Editors' Suggestion

All-Order Structure of Static Gravitational Interactions and the Seventh Post-Newtonian Potential

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We present a closed formula for the computation of static post-Newtonian corrections to the two-body gravitational dynamics at any odd order, assuming the lower-order results are known. The formula is derived within a correlation function framework and exploits the $\mathbb{Z}_2$ symmetry of the static sector, leading to a novel theoretical interpretation of the factorization theorem. As an application, we compute the gravitational interaction of two compact coalescing objects at the seventh post-Newtonian order in the static limit, which receives contributions from seven-loop graphs at order $\mathcal{O}(G_N^8 v^0)$, and find complete agreement with the results obtained using the diagrammatic approach of the factorization theorem.

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Introduction—The gravitational dynamics of coalescing binary systems is one of the most compelling problems in fundamental physics. It underlies the generation of gravitational waves, which encode information about the internal structure of compact objects before and after coalescence. Precise theoretical predictions for such systems offer a unique opportunity to understand the distributions of black hole masses and spins [1], constraints on the neutron-star equation of state [2], measurements of the Hubble-Lemaître parameter [3–5], and constraints on the theory of general relativity [6–9]. Systematic biases are already visible in the current gravitational-wave analyses [10–12], which call for ever more precise theoretical templates of the binary dynamics and emitted waveform.

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The modeling of compact binary dynamics relies on a combination of complementary approaches. Numerical relativity [13–15] is required in the strongly nonlinear merger regime, whereas the inspiral phase admits a systematic analytic treatment within the post-Newtonian (PN) expansion [16–30], valid for slowly moving and widely separated bodies. For bound binaries, observables are organized as a series in powers of v/c, and as a result, at kPN rder one encounters contributions of the form $G_N^l (v^2)^{k+1-l}$, with $0 \leq l \leq k+1$, where G_N is the Newton constant. The post-Minkowskian approximation [31–70] and the gravitational self-force (SF) framework [71–84] further extend the analytic description to weak-field configurations and small mass ratios. The most commonly used approaches to build complete inspiral-merger-ringdown waveform models of compact binaries are the phenomenological methods [85–87] and effective-one-body formalism [88–92].

The PN program began with the 1PN results of Refs. [16,17], and higher-order calculations have required increasingly sophisticated analytical and computational techniques. The current state of the art includes conservative dynamics at 4PN order [24–28] and radiation at 4.5PN order [29,30]. Pushing beyond this frontier is necessary to match the accuracy demanded by current and future gravitational-wave detectors [10,12,93,94].

Major progress in the past decade has been achieved within the effective field theory (EFT) approach [95],

which reformulates the PN expansion in terms of classical Feynman diagrams for conservative dynamics [96–120] and radiative emission [121–127]. Combined with modern multiloop methods [99,128], this framework led to the complete conservative dynamics at 4PN [98–100], and recently, at 5PN [102–104,107–109,129,130]. At 6PN, the order $O(G_N^4)$ contributions [105], partial radiative contributions [131], and further local and nonlocal dynamical information [132–135] have been computed.

In this approach, PN calculations are equivalent to evaluating multiloop massless two-point Feynman integrals, with the static sector corresponding to the highest-loop diagrams at each PN order. Very recently, this perspective enabled us to compute the static two-body effective interaction potential at the order $\mathcal{O}(G_N^7 v^0)$ (6PN) [136] using the development of novel techniques and computational tools for high-loop integrals.

In this Letter, we report two main results. First, we derive a closed formula for static PN corrections at arbitrary PN order in terms of gravitational correlation functions. This leads to a novel theoretical interpretation for the odd-PN diagrammatic factorization theorem observed in Ref. [102] as a consequence of an accidental symmetry of the static sector. Second, we obtain the first result for the static two-body effective interaction at order $\mathcal{O}(G_N^8 v^0)$, corresponding to 7PN. The 7PN result is derived independently using both the earlier diagrammatic factorization and the present formalism, and the two computations agree exactly. In addition to extending the PN-EFT frontier to seven loops, these results reveal a simple all-order structure in the static sector and open a direct route to further higher orders.

Conventions: The mostly minus metric, with signature $(+,-,-,-)$, is employed throughout this Letter, and multiloop integrals are defined within the dimensional regularization scheme, considering $d = 3 + \epsilon$ spatial dimensions, with ϵ being the analytic regulator.

Gravitational dynamics in the static sector—The conservative dynamics of a binary system of coalescing compact objects is characterized by a well-defined hierarchy of scales: the size of the compact objects R_s is much smaller than the orbital separation r, which, in turn, is much smaller than the wavelength of the emitted gravitational radiation λ: $R_s \ll r \ll \lambda$. The first inequality defines a weak-coupling expansion in the ratio R_s/r, while the second corresponds to a small-velocity expansion, since $r/\lambda \sim v \ll 1$.

Within the post-Newtonian approximation, the expansion parameter is the typical velocity v of the binary system, which is related to the gravitational potential through the virial theorem, $v^2 \sim G_N M/r$. Accordingly, one has $G_N M/r \sim v^2 \ll 1$, where $M = m_1 + m_2$ denotes the total mass of the system.

In this Letter, we focus on the static contributions to the gravitational potential, corresponding to terms of order $\mathcal{O}(G_N^{k+1} v^0)$ at the kth PN order. We adopt the effective field theory framework for the PN expansion, as introduced in [95], and analyze the all-order structure in terms of Feynman diagrams and correlation functions.

The bulk action governing the gravitational dynamics is given by the gauge-fixed Einstein-Hilbert action:

$$S_{\rm bulk} = S_{\rm EH} + S_{\rm GF} = -\frac{c^4}{16\pi G_d}\int \mathrm{d}^{d+1}x\sqrt{|g|}\left(R - \frac{1}{2}g_{\mu\nu}\Gamma^\mu\Gamma^\nu\right), \quad (1)$$

where $G_d = G_N(\sqrt{4\pi e^{\gamma_E}}R_0)^{(d-3)}$ is the d-dimensional gravitational coupling, with γ_E the Euler-Mascheroni constant and R_0 an arbitrary length scale, with the gauge-fixing term enforcing harmonic gauge ($\Gamma^\mu = g^{\nu\rho}\Gamma^\mu_{\nu\rho} = 0$). The compact objects are described in terms of point-particle worldline actions:

$$S_{\rm pp} = \sum_{a=1,2}\int \mathrm{d}\tau\left(-m_a c\sqrt{g_{\mu\nu}u^\mu_{(a)}u^\nu_{(a)}}\right), \quad (2)$$

where $u^\mu_{(a)} = \mathrm{d}x^\mu_{(a)}/\mathrm{d}\tau$ is the four-velocity. In the weak field approximation, we expand the metric around the Minkowski background, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ the graviton field. The nonrelativistic limit splits the graviton field into potential modes $H_{\mu\nu}$, which are short range and off shell and scale as $(k_0, \mathbf{k}) \sim (v/r, 1/r)$, and radiation modes $\bar{h}_{\mu\nu}$, which are long range and on shell and scale as $(k_0, \mathbf{k}) \sim (v/r, v/r)$. In the following, we focus on the conservative static contributions to the potential, which are sourced by the potential modes, and therefore restrict attention to the $H_{\mu\nu}$ field.

We decompose the potential gravitons $H_{\mu\nu}$ using the Kaluza-Klein (KK) decomposition of the metric ($g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}$) introduced in [137,138] as

$$g_{\mu\nu} = e^{2\boldsymbol{\phi}/c^2}\begin{pmatrix} 1 & -\boldsymbol{A}_j/c^2 \\ -\boldsymbol{A}_i/c^2 & -e^{-c_d\boldsymbol{\phi}/c^2}\boldsymbol{\gamma}_{ij} + \boldsymbol{A}_i\boldsymbol{A}_j/c^4 \end{pmatrix}, \quad (3)$$

where $c_d = 2(d-1)/(d-2)$ and $\boldsymbol{\gamma}_{ij} = \delta_{ij} + \boldsymbol{\sigma}_{ij}/c^2$. This parametrization rewrites the metric in terms of a scalar field $\boldsymbol{\phi}$, a d-dimensional vector $\boldsymbol{A}_i$, and a d-dimensional symmetric tensor $\boldsymbol{\sigma}_{ij}$. Substituting the parametrization into the bulk action (1), and expanding it in the static limit, the $\boldsymbol{A}_i$ field appears only quadratically, and therefore, contributes only through quantum graviton loop corrections, which are discarded in the classical limit.

The action in the static limit reduces to [139]

$$S^{(v^0)}_{\rm bulk} = -\frac{c^4}{16\pi G_d}\int \mathrm{d}^{d+1}x\sqrt{\boldsymbol{\gamma}}\left(\frac{c_d}{2c^4}\boldsymbol{\gamma}^{ij}\partial_i\boldsymbol{\phi}\partial_j\boldsymbol{\phi} - R[\boldsymbol{\gamma}] + \frac{1}{2}|\Gamma_i[\boldsymbol{\gamma}]|^2\right). \quad (4)$$

Analogously, substituting the KK variables in the point-particle action, and imposing the static limit $u_a^\mu = (1, \mathbf{0})$, we obtain

$$S_{\rm pp}^{(v^0)} = \sum_{a=1,2} \int \mathrm{d}t \left(-m_a c^2 e^{\boldsymbol{\phi}(x_{(a)})/c^2}\right), \tag{5}$$

where only the $\boldsymbol{\phi}$ field appears.

An effective action that depends only on the worldline parameters can be obtained by integrating out the potential gravitational degrees of freedom of the theory. In the static limit, this reads as

$$\begin{aligned} Z[\{\boldsymbol{x}_{(a)}^\mu\}] &= e^{\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t(\mathcal{T}-\mathcal{V}_{\rm eff})(\{\boldsymbol{x}_{(a)}^\mu\})} \\ &= \frac{1}{Z_0}\int \mathrm{D}[\boldsymbol{\phi}, \boldsymbol{\sigma}_{ij}] e^{\frac{\mathrm{i}}{\hbar}(S_{\rm pp}+S_{\rm bulk})}, \end{aligned} \tag{6}$$

with $S_{\rm pp} = S_{\rm pp}[\{\boldsymbol{x}_a^\mu\}, \boldsymbol{\phi}, \boldsymbol{\sigma}_{ij}]$, $S_{\rm bulk} = S_{\rm bulk}[\boldsymbol{\phi}, \boldsymbol{\sigma}_{ij}]$, and

$$Z_0 = \int \mathrm{D}[\boldsymbol{\phi}, \boldsymbol{\sigma}_{ij}] e^{\frac{\mathrm{i}}{\hbar}S_{\rm bulk}}. \tag{7}$$

In Eq. (6), $\mathcal{T}$ denotes the kinetic term density, while $\mathcal{V}_{\rm eff}$ is the effective potential density. Throughout, we use calligraphic letters to denote quantities defined at the level of the time integrand. The functional integration can be performed perturbatively, expressing the effective action in terms of one-particle irreducible classical connected Feynman diagrams:

$$\mathcal{V}_{\rm eff} = \mathbf{i} \lim_{d\to 3} \int \frac{d^d\mathbf{p}}{(2\pi)^d}\, e^{\mathbf{i}\mathbf{p}\cdot(\mathbf{x}_{(1)}-\mathbf{x}_{(2)})} \times$$  . (8)

In the static sector, the time dependence trivializes; hence, here and in the remainder of this Letter, we assume the overall time integral is understood. We also set $c = 1$ for convenience.

Factorization from Feynman graphs—As proved in Ref. [102], static diagrams at odd PN orders can be factorized in terms of products of lower PN order diagrams. It is useful to distinguish between *prime* and *factorizable* static graphs. A prime graph (p.g.) contains only linear matter–$\boldsymbol{\phi}$ couplings of the form $m\boldsymbol{\phi}$, namely, each $\boldsymbol{\phi}$ field emitted from the bulk couples individually to a worldline. Instead, a factorizable graph contains at least one composite worldline vertex $m\boldsymbol{\phi}^n$ with $n > 1$, and can, therefore, be obtained by sewing together two or more lower-order subgraphs at that vertex (see Fig. 1 of Ref. [102] for examples). Reference [102] proves that static prime graphs exist only at even PN orders; equivalently, all odd-PN static graphs are factorizable. At 7PN, therefore, no new prime static graph appears. The full answer is obtained by combining lower-order even-PN building blocks. As in the 5PN analysis of Ref. [102], the factorized contributions are organized according to their sewing pattern. Importantly, each factorized graph is not given by a simple product of lower-order contributions: when two subgraphs are sewn together, one must also include the factor associated with the new composite worldline vertex and the appropriate combinatorial weight. By the factorization theorem, the contribution of a factorizable graph $\mathcal{G}$ to the kPN potential can be expressed in terms of the product of contributions from lower-PN order subgraphs $\mathcal{G}_i$ as

$$\mathcal{V}_k^{\mathcal{G}} = \mathcal{C}_k \prod_{k_i} \frac{\mathcal{V}_{k_i}^{\mathcal{G}_i}}{\mathcal{C}_{k_i}}, \tag{9}$$

with $\sum_i k_i = k - 1$, where $\mathcal{C}_k$ and $\mathcal{C}_{k_i}$ are the symmetry factors of $\mathcal{G}$ and $\mathcal{G}_i$, respectively, as proposed in [102]. At the considered 7PN order, we remark that the determination of $\mathcal{C}_k$ requires the generation of the complete set of 3842 contributing 7-loop diagrams computed using PNTHR: Post-Newtonian Toolkit for Hamiltonian and Radiation [140]. Thus, owing to Eq. (9), the 7PN static contribution is obtained by summing all admissible sewings of lower-order even-PN prime sectors, with the appropriate $\mathcal{C}$ factors included. Accordingly, the 7PN correction to the effective potential is organized into six factorization classes:

$$\begin{aligned} \mathcal{V}_{\rm 7PN}^{G_N^8} = {} & \mathcal{V}_{\rm 0PN\times 6PN} + \mathcal{V}_{\rm 2PN\times 4PN} + \mathcal{V}_{(\rm 0PN)^3\times 4PN} \\ & + \mathcal{V}_{(\rm 0PN)^2\times(2PN)^2} + \mathcal{V}_{(\rm 0PN)^5\times 2PN} + \mathcal{V}_{(\rm 0PN)^8}, \end{aligned} \tag{10}$$

and can be shown graphically as

$\mathcal{V}_{\rm 7PN}^{G_N^8} =$

(11)

where the black, dashed, and grey lines represent the massive bodies and the ϕ and σ fields, respectively. The number of diagrams in each factorized class is reported in Table I and grouped according to their nth-order self-force (nSF) contribution, which corresponds to the terms proportional to $m_1^{8-n} m_2^{1+n}$ [141].

The prime graphs appearing on the right-hand side of Eqs. (10) and (11) reduce to Euclidean multiloop massless two-point integrals in $d = 3 + \epsilon$ dimensions [99], with even loop order $\ell = \{0, 2, 4, 6\}$. The topologies required up to $V_{\rm 6PN}$ were already computed in Ref. [136], so the

TABLE I. Number of diagrams contributing to the 7PN static potential, displayed according to the self-force contribution (rows) and grouped in factorization classes (column) shown in (10), where the number in the product denotes the diagrams at a PN order.

SF	0×6	2×4	$0^3\times 4$	$0^2\times 2^2$	$0^5\times 2$	0^8	Diagrams
0	28	5	5	1	1	1	41
1	218	52	80	23	20	4	397
2	486	135	354	136	117	14	1242
3	702	203	672	288	269	28	2162
Total	1434	395	1111	448	407	47	3842

present calculation can be carried out within the same computational framework. The corresponding integration-by-parts (IBP) reductions [142–144] are performed analytically with the code PRISM [145], combining spanning cuts [146–148], syzygy-based methods [149–152] implemented with Singular [153], improved seeding algorithms [150,151,154], and finite-field reconstruction through FiniteFlow [155]. Analytic expressions for the relevant master integrals at 4PN and 6PN can be found, respectively, in Refs. [99,136].

Factorization from correlation functions—Building on a proposed computational framework based on correlation functions [156], we derive the effective static potential in terms of connected gravitational correlators. We show that the factorization theorem is a consequence of the $\mathbb{Z}_2$ symmetry of the static sector.

Correlation functions framework: The effective action of the system can be evaluated as the logarithm of the full path integral in Eq. (6). The latter can be recast as the bulk expectation value of the operator $\mathcal{O}=e^{(\mathbf{i}/\hbar)\mathcal{S}_{\rm pp}}$,

$$\big(\mathcal{T}-\mathcal{V}_{\rm eff}[\{\boldsymbol{x}^{\mu}_{(a)}\}]\big)=-\mathbf{i}\hbar\log\big(\langle e^{\frac{\mathbf{i}}{\hbar}\mathcal{S}_{\rm pp}}\rangle_{\rm bulk}\big),\tag{12}$$

where we define the bulk expectation value of a generic operator $\mathcal{O}[\boldsymbol{\phi},\boldsymbol{\sigma}_{ij}]$ as the path integral over the gravitational fields,

$$\langle\mathcal{O}[\boldsymbol{\phi},\boldsymbol{\sigma}_{ij}]\rangle_{\rm bulk}=\frac{1}{Z_0}\int \mathrm{D}[\boldsymbol{\phi},\boldsymbol{\sigma}_{ij}]\mathcal{O}[\boldsymbol{\phi},\boldsymbol{\sigma}_{ij}]e^{\frac{\mathbf{i}}{\hbar}\mathcal{S}_{\rm bulk}}.\tag{13}$$

In the static sector, the time dependence drops out, and the path integral is evaluated over fields on a d-dimensional Euclidean space. We note that the point-particle action of Eq. (5) localizes the fields $\boldsymbol{\phi}(x)$ to be evaluated at either the worldline position $\boldsymbol{x}_{(1)}$ or $\boldsymbol{x}_{(2)}$. Hence, we define $\boldsymbol{\phi}_a=\boldsymbol{\phi}(\boldsymbol{x}_{(a)})$ and the composite operators $\Phi_1\coloneqq e^{\boldsymbol{\phi}_1}$, $\Phi_2\coloneqq e^{\boldsymbol{\phi}_2}$, so that the point-particle action reads $\mathcal{S}_{\rm pp}=-(m_1\Phi_1+m_2\Phi_2)$. We evaluate the bulk expectation value in Eq. (12) perturbatively, performing a series expansion to recover a polynomial expression in the gravitational fields:

$$\langle e^{\frac{\mathbf{i}}{\hbar}\mathcal{S}_{\rm pp}}\rangle_{\rm bulk}=\sum_{a,b\geq 0}\left(-\frac{\mathbf{i}}{\hbar}\right)^{a+b}\frac{m_1^a}{a!}\frac{m_2^b}{b!}\langle\Phi_1^a\Phi_2^b\rangle_{\rm bulk}.\tag{14}$$

The expectation value of $\langle\Phi_1^a\Phi_2^b\rangle_{\rm bulk}$ can be further expanded perturbatively. By linearity of the expectation value, the effective static post-Newtonian action is then entirely expressed in terms of gravitational correlation functions of the $\boldsymbol{\phi}$ field.

We introduce the connected correlation functions Γ_{n_1,n_2} of the $\boldsymbol{\phi}$ field, evaluated at the worldline positions,

$$\Gamma_{n_1,n_2}=\left(\frac{\mathbf{i}}{\hbar}\right)^{n_1+n_2-1}\langle\boldsymbol{\phi}_1^{n_1}\boldsymbol{\phi}_2^{n_2}\rangle_c=$$



, (15)

normalized such that its leading order classical scaling is the real quantity $\Gamma_{n_1,n_2}\propto\hbar^0(G_N/r)^{n_1+n_2-1}$. The connected correlation functions are diagrammatically represented as sums over sets of prime graphs, i.e., the sum of connected Feynman diagrams with n_1 and n_2 $\boldsymbol{\phi}$ legs sourced from the worldlines via linear operators at positions $\boldsymbol{x}_{(1)}$ and $\boldsymbol{x}_{(2)}$, respectively.

To evaluate the quantity $\langle\Phi_1^a\Phi_2^b\rangle_{\rm bulk}$, we introduce the linear sources J_1 and J_2 and the generating functional of the bulk connected correlation functions

$$\begin{aligned}W_{\rm bulk}(J_1,J_2)&=-\mathbf{i}\hbar\log\left\langle\exp\left[\frac{\mathbf{i}}{\hbar}\left(J_1\boldsymbol{\phi}_1+J_2\boldsymbol{\phi}_2\right)\right]\right\rangle_{\rm bulk}\\&=\sum_{n_1,n_2\geq 1}\frac{J_1^{n_1}J_2^{n_2}}{n_1!n_2!}\Gamma_{n_1,n_2},\end{aligned}\tag{16}$$

which have been directly collapsed to the localized worldlines positions. The sum in $W_{\rm bulk}$ is restricted to $n_a>0$ to remove self-energies and vanishing 1-point functions. Using the identities $\langle O[\phi]\rangle=O[-\mathbf{i}\hbar(\delta/\delta J)]Z[J]|_{J=0}$ and $\exp[\alpha(\partial/\partial J)]Z(J)=Z(J+\alpha)$, with $Z=\exp(\mathbf{i}/\hbar W_{\rm bulk}[J])$ [157,158], it holds:

$$\langle\Phi_1^a\Phi_2^b\rangle_{\rm bulk}=\exp\left(\frac{\mathbf{i}}{\hbar}W_{\rm bulk}(-\mathbf{i}\hbar a,-\mathbf{i}\hbar b)\right).\tag{17}$$

The expression of the bulk expectation value in Eq. (14), then, can be expressed entirely in terms of the connected gravitational correlation functions Γ_{n_1,n_2}.

$\mathbb{Z}_2$ symmetry of the static sector: Employing the KK parametrization of the metric [137,138], the classical d-dimensional static bulk action of Eq. (4) is manifestly invariant under a $\mathbb{Z}_2$ symmetry of the $\boldsymbol{\phi}$ gravitational field [159], i.e., the action is invariant under the discrete transformation $\boldsymbol{\phi}\to-\boldsymbol{\phi}$. This $\mathbb{Z}_2$ symmetry is a subgroup of $\mathrm{SL}(2,\mathbb{R})$ symmetry of the static Einstein-Hilbert action [159–166] in the $d=3$ spatial dimensions.

We assume the Minkowski vacuum to preserve the symmetry, $\langle\boldsymbol{\phi}\rangle=0$. Then, the $\mathbb{Z}_2$ symmetry implies

selection rules for the correlation functions. Specifically, only correlation functions with an even total number of $\boldsymbol{\phi}$ fields can be nonvanishing:

$$\Gamma_{n_1,n_2} = 0 \quad \text{if } n_1 + n_2 \in 2\mathbb{N} + 1. \tag{18}$$

This accidental symmetry of the bulk gravitational action, which also underlies the vanishing of black hole tidal Love numbers [159,166], strictly constrains the static potential for binary black holes to all post-Newtonian orders.

All-order structure of the static potential: The all-order effective action can be obtained from Eqs. (12), (14), and (17), enforcing the $\mathbb{Z}_2$ symmetry:

$$\mathcal{S}_{\text{eff}} = \mathcal{T} - \mathcal{V}_{\text{eff}} = -\mathbf{i}\hbar \log\left[\sum_{a,b\geq 0}\left(-\frac{\mathbf{i}}{\hbar}\right)^{a+b}\frac{m_1^a}{a!}\frac{m_2^b}{b!} \times \exp\left(\sum_{\substack{n_1,n_2\geq 1\\ n_1+n_2 \text{ even}}} (-\mathbf{i}\hbar)^{n_1+n_2-1}\frac{a^{n_1}}{n_1!}\frac{b^{n_2}}{n_2!}\Gamma_{n_1,n_2}\right)\right]. \tag{19}$$

This formula represents the first main result of this Letter. It establishes the exact all-order structure of the static post-Newtonian effective action as a decomposition into fundamental *connected correlation functions* Γ_{n_1,n_2}. The classical static potential at kPN order, $\mathcal{V}_{k\text{PN}}$ at $\mathcal{O}(G_N^{k+1})$, is a mass polynomial of order $\mathcal{O}(m_i^{k+2})$ (for $i = 1, 2$) [167], which can be obtained by taking a series expansion in the masses and then evaluating the $\hbar \to 0$ limit:

$$\mathcal{V}_{k\text{PN}}^{G_N^{k+1}} = \sum_{l=1}^{k+1} m_1^l m_2^{k-l+2}\frac{\hat{\mathcal{V}}_{l,k-l+2}}{l!(k-l+2)!},$$
$$\hat{\mathcal{V}}_{r,s} = -\lim_{\hbar\to 0}\frac{\partial^{r+s}}{\partial m_1^r \partial m_2^s}(\mathcal{S}_{\text{eff}})\Big|_{m_1=0,m_2=0}. \tag{20}$$

Interested readers may find the expressions of $\mathcal{V}_{k\text{PN}}^{G_N^{k+1}}$, for $k \leq 7$, in terms of Γ_{n_1,n_2} in the Supplemental Material [168].

Factorization theorem from $\mathbb{Z}_2$ symmetry: The factorization theorem admits a reformulation in terms of correlation functions based on Eq. (20) and its underlying $\mathbb{Z}_2$ symmetry. Each coefficient $\hat{\mathcal{V}}$ in Eq. (20) can be written as $\hat{\mathcal{V}}_{l,k-l+2} = c_{l,k-l+2}\Gamma_{l,k-l+2} + \sum_\alpha c_\alpha \prod_{i=1}^{N_\alpha}\Gamma_{n_{1,i}^{(\alpha)},n_{2,i}^{(\alpha)}}$, with $\sum_{i=1}^{N_\alpha}(n_{1,i}^{(\alpha)} + n_{2,i}^{(\alpha)}) = k + 1 + N_\alpha$, where N_α is the number of factorized correlators. Here, the correlator $\Gamma_{l,k-l+2}$ appears linearly and corresponds to *prime* graphs, while the products of lower-order correlators $\Gamma_{n_{1,i}^{(\alpha)},n_{2,i}^{(\alpha)}}$ encode *factorizable* graphs generated by the composite worldline operators. For odd k, the prime correlators $\Gamma_{l,k-l+2}$ vanish identically due to the $\mathbb{Z}_2$ selection rule in Eq. (18). As a result, each $\hat{\mathcal{V}}_{l,k-l+2}$, and thus, the full classical odd-PN static potential $\hat{\mathcal{V}}_{k\text{PN}}^{G_N^{k+1}}$, depends exclusively on products of correlators appearing at lower even orders. This identifies the $\mathbb{Z}_2$ symmetry as the underlying origin of the factorization theorem of Ref. [102] and explains why odd-PN static contributions are fully determined by lower-order data.

TABLE II. Rescaled $\bar{\Gamma}_{n_1,n_2}$ correlators in $d = 3$ spatial dimensions. They are related to Γ_{n_1,n_2} via $\Gamma_{n_1,n_2} = (G_N/r)^{n_1+n_2-1}\bar{\Gamma}_{n_1,n_2}$, with $\bar{\Gamma}_{n_1,n_2} = \bar{\Gamma}_{n_2,n_1}$.

PN	0SF	1SF	2SF	3SF
0	$\bar{\Gamma}_{1,1} = 1$			
2	$\bar{\Gamma}_{1,3} = 2$	$\bar{\Gamma}_{2,2} = 8$		
4	$\bar{\Gamma}_{1,5} = 24$	$\bar{\Gamma}_{2,4} = 160$	$\bar{\Gamma}_{3,3} = 504$	
6	$\bar{\Gamma}_{1,7} = 720$	$\bar{\Gamma}_{2,6} = 6400$	$\bar{\Gamma}_{3,5} = 31\,760$	$\bar{\Gamma}_{4,4} = 83\,456$

Known correlation functions: Table II reports the known nonzero correlators obtained by matching Eq. (20) to the known static potentials through 6PN [99,102,107,136]. Notably, all correlators known thus far are positive integers that increase with PN order.

All-order value of 0SF correlators: The 0SF correlators $\Gamma_{1,n} = \Gamma_{n,1}$ are fixed to all orders by matching Eq. (20) to the known test-particle limit, $\mathcal{V}^{m_1\ll m_2} = m_1\sqrt{g_{00}} = m_1\sqrt{(1-x)/(1+x)}$, with $x = G_N m_2/r$, yielding

$$\Gamma_{1,n} = \quad \overset{\bullet}{\underset{n}{\cdots}} \quad = \left(\frac{G_N}{r}\right)^n (n-1)! \quad \text{for } n \text{ odd}. \tag{21}$$

Interested readers may find the derivation of this formula in the Supplemental Material [168].

7PN static potential—The 7PN static contribution $\mathcal{O}(G_N^8)$ of the two-body gravitational potential receives contributions from 3842 7-loop PN-EFT diagrams (up to $m_1 \leftrightarrow m_2$ exchange). Nevertheless, owing to the factorization theorem, it can be determined either using Eqs. (10) and (11), in terms of up to 6-loop diagrams, or, more directly, via correlation functions using Eq. (20) and Table II, both yielding

$$\mathcal{V}_{7\text{PN}}^{G_N^8} = \frac{G_N^8}{r^8}\left(\frac{35}{128}m_1 m_2^8 + \frac{248}{9}m_1^2 m_2^7 + \frac{1059}{2}m_1^3 m_2^6 + \frac{89383}{36}m_1^4 m_2^5 + (1\leftrightarrow 2)\right). \tag{22}$$

This remarkably compact formula represents the second main result of this Letter. The coefficient of $m_1 m_2^8$ agrees with the expansion of the test-particle limit above, providing a direct check of Eq. (22).

Beyond 7PN—The master formula in Eq. (20) can be used to derive the static contributions to higher PN orders in terms of correlation functions. The 8PN and 9PN static potentials depend on four currently undetermined

correlators, which require the evaluation of 8-loop integrals. In general, at even $(2k)$-PN order, $k+1$ new prime correlation functions appear, reducing to k via Eq. (21). Genuinely new prime correlators enter only at even PN orders, while odd-PN static potentials are fixed recursively by lower-order correlators.

Interested readers may find the expressions of the 8PN and 9PN static potentials in the Supplemental Material [168].

Conclusion—We investigated the all-order structure of the post-Newtonian static gravitational potential within a field-theoretic framework. Exploiting the $\mathbb{Z}_2$ symmetry of the static sector, we derived a closed formula for the generic contribution at nPN order in terms of correlation functions. We showed that any odd-order term is entirely determined by lower-order data, thereby providing a novel theoretical formulation of the factorization theorem [102].

Our results yield both conceptual and computational advantages. In its original formulation, the factorization theorem allows one to determine the odd nth-order contribution by avoiding the explicit evaluation of the corresponding n-loop diagrams, reducing the problem to lower-order [up to $(n-1)$-loop] computations. On the other hand, our closed formula, expressing odd-order contributions directly and solely in terms of known lower-order correlators (which are intrinsically sums of diagrams), eliminates the need for diagrammatic input altogether.

As a concrete application, we have computed, for the first time, the static gravitational potential at 7PN order and explicitly verified the consistency between our closed-form approach and the diagrammatic implementation of the factorization theorem up to this order.

More generally, the formalism developed here provides an efficient and systematic organization of static PN corrections at arbitrarily high orders. Once the lower-order connected correlators are known, all odd-PN contributions follow recursively, while genuinely new information enters only through even-PN correlators. This structure imposes strong constraints on higher-order terms and clarifies the origin of the remarkable predictive power observed in the static sector. The correlator decomposition also suggests the possibility of resumming the static sector, as exemplified by Eq. (21), which resums the test-particle limit.

Our findings further suggest that the static post-Newtonian expansion is governed by an underlying algebraic structure in which symmetry and correlation functions organize the multiloop, yet classical, two-body dynamics in a highly constrained and unexpectedly compact form.

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