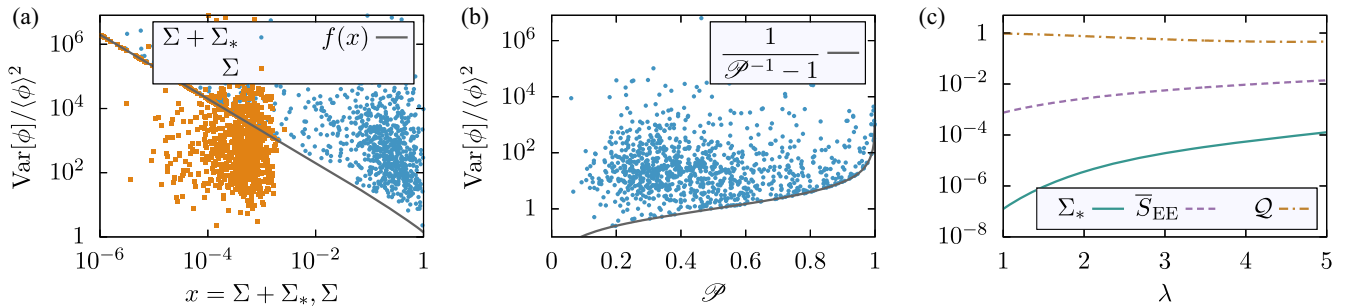


FIG. 2. Numerical illustration of the main results (8) and (10) for a qubit interacting with a finite-dimensional environment. (a) Blue circles and orange squares represent the relative fluctuation of current-type observables plotted against $\Sigma + \Sigma_*$ and Σ , respectively. The solid line depicts the lower-bound function $f(x)$. (b) Blue circles represent the relative fluctuation of generic observables plotted against $\mathcal{P}$. The solid line shows the lower bound $1/(\mathcal{P}^{-1} - 1)$. (c) Solid, dashed, and dash-dotted lines represent the asymmetry Σ_* , the entanglement entropy $\bar{S}_{\text{EE}}$, and the quality factor $\mathcal{Q}$, respectively, as functions of the coupling strength λ . Parameters are chosen as follows: the environment dimension d_E is a random integer in the range $[2, 5]$; the eigenvalues of the environmental Hamiltonian H_E are sampled from $[0, 0.1]$; V_S and V_E are random Hermitian operators with matrix elements in $[-1 - i, 1 + i]$; and fixed parameters are $\omega_z = 1$, $\omega_x = 0.1$, $\lambda = 5$, $\beta = 1$, and $T = 5$.

constraints emerge for widely studied observables, such as counting observables in conventional TUR and KUR formulations, potentially revealing a hierarchy of precision limits. Finally, extending our results to incorporate active quantum feedback control may provide guiding principles for the design of quantum clocks, sensors, and other precision devices [79].

Note added—Recently, we became aware that results related to Eq. (10) have been derived in Ref. [80] using the Petrov inequality and in Ref. [81] using a fluctuation theorem approach.

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Data availability—The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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End Matter

Appendix A: Proof of Eqs. (8) and (10)—For any observable ϕ that satisfies the time-antisymmetry condition, its first and second moments can be expressed as

$$\langle \phi \rangle = \frac{1}{2} \sum_{\gamma} \phi(\gamma) [\mathbb{P}(\gamma) - \mathbb{P}(\tilde{\gamma})], \quad (\text{A1})$$

$$\langle \phi^2 \rangle = \frac{1}{2} \sum_{\gamma} \phi(\gamma)^2 [\mathbb{P}(\gamma) + \mathbb{P}(\tilde{\gamma})]. \quad (\text{A2})$$

Applying the Cauchy-Schwarz inequality to these expressions yields

$$\begin{aligned} \langle \phi \rangle^2 &\leq \frac{1}{4} \sum_{\gamma} \frac{[\mathbb{P}(\gamma) - \mathbb{P}(\tilde{\gamma})]^2}{\mathbb{P}(\gamma) + \mathbb{P}(\tilde{\gamma})} \sum_{\gamma} \phi(\gamma)^2 [\mathbb{P}(\gamma) + \mathbb{P}(\tilde{\gamma})] \\ &= \ell \langle \phi^2 \rangle, \end{aligned} \quad (\text{A3})$$

where we define

$$\ell := \frac{1}{2} \sum_{\gamma} \frac{[\mathbb{P}(\gamma) - \mathbb{P}(\tilde{\gamma})]^2}{\mathbb{P}(\gamma) + \mathbb{P}(\tilde{\gamma})}. \quad (\text{A4})$$

From Eq. (A3), we obtain the following lower bound on the relative fluctuation of the observable ϕ :

$$\frac{\text{Var}[\phi]}{\langle \phi \rangle^2} \geq \ell^{-1} - 1. \quad (\text{A5})$$

We only need to upper bound ℓ in terms of the entropy production Σ and the forward-backward asymmetry Σ_* . To this end, note that the entropy production can be calculated as

$$\begin{aligned} \Sigma &= \sum_{\gamma} \mathbb{P}(\gamma) \ln \frac{\mathbb{P}(\gamma)}{\tilde{\mathbb{P}}(\tilde{\gamma})} \\ &= \sum_{\gamma} \mathbb{P}(\gamma) \ln \frac{\mathbb{P}(\gamma)}{\mathbb{P}(\tilde{\gamma})} - \sum_{\gamma} \mathbb{P}(\gamma) \ln \frac{\tilde{\mathbb{P}}(\tilde{\gamma})}{\mathbb{P}(\tilde{\gamma})} \\ &= \sum_{\gamma} \mathbb{P}(\gamma) \ln \frac{\mathbb{P}(\gamma)}{\mathbb{P}(\tilde{\gamma})} - \sum_{\gamma} \mathbb{P}(\gamma) \ln \frac{\mathbb{P}(\gamma)}{\tilde{\mathbb{P}}(\tilde{\gamma})} \\ &= \sum_{\gamma} \mathbb{P}(\gamma) \ln \frac{\mathbb{P}(\gamma)}{\mathbb{P}(\tilde{\gamma})} - \Sigma_*. \end{aligned} \quad (\text{A6})$$

Here, we use the equality $\tilde{\mathbb{P}}(\tilde{\gamma})/\mathbb{P}(\tilde{\gamma}) = \mathbb{P}(\gamma)/\tilde{\mathbb{P}}(\gamma)$ (see Appendix B) to get the third line. Therefore, we obtain

$$\begin{aligned} \Sigma + \Sigma_* &= \sum_{\gamma} \mathbb{P}(\gamma) \ln \frac{\mathbb{P}(\gamma)}{\mathbb{P}(\tilde{\gamma})} \\ &= \frac{1}{2} \sum_{\gamma} [\mathbb{P}(\gamma) - \mathbb{P}(\tilde{\gamma})] \ln \frac{\mathbb{P}(\gamma)}{\mathbb{P}(\tilde{\gamma})}. \end{aligned} \quad (\text{A7})$$

To proceed, we define

$$\sigma(\gamma) := [\mathbb{P}(\gamma) - \mathbb{P}(\tilde{\gamma})] \ln \frac{\mathbb{P}(\gamma)}{\mathbb{P}(\tilde{\gamma})}, \quad (\text{A8})$$

$$a(\gamma) := \mathbb{P}(\gamma) + \mathbb{P}(\tilde{\gamma}). \quad (\text{A9})$$

Evidently, these quantities satisfy the relations $\sum_{\gamma} \sigma(\gamma) = 2(\Sigma + \Sigma_*)$ and $\sum_{\gamma} a(\gamma) = 2$. We now derive an upper bound on ℓ in terms of $\Sigma + \Sigma_*$. By algebraic manipulations and an application of Jensen's inequality, we can show that

$$\begin{aligned} \ell &= \frac{1}{2} \sum_{\gamma} \frac{[\mathbb{P}(\gamma) - \mathbb{P}(\tilde{\gamma})]^2}{\mathbb{P}(\gamma) + \mathbb{P}(\tilde{\gamma})} \\ &= \frac{1}{2} \sum_{\gamma} \frac{\sigma(\gamma)^2}{4a(\gamma)} \Phi \left[\frac{\sigma(\gamma)}{2a(\gamma)} \right]^{-2} \\ &\leq \frac{1}{4} (\Sigma + \Sigma_*)^2 \Phi \left(\frac{\Sigma + \Sigma_*}{2} \right)^{-2}. \end{aligned} \quad (\text{A10})$$

Here, we exploit the convexity of $(x^2/y)\Phi(x/2y)^{-2}$ to obtain the last line. Combining Eqs. (A5) and (A10) leads to the desired relation (8):

$$\frac{\text{Var}[\phi]}{\langle \phi \rangle^2} \geq f(\Sigma + \Sigma_*). \quad (\text{A11})$$

According to Proposition 1 in Ref. [62], the lower-bound function can also be alternatively expressed as

$$f(\Sigma + \Sigma_*) = \text{csch}^2 \left[\Phi \left(\frac{\Sigma + \Sigma_*}{2} \right) \right] \geq \frac{2}{e^{\Sigma + \Sigma_*} - 1}. \quad (\text{A12})$$

Next, we prove the relation (10). Since $\phi(\gamma) = 0$ for any $\gamma \in \mathcal{I}$, the first and second moments of generic observable ϕ can be calculated as

$$\langle \phi \rangle = \sum_{\gamma \notin \mathcal{I}} \phi(\gamma) \mathbb{P}(\gamma), \quad (\text{A13})$$

$$\langle \phi^2 \rangle = \sum_{\gamma \notin \mathcal{I}} \phi(\gamma)^2 \mathbb{P}(\gamma). \quad (\text{A14})$$

Applying the Cauchy-Schwarz inequality, the first moment can be upper bounded by the second moment as

$$\begin{aligned} \langle \phi \rangle^2 &= \left[\sum_{\gamma \notin \mathcal{I}} \phi(\gamma) \mathbb{P}(\gamma) \right]^2 \\ &\leq \left[\sum_{\gamma \notin \mathcal{I}} \mathbb{P}(\gamma) \right] \left[\sum_{\gamma \notin \mathcal{I}} \phi(\gamma)^2 \mathbb{P}(\gamma) \right] \\ &= (1 - \mathcal{P}) \langle \phi^2 \rangle. \end{aligned} \quad (\text{A15})$$

By transforming Eq. (A15), the bound (10) can be readily obtained.

Appendix B: Proof of $\tilde{\mathbb{P}}(\tilde{\gamma})/\mathbb{P}(\tilde{\gamma}) = \mathbb{P}(\gamma)/\tilde{\mathbb{P}}(\gamma)$ — Noting that $\tilde{\gamma} = \{m, (\mu_N, \nu_N), \dots, (\mu_1, \nu_1), n\}$, the path probabilities can be explicitly expressed as follows:

$$\mathbb{P}(\gamma) = p_n |\langle m | M_{\mu_N \nu_N} \dots M_{\mu_1 \nu_1} | n \rangle|^2, \quad (\text{B1})$$

$$\mathbb{P}(\tilde{\gamma}) = p_m |\langle n | M_{\nu_1 \mu_1} \dots M_{\nu_N \mu_N} | m \rangle|^2, \quad (\text{B2})$$

$$\tilde{\mathbb{P}}(\tilde{\gamma}) = p_m |\langle n | \Theta_S^\dagger \tilde{M}_{\nu_1 \mu_1} \dots \tilde{M}_{\nu_N \mu_N} \Theta_S | m \rangle|^2, \quad (\text{B3})$$

$$\tilde{\mathbb{P}}(\gamma) = p_n |\langle m | \Theta_S^\dagger \tilde{M}_{\mu_N \nu_N} \dots \tilde{M}_{\mu_1 \nu_1} \Theta_S | n \rangle|^2. \quad (\text{B4})$$

In addition, the Kraus operators in the backward process are related to those in the forward process as

$$\begin{aligned} \Theta_S^\dagger \tilde{M}_{\mu \nu} \Theta_S &= \sqrt{p_\nu} \Theta_S^\dagger \langle \mu | \Theta_E^\dagger \tilde{U} \Theta_E | \nu \rangle \Theta_S \\ &= \sqrt{p_\nu} \langle \mu | U^\dagger | \nu \rangle \\ &= \sqrt{p_\nu / p_\mu} M_{\nu \mu}^\dagger. \end{aligned} \quad (\text{B5})$$

Here, we use the fact $\tilde{U} = \Theta U^\dagger \Theta^\dagger$ to obtain the second line. Using this relation, the path probabilities can be simplified further as follows:

$$\begin{aligned} \tilde{\mathbb{P}}(\tilde{\gamma}) &= p_m |\langle n | \Theta_S^\dagger \tilde{M}_{\nu_1 \mu_1} \dots \tilde{M}_{\nu_N \mu_N} \Theta_S | m \rangle|^2 \\ &= p_m \frac{p_{\mu_1} \dots p_{\mu_N}}{p_{\nu_1} \dots p_{\nu_N}} |\langle n | M_{\mu_1 \nu_1}^\dagger \dots M_{\mu_N \nu_N}^\dagger | m \rangle|^2 \\ &= p_m \frac{p_{\mu_1} \dots p_{\mu_N}}{p_{\nu_1} \dots p_{\nu_N}} |\langle m | M_{\mu_N \nu_N} \dots M_{\mu_1 \nu_1} | n \rangle|^2 \\ &= \frac{p_m}{p_n} \frac{p_{\mu_1} \dots p_{\mu_N}}{p_{\nu_1} \dots p_{\nu_N}} \mathbb{P}(\gamma), \end{aligned} \quad (\text{B6})$$

$$\begin{aligned} \tilde{\mathbb{P}}(\gamma) &= p_n |\langle m | \Theta_S^\dagger \tilde{M}_{\mu_N \nu_N} \dots \tilde{M}_{\mu_1 \nu_1} \Theta_S | n \rangle|^2 \\ &= p_n \frac{p_{\nu_1} \dots p_{\nu_N}}{p_{\mu_1} \dots p_{\mu_N}} |\langle m | M_{\nu_N \mu_N}^\dagger \dots M_{\nu_1 \mu_1}^\dagger | n \rangle|^2 \\ &= p_n \frac{p_{\nu_1} \dots p_{\nu_N}}{p_{\mu_1} \dots p_{\mu_N}} |\langle n | M_{\nu_1 \mu_1} \dots M_{\nu_N \mu_N} | m \rangle|^2 \\ &= \frac{p_n}{p_m} \frac{p_{\nu_1} \dots p_{\nu_N}}{p_{\mu_1} \dots p_{\mu_N}} \mathbb{P}(\tilde{\gamma}). \end{aligned} \quad (\text{B7})$$

Therefore, we can verify that $\tilde{\mathbb{P}}(\tilde{\gamma})\tilde{\mathbb{P}}(\gamma) = \mathbb{P}(\gamma)\mathbb{P}(\tilde{\gamma})$, which immediately derives $\tilde{\mathbb{P}}(\tilde{\gamma})/\mathbb{P}(\tilde{\gamma}) = \mathbb{P}(\gamma)/\tilde{\mathbb{P}}(\gamma)$.