Quantum Algorithms for Representation-Theoretic Multiplicities

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Kostka, Littlewood-Richardson, Plethysm, and Kronecker coefficients are the multiplicities of irreducible representations in the decomposition of representations of the symmetric group that play an important role in representation theory, geometric complexity, and algebraic combinatorics. We give quantum algorithms for computing these coefficients whenever the ratio of dimensions of the representations is polynomial. We show that there is an efficient classical algorithm for computing the Kostka numbers under this restriction and conjecture the existence of an analogous algorithm for the Littlewood-Richardson coefficients. We argue why such classical algorithm does not straightforwardly work for the Plethysm and Kronecker coefficients and conjecture that our quantum algorithms lead to superpolynomial speedups. The conjecture about Kronecker coefficients was disproved by Panova [Polynomial time classical versus quantum algorithms for representation theoretic multiplicities, arXiv:2502.20253] with a classical algorithm which, if optimal, points to a $\mathcal{O}(n^{4+2k})$ vs $\tilde{\Omega}(n^{4k^2+1})$ polynomial gap in quantum vs classical computational complexity for an integer parameter k.

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Introduction—Is there a computational problem that has an efficient quantum algorithm but no efficient randomized algorithm? Quantum computing is driven by the belief that the answer is yes, still largely because "... our best description of the physical world is quantum-mechanical and quantum mechanics seems hard to simulate by classical computers..." [1]—but supporting this belief rigorously faces significant challenges in computational complexity theory. Here we identify a class of problems in representation theory that admit efficient quantum algorithms, study what makes these problems intractable classically, and find a parameter regime with a potential quantum speedup.

Let G be a finite group. A unitary representation R of G on a finite-dimensional vector space $\mathcal{H} = \mathbb{C}^d$ is a function R from G to the group of unitary matrices on $\mathcal{H}$ such that $R(g)R(h) = R(gh)$. For any G representation R there exists a unitary U, such that for all $g \in G$:

$$UR(g)U^\dagger = \bigoplus_\alpha I_{\mathbf{mult}(r_G^\alpha, R)} \otimes r_G^\alpha(g), \tag{1}$$

where r_G^α is an irreducible G representation (irrep) labeled by α. The number of times each irrep r_G^α appears in R is the *multiplicity* $\mathbf{mult}(r_G^\alpha, R)$. We denote $d_R = \dim(R)$ and $d_\alpha = \dim(r_G^\alpha)$ the dimension of the vector spaces on which R and r_G^α act on, use $R{\downarrow}_H^G$ for the restriction of a G representation from G to $H \subseteq G$, $\times$ for the direct product, and $\wr$ for the wreath product. Let S_n be the symmetric group on n elements. Its irreps are labeled by integer partitions $\lambda \vdash n$. The algorithms compute as follows.

Kostka numbers: the multiplicity of the trivial S_μ irrep in the restriction of an S_n irrep:

$$K_\nu^\mu = \mathbf{mult}(r_{S_\mu}^{\mathrm{triv}}, r_{S_n}^\nu{\downarrow}_{S_\mu}^{S_n}), \tag{2}$$

where $\nu, \mu \vdash n$ and $S_\mu = \times_i S_{\mu_i}$ is a Young subgroup. K_ν^μ also counts the number of semistandard Young tableaus of shape ν and content μ. They play a role in quantum tomography (see, for example, Ref. [2]).

Littlewood-Richardson coefficients: the multiplicity of $S_a \times S_b$ irreps in an S_n irrep for $a + b = n$:

$$c_{\lambda\mu}^\nu = \mathbf{mult}(r_{S_a}^\lambda \otimes r_{S_b}^\mu, r_{S_n}^\nu{\downarrow}_{S_a \times S_b}^{S_n}). \tag{3}$$

By Schur-Weyl duality [3,4] this is also a multiplicity of SU(d) irreps in the tensor product of two other SU(d) irreps. The coefficient appears in angular momentum recoupling theory (for $d = 2$; see, for example, Refs. [5–7]) or nuclear and particle physics (for example, for $d = 3$, see [8]).

Plethysm coefficients: the multiplicity of irreps of the wreath product subgroup $S_c \wr S_d \subset S_n$ for $cd = n$ in an S_n irrep:

$$a_{\lambda\mu}^\nu = \mathbf{mult}(r_{S_c \wr S_d}^{\lambda\mu}, r_{S_n}^\nu{\downarrow}_{S_c \wr S_d}^{S_n}), \tag{4}$$

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where $\lambda, \mu, \nu \vdash c, d, n$. It is also the multiplicity of SU(d) irreps in the composition of a pair of them. They appear in the quantum marginal problem for distinguishable particles (Ref. [9], Sec. VII).

Our construction also subsumes the algorithm for Kronecker coefficients which previously appeared in Ref. [10]. Kronecker coefficients are multiplicities of S_n irreps in the tensor products of S_n irreps:

$$g_{\lambda\mu\nu} = \mathbf{mult}(r^{\nu}_{S_n}, r^{\lambda}_{S_n} \otimes r^{\mu}_{S_n}), \tag{5}$$

and play a role in the quantum marginal problem [4,11].

Computation of Plethysm and Kronecker coefficients is worst-case sharp-P-hard if we treat n in S_n as the input size [12,13] and it has been conjectured that the same is true for the Littlewood-Richardson (LR) and Kostka coefficients (Ref. [14], Conjecture 5.11, and Ref. [15]). We give quantum algorithms that are efficient on nontrivial subsets of inputs to these problems and analyze their performance.

Quantum algorithms—We now state our main results. We invite the reader to consult Supplemental Material [17], where we include background concepts, full proofs, and other relevant information, which rely on Refs. [18–29].

Let G be a finite group and R be a G representation on $\mathbb{C}[G]$ with orthonormal basis $\{|g\rangle\}_{g\in G}$ acting as

$$R(h)|g\rangle = |hg\rangle, \quad \text{for all } h, g \in G. \tag{6}$$

This is the left-regular representation. A transformation U that maximally block diagonalizes this representation is the G-quantum Fourier transform (G-QFT), defined as

$$\langle \alpha, i, j|U|g\rangle = \sqrt{\frac{d_\alpha}{|G|}}\langle i|r^{\alpha}_{G}(g)|j\rangle, \tag{7}$$

where $r^{\alpha}_{G}(g)$ is a G irrep labeled by α and $\langle i|r^{\alpha}_{G}(g)|j\rangle$ for $i, j \in [d_\alpha]$ is its i, j th matrix element. Efficient quantum algorithms for G-QFT are known for $G = S_n$ [30] and some other finite groups [31]. We use QFT over the Young subgroup $S_\mu = \times_i S_{\mu_i}$ and the wreath product group $S_a \wr S_b$ from Ref. [31].

Our algorithm also requires efficient preparation of states of the form:

$$\rho^{\alpha}_{G} = U^{\dagger}\left(|\alpha\rangle\langle\alpha| \otimes \frac{I_{d_\alpha}}{d_\alpha} \otimes \frac{I_{d_\alpha}}{d_\alpha}\right)U = \frac{\Pi^{G}_{\alpha}}{d^2_\alpha}, \tag{8}$$

where I_{d_α} is the $d_\alpha \times d_\alpha$ identity matrix and Π^{G}_{α} is the projector onto the maximally isotypic subspace α in the regular representation of G [32]. A G-QFT is sufficient to prepare this state.

Theorem 1—Let $H \subseteq G$. Let $\mathcal{H}$ be the size of a quantum circuit for H-QFT, $\mathcal{R}$ be the size of the circuit to prepare ρ^{α}_{G} in Eq. (8), and $\mathcal{D}$ be the size of a circuit that implements a controlled action of G-regular representation. Then there is an $\mathcal{O}((\mathcal{H}+\mathcal{R}+\mathcal{D})(d_\alpha/d_\beta)^2)$ quantum algorithm that, given a pair of irrep labels (α, β) as an input, computes $\mathbf{mult}(r^{\beta}_{H}, r^{\alpha}_{G}{\downarrow}^{G}_{H})$ with probability at least 2/3 [33].

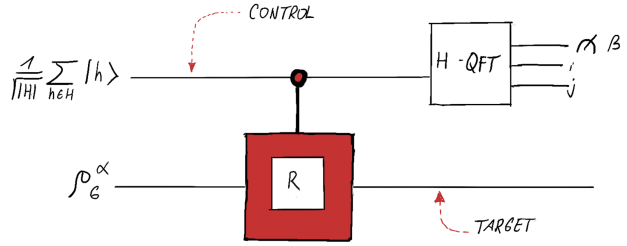


FIG. 1. The quantum algorithm in Theorem 1. The algorithm initializes the state from Eq. (8), applies a G-regular representation controlled by a uniform superposition over H, and weak Fourier samples the control register.

Proof—The algorithm is a variation on Harrow's generalized phase estimation [34] shown in Fig. 1. Let G be a finite group and let $H \subseteq G$. It uses two registers: a target register $\mathcal{T}$ and a control register $\mathcal{C}$. ρ^{α}_{G} from Eq. (8) is input to the target register and a uniform superposition over $H \subseteq G$ to the control, prepared by applying an inverse H-QFT to the state $|\text{triv}, 1, 1\rangle$, where triv labels the trivial irrep of H. The algorithm applies a controlled action of the G-regular representation R to ρ^{α}_{G}. Since R is controlled by a register that contains a superposition over the subgroup H, this acts as a restricted representation $R{\downarrow}^{G}_{H}$ and decomposes into irreducible representations of H. The algorithm concludes by measuring the H irrep label on the control register. All of this can be described by

$$E_\beta = \frac{1}{\sqrt{|H|}}\sum_{h\in H}\left[(|\beta\rangle\langle\beta| \otimes I_{d_\beta} \otimes I_{d_\beta})V|h\rangle\right]_{\mathcal{C}} \otimes \left[R(h)\right]_{\mathcal{T}}, \tag{9}$$

where V is the H-QFT and β labels inequivalent irreps of H. A measurement outcome β appears with probability

$$p_\alpha(\beta) = \mathrm{Tr}\left[E_\beta \rho^{\alpha}_{G} E^{\dagger}_{\beta}\right] = \mathbf{mult}(r^{\beta}_{H}, r^{\alpha}_{G}{\downarrow}^{G}_{H})\frac{d_\beta}{d_\alpha}. \tag{10}$$

Because $\mathbf{mult}(r^{\beta}_{H}, r^{\alpha}_{G}{\downarrow}^{G}_{H})$ is an integer, we can get its exact value in $\mathcal{O}[(d_\alpha/d_\beta)^2]$-many shots with high probability by the Chernoff-Hoeffding bound (see Supplemental Material [17]). The algorithm computes $\mathbf{mult}(r^{\beta}_{H}, r^{\alpha}_{G}{\downarrow}^{G}_{H})$ in $\mathcal{O}((\mathcal{H}+\mathcal{R}+\mathcal{D})(d_\alpha/d_\beta)^2)$ time. ∎

See Table I for choices of H and G where the quantum algorithm in Theorem 1 computes the coefficients in Eqs. (2)–(5). Almost all the cases considered have the same ambient group $G = S_n$, and since $\log|S_n| = \log n! \approx n\log n$, we parametrize the complexity of the algorithms below by n and the dimensions of S_n-irreducible

TABLE I. Quantum algorithms for branching coefficients. Choices of groups $H \subseteq G$, H representation R, and an H irrep r_H^α for which the algorithm in Theorem 1 computes the studied multiplicity coefficients. Here, $n, a, b, c, d \in \mathbb{N}$ where $a + b = n$ (Littlewood-Richardson) and $cd = n$ (Plethysm). The cost $\dim(R)/\dim(r_H^\alpha)$ is the number of samples required to (exactly) compute the coefficient.

Problem	Input	Group G	Subgroup $H \subseteq G$	H rep R	H irrep r_H^α	Output $\mathbf{mult}(r_H^\alpha, R)$	Cost $\mathcal{O}(\dim(R)/\dim(r_H^\alpha))$
Kostka	$\nu, \mu \vdash n$	S_n	$S_\mu := \times_i S_{\mu_i}$	$(r^\nu_{S_n})\downarrow^{S_n}_{S_\mu}$	$\bigotimes_i r^{(\mu_i)}_{S_{\mu_i}}$	K^μ_ν	$\mathcal{O}(d_\nu)$
Littlewood	$\nu \vdash n$ and $\lambda, \mu \vdash a, b$	S_n	$S_a \times S_b$	$(r^\nu_{S_n})\downarrow^{S_n}_{S_a \times S_b}$	$r^\lambda_{S_a} \otimes r^\mu_{S_b}$	$c^\nu_{\lambda\mu}$	$\mathcal{O}(d_\nu/d_\lambda d_\mu)$
Plethysm	$\nu \vdash n$ and $\lambda, \mu \vdash c, d$	S_n	$S_c \wr S_d$	$(r^\nu_{S_n})\downarrow^{S_n}_{S_c \wr S_d}$	$r^\lambda_{S_c} \wr r^\mu_{S_d}$	$a^\nu_{\lambda\mu}$	$\mathcal{O}(d_\nu/d_\lambda^d d_\mu)$
Kronecker [10]	$\nu, \lambda, \mu \vdash n$	$S_n \times S_n$	S_n	$(r^\lambda_{S_n} \otimes r^\mu_{S_n})\downarrow^{S_n \times S_n}_{S_n}$	$r^\nu_{S_n}$	$g_{\lambda\mu\nu}$	$\mathcal{O}(d_\lambda d_\mu/d_\nu)$

representations. These can be computed in $\mathcal{O}(n^2)$ using the hook-length formula (see Supplemental Material [17]). We also assume that the S_n-QFT can be implemented in gate complexity $\mathcal{O}(n^4)$ [35].

Kostka numbers—The Kostka number K^μ_ν is the multiplicity of the trivial irreducible representation in the restriction of an S_n irrep $r^\nu_{S_n}$ to S_μ, $K^\mu_\nu = \mathbf{mult}(r^{\mathrm{triv}}_{S_\mu}, r^\nu_{S_n}\downarrow^{S_n}_{S_\mu})$. A quantum algorithm for the Kostka numbers fixes $H = S_\mu \subseteq G = S_n$ for an integer partition $\mu \vdash n$, where S_μ is the Young's subgroup $S_\mu := \times_i S_{\mu_i}$. An efficient S_μ-QFT can be setup from a tensor product of S_{μ_i}-QFTs for each factor in $\mu \vdash n$. Theorem 1 then gives the label of the trivial irreducible representation of S_μ with probability [cf. Eq. (10)]

$$p_{\nu,\mu}(\mathrm{triv}) = K^\mu_\nu / d_\nu. \tag{11}$$

This suffices to compute the Kostka number in $\mathcal{O}(n^4 d_\nu^2)$. Computing Kostka numbers K^μ_ν on an input μ, ν in *unary* is conjectured to be sharp-P-complete in Ref. [14], Conjecture 5.11, and one could hope that restricting $d_\nu \leq \mathrm{poly}(n)$ remains classically hard. We show that this is not the case:

Theorem 2—There is a deterministic algorithm that on an input $\mu, \nu \vdash n$ in *unary*, such that $d_\nu \leq \mathrm{poly}(n)$, computes the Kostka number K^μ_ν in polynomial time in n.

See Supplemental Material for a proof [17]. We use that K^μ_ν is the number of semistandard Young tableaus of shape ν and content μ and there will be at most polynomially many such tableaus in the regime where the quantum algorithm from Theorem 1 is efficient.

Littlewood-Richardson coefficients—The quantum algorithm for LR coefficients fixes $H = S_a \times S_b \subseteq G = S_n$ for $a + b = n$. The irreducible representations of $S_a \times S_b$ are a tensor product of irreducibles $r^\lambda_{S_a} \otimes r^\mu_{S_b}$, labeled by pairs $\alpha = (\lambda, \mu)$ with $\lambda, \mu \vdash a, b$. We have $c^\nu_{\lambda\mu} = \mathbf{mult}(r^\lambda_{S_a} \otimes r^\mu_{S_b}, r^\nu_{S_n}\downarrow^{S_n}_{S_a \times S_b})$, and from Theorem 1, the algorithm samples from

$$p_\nu(\lambda, \mu) = c^\nu_{\lambda\mu} \frac{d_\lambda d_\mu}{d_\nu}. \tag{12}$$

This computes the LR coefficient in $\mathcal{O}\big(n^4 (d_\nu/d_\lambda d_\mu)^2\big)$ gate complexity. LR coefficients count combinatorial objects called LR tableaus. A skew shape ν/λ is a diagram where λ is subtracted from ν and a skew semistandard Young tableau is a skew shape filled with integers such that the entries are nondecreasing along every row and strictly increasing along every column. The LR table is a Young table with a skew of shape ν/λ whose row reading is compared to μ. The LR coefficient $c^\nu_{\lambda\mu}$ is the number of such tableaus; this is called the LR rule. See [36–38] and Supplemental Material [17] for details. The quantum algorithm in Theorem 1 is only efficient for $c^\nu_{\lambda\mu} \leq \mathrm{poly}(n)$, in which case it counts only up to $\mathrm{poly}(n)$ many LR tableaus. In analogy to Theorem 2, we expect that there is a classical algorithm that efficiently computes LR coefficients under this constraint.

Hypothesis 1—There is a polynomial-time randomized algorithm for computing LR coefficients $c^\nu_{\lambda\mu}$ for $\nu \vdash n$, $\lambda \vdash a$, and $\mu \vdash b$ for $a + b = n$ and $d_\nu/d_\lambda d_\mu \leq \mathrm{poly}(n)$.

See also [39]. If true, this rules out superpolynomial speedups for computing LR coefficients by Theorem 1 [40].

Plethysm coefficients—The algorithm for Plethysm coefficients sets $H = S_c \wr S_d \subseteq G = S_n$ for $cd = n$. An efficient implementation of the QFT over this group is known for $|S_c| \leq \mathrm{poly}(n)$ [31]. The quantum algorithm in Theorem 1 then samples efficiently from a distribution:

$$p_\nu(\lambda, \mu) = a^\nu_{\lambda\mu} \frac{d_\lambda d_\mu^d}{d_\nu}, \tag{13}$$

where $a^\nu_{\lambda\mu} = \mathbf{mult}(r^{\lambda,\mu}_{S_c \wr S_d}, r^\nu_{S_n}\downarrow^{S_n}_{S_c \wr S_d})$ is the Plethysm coefficient. It is not known if Plethysm coefficients count a well-defined set of combinatorial objects (Problem 9 on Stanley's list [41]), and the simulation strategy of Theorem 2 does not straightforwardly apply [13,42,43]. The classical complexity of computing the Plethysm

coefficients was recently studied in Ref. [13] where it was shown that the problem is in P if the number of parts $\ell(\lambda)$ is constant and $\mu = \mathrm{triv}$, but becomes sharp-P-hard even for fixed $d = 3$ and $\ell(\lambda)$ unrestricted. While this suggests that the complexity of the problem is controlled by the parameter c, its complexity for $|S_c| = \mathrm{poly}(n)$ $[c \sim \log(n)]$ remains, to our best knowledge, open. We thus state the following.

Conjecture 1 (see also [39])—There is no polynomial-time randomized algorithm that (with high probability) computes $a^{\nu}_{\lambda\mu}$ for $\lambda \vdash c$ and $\mu \vdash d$, such that $cd = n$, $c \leq \mathrm{poly}[\log(n)]$ and $d_\lambda d_\mu^d / d_\nu \geq 1/\mathrm{poly}(n)$.

If true, the quantum algorithm for Plethysm coefficients gives a superpolynomial quantum speedup. The fact that an analog of Theorem 2 does not apply here supports this, but the constraints on the conjectured computationally hard regime for the problem can make the restricted computational problem significantly easier. We have little intuition whether the conjecture is true or false, besides the obstructions proved in [39,44].

Kronecker coefficients—Given $\lambda, \mu, \nu \vdash n$, the Kronecker coefficient $g_{\lambda\mu\nu}$ is defined as $g_{\lambda\mu\nu} = \mathbf{mult}(r^{\lambda}_{S_n} \otimes r^{\mu}_{S_n}, r^{\nu}_{S_n})$. The algorithm in Ref. [45] is a special case of Theorem 1, for the choice $H = S_n \subseteq G = S_n \times S_n$. It samples from the output distribution,

$$p_{\lambda\mu}(\lambda) = \frac{d_\nu}{d_\lambda d_\mu} g_{\lambda\mu\nu}, \tag{14}$$

and computes $g_{\lambda\mu\nu}$ (with high constant probability) in time $\mathcal{O}(n^4 (d_\lambda d_\mu / d_\nu)^2)$. It remains a major open question if the Kronecker coefficients count some simple set of combinatorial objects (Stanley's Problem 10 [41]), which again precludes straightforward application of the simulation strategy of Theorem 2. The classical complexity of computing the Kronecker coefficients was studied in Ref. [43] which identified the extensive number of parts as a potential obstruction to their efficient computation, showing that $g_{\lambda\mu\nu}$ can be computed in time:

$$\mathcal{O}\big(L \log(N)\big) + \big(L \log(M)\big)^{\mathcal{O}(L^3 \log L)}, \tag{15}$$

where $L = \max(\ell(\lambda), \ell(\mu), \ell(\nu))$, $N = \max(\lambda_1, \mu_1, \nu_1)$ is the largest length of the first row and $M = \max(\lambda_2, \mu_2, \nu_2)$ is the largest length of the second row among these partitions (Ref. [43], Theorem 3.5). For fixed L, the algorithms of Refs. [46,47] compute the Kronecker coefficient in polynomial time. The algorithm in Ref. [48] computes Kronecker coefficient by reduction to vector partition function problem and all of these algorithms are based on Barvinok's algorithm for counting the lattice points in convex polytopes (Refs. [49–55]). The quantum algorithm in Theorem 1 is for all practical purposes outperformed by such algorithms on some inputs. For example, Ref. [48], Example 3.4, computes $g_{\lambda\mu\nu} = 391$ for $\lambda = [140^2]$, $\mu = [70^4]$, $\nu = [57^3, 33^3, 10]$ for which $d_\nu / (d_\lambda d_\mu) \approx 1.8 \times 10^{-31}$ on which the quantum algorithm in Theorem 1 requires an order of 10^{31} samples. On the other hand, it runs in polynomial when $d_\mu \leq \mathrm{poly}(n)$ and $\lambda = \nu$ is arbitrary, and we give an explicit construction of a family of such inputs in Supplemental Material [17].

We took the length obstruction in Eq. (15) as a support for the conjecture of Ref. [10] that the algorithm for the Kronecker coefficients outperforms classical algorithms for triples of partitions with large number of parts. This led us to state the following.

Conjecture 2 (disproved in [39])—There is no polynomial-time randomized algorithm that (with high probability) computes $g_{\lambda\mu\nu}$ for $\mu, \nu, \lambda \vdash n$ and $d_\mu \leq \mathrm{poly}(n)$.

However, Panova disproved this conjecture by extending the result of Ref. [46] by showing that whenever $d_\mu \leq n^k$, there is a $\tilde{\mathcal{O}}(n^{4k^2+1})$ classical algorithm for the problem [39], Theorem 1. A more general form of this conjecture (discussed in [10]) narrowly escapes this disproof but requires fine-tuned inputs, so we take Panova's result as a strong evidence against superpolynomial quantum speedups for this problem [56]. The gate complexity of our algorithm is $\mathcal{O}(n^4 (d_\mu d_\lambda / d_\nu)^2)$ so that a $\mathcal{O}(n^{4+2k})$ vs $\tilde{\Omega}(n^{4k^2+1})$ polynomial speedup for $d_\lambda = d_\nu$ and $d_\mu \leq n^k$ is possible, assuming that the scaling in Ref. [39], Theorem 1, is tight. As this asks for additional scrutiny, we pose the following.

Question 1—Is there a classical algorithm that computes $g_{\mu\nu\lambda}$ in time $\mathcal{O}(n^{2k+4})$ whenever $d_\mu \leq n^k$?

See Supplemental Material for an explicit construction of a nontrivial family of inputs on which our algorithm is efficient, additional analysis, discussion, and numerical study [17].

Induction algorithm—Frobenius's reciprocity [3],

$$\mathbf{mult}(r^{\beta}_H, r^{\alpha}_G{\downarrow^G_H}) = \mathbf{mult}(r^{\alpha}_G, r^{\beta}_H{\uparrow^G_H}), \tag{16}$$

suggests an alternative to Theorem 1 that computes multiplicities by Fourier sampling over induced representations.

Theorem 3—Let $H \subseteq G$. If there is a quantum circuit for G-QFT with $\mathcal{G}$ many gates, a quantum circuit with $\mathcal{R}$ many gates for preparing ρ^{β}_H in Eq. (8), then there is a $\mathcal{O}((\mathcal{G} + \mathcal{R})(d_\beta |G| / d_\alpha |H|)^2)$ quantum algorithm that, given a pair of irrep labels (α, β) as an input, computes $\mathbf{mult}(r^{\alpha}_G, r^{\beta}_H{\uparrow^G_H})$ with probability at least $2/3$.

Proof—See Fig. 2. The algorithm uses two registers, an H-regular rep register $\mathcal{S} = \mathbb{C}[H]$ and a coset register $\mathcal{V} = \mathbb{C}[G/H]$. It prepares ρ^{β}_H from Eq. (8) in $\mathcal{S}$ and $|H|\mathbb{I}/|G|$ in $\mathcal{V}$ and applies an "embedding" unitary U_E to $\mathcal{S} \otimes \mathcal{V}$: $U_E(|t\rangle_{\mathcal{V}} \otimes |h\rangle_{\mathcal{S}}) = |th\rangle$, where t is a representative of the coset of H in G and th is an element of G. This unitary can be implemented efficiently as long as the multiplication of the group elements in G is efficient.

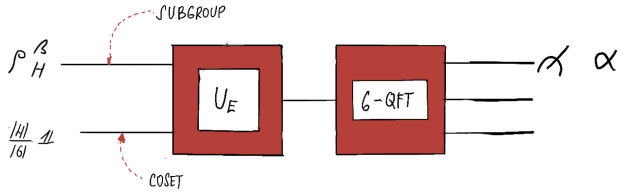


FIG. 2. The quantum algorithm in Theorem 3. The algorithm applies the embedding unitary U_E to the subgroup and coset register and weak Fourier samples.

The algorithm applies a G-QFT to $\mathcal{S} \otimes \mathcal{V}$ and measures the irrep label. An outcome α appears with probability:

$$q_\beta(\alpha) = \frac{|H| d_\alpha}{|G| d_\beta} \mathbf{mult}(r_G^\alpha, r_H^\beta \uparrow_H^G). \tag{17}$$

Following the analysis in Theorem 1, we can find $\mathbf{mult}(r_G^\alpha, r_H^\beta \uparrow_H^G)$ in $\mathcal{O}((\mathcal{G}+\mathcal{R})(d_\beta |G|/d_\alpha |H|)^2)$ with high probability. See Supplemental Material for details [17]. ∎

The sampling cost of the restriction algorithm in Theorem 1 is d_α/d_β, compared to $(d_\beta|G|)/(d_\alpha|H|)$ in Theorem 3 and the restriction is preferable whenever $(d_\alpha/d_\beta)^2 < |G|/|H|$. For $\beta = \text{triv}$, Eq. (17) is the distribution of weak Fourier sampling from the hidden subgroup state, appearing in the context of the hidden subgroup problem (HSP). The computational task in HSP is of course different from computing the multiplicity but the connection suggests generalizations of HSP to $\beta \neq \text{triv}$ we explore in future work.

Discussion—We gave quantum algorithms for Kostka, Littlewood-Richardson, Kronecker, and Plethysm coefficients, efficient on a nontrivial subset of inputs. We gave a polynomial-time classical algorithm for computing of the Kostka numbers K_ν^μ whenever $d_\nu \leq \text{poly}(n)$, ruling out a superpolynomial quantum speedup for this problem and conjectured existence of a similar classical algorithm for the Littlewood-Richardson coefficients. We discussed why such a classical algorithm does not straightforwardly extend to the Plethysm and Kronecker coefficients and conjectured that our quantum algorithms efficiently solve problems that are hard for classical algorithms.

Going against our thesis is a result in Ref. [57] in which Jordan presented a quantum algorithm for approximating S_n characters to a similar precision as our Theorem 1 and gave an efficient classical algorithm based on the Roichman's rule [58]. Our Conjecture 2 followed a similar fate and was disproved by an algorithm in Ref. [39], Theorem 1. The run-time of this algorithm is $\tilde{\mathcal{O}}(n^{4k^2+1})$ which suggests a possible $\mathcal{O}(n^{4+2k})$ vs $\tilde{\Omega}(n^{4k^2+1})$ speedup on inputs with $d_\nu = d_\lambda$ and $d_\mu \leq n^k$. It is thus important to address Question 1 and understand if the large polynomial gap for computing Kronecker coefficient closes.

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[1] R. P. Feynman, Simulating physics with computers, Int. J. Theor. Phys. **21**, 467 (1982).
[2] J. Haah, A. W. Harrow, Z. Ji, X. Wu, and N. Yu, Sample-optimal tomography of quantum states, IEEE Trans. Inf. Theory **63**, 5628 (2017).
[3] W. Fulton and J. Harris, *Representation Theory: A First Course* (Springer, New York, 1991).
[4] M. Christandl and G. Mitchison, The spectra of quantum states and the Kronecker coefficients of the symmetric group, Commun. Math. Phys. **261**, 789 (2005).
[5] E. Wigner, *Group Theory: And its Application to the Quantum Mechanics of Atomic Spectra*, Pure and Applied Physics (Elsevier Science, New York, 1931).
[6] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover Books on Mathematics (Dover Publications, New York, 1950).
[7] R. Pauncz, *Spin Eigenfunctions: Construction and Use* (Springer US, New York, 1979).
[8] M. Gell-Mann, The eightfold way: A theory of strong interaction symmetry, 10.2172/4008239 (1961).
[9] M. Christandl, B. Doran, S. Kousidis, and M. Walter, Eigenvalue distributions of reduced density matrices, Commun. Math. Phys. **332**, 1 (2014).
[10] S. Bravyi, A. Chowdhury, D. Gosset, V. Havlicek, and G. Zhu, Quantum complexity of the Kronecker coefficients, PRX Quantum **5**, 010329 (2024).
[11] M. Christandl, A. W. Harrow, and G. Mitchison, Nonzero Kronecker coefficients and what they tell us about spectra, Commun. Math. Phys. **270**, 575 (2007).
[12] P. Bürgisser and C. Ikenmeyer, The complexity of computing Kronecker coefficients, in *20th Annual International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2008), 2008, Viña del Mar, Chile* (2008), pp. 357–368, 10.46298/dmtcs.3622.
[13] N. Fischer and C. Ikenmeyer, The computational complexity of plethysm coefficients, Comput. Complex. **29**, 8 (2020).
[14] G. Panova, Computational complexity in algebraic combinatorics, arXiv:2306.17511.
[15] The computation of Littlewood-Richardson coefficients (to which the Kostka number computation reduces parsimoniously) is sharp-P-hard for *binary* encoding by a reduction to Knapsack [16]. Binary encoding encodes the length of each row in the input integer partition in binary; this means that for a fixed input size n, the order of the underlying symmetric group ranges from S_n to S_{2^n}. Fixing the order of S_n, as we do here, is equivalent to using unary encoding

of the input. It is open if Kostka and LR coefficients are sharp-P-hard with unary encoding. See [14].
[16] H. Narayanan, On the complexity of computing Kostka numbers and Littlewood-Richardson coefficients, J. Algebraic Combinatorics **24**, 347 (2006).
[17] See Supplemental Material at http://link.aps.org/supplemental/10.1103/k5tx-xtr3 for full proofs, additional background, and supplementary figures.
[18] M. Clausen, Fast generalized Fourier transforms, Theor. Comput. Sci. **67**, 55 (1989).
[19] P. Diaconis and D. N. Rockmore, Efficient computation of the Fourier transform on finite groups, J. Am. Math. Soc. **3**, 297 (1990).
[20] K. D. Mulmuley, The GCT program toward the P vs. NP problem, Commun. ACM **55**, 98 (2012).
[21] R. Goodman and N. R. Wallach, *Symmetry, Representations, and Invariants* (Springer, New York, 2009), Vol. 255.
[22] D. Grinko, A. Burchardt, and M. Ozols, Efficient quantum circuits for port-based teleportation, arXiv:2312.03188.
[23] D. Grinko, A. Burchardt, and M. Ozols, Gelfand-Tsetlin basis for partially transposed permutations, with applications to quantum information, arXiv:2310.02252.
[24] R. Howe and S. T. Lee, Why should the Littlewood–Richardson rule be true?, Bull. Am. Math. Soc. **49**, 187 (2012).
[25] A. Klyachko, Quantum marginal problem and representations of the symmetric group, arXiv:quant-ph/0409113.
[26] H. Krovi, An efficient high dimensional quantum Schur transform, Quantum **3**, 122 (2019).
[27] J.-P. Serre *et al.*, *Linear Representations of Finite Groups* (Springer, New York, 1977), Vol. 42.
[28] J. Stembridge, On the eigenvalues of representations of reflection groups and wreath products, Pac. J. Math. **140**, 353 (1989).
[29] J. R. Stembridge, A concise proof of the Littlewood-Richardson rule, Electron. J. Comb. **9**, N5 (2002).
[30] R. Beals, Quantum computation of Fourier transforms over symmetric groups, in *Proceedings of the Twenty-Ninth Annual ACM Symposium on Theory of Computing* (ACM, New York, NY, 1997), pp. 48–53.
[31] C. Moore, A. Russell, and P. Sniady, On the impossibility of a quantum sieve algorithm for graph isomorphism, in *Proceedings of the Thirty-Ninth Annual ACM Symposium on Theory of Computing (STOC07)* (ACM, New York, NY, 2007).
[32] The maximally isotypic subspace of a given type is the subspace of $\mathcal{H}$ containing all irreps of such type.
[33] The irreps α, β can be labeled with $O(\log|G|)$-many bits each, so that the input size n relates nontrivially to the order of the group G. Since there are at most $|G|$ distinct irreducible representations, such labeling always exists.
[34] A. W. Harrow, Applications of coherent classical communication and the Schur transform to quantum information theory, arXiv:quant-ph/0512255.
[35] Y. Kawano and H. Sekigawa, Quantum Fourier transform over symmetric groups—Improved result, J. Symb. Comput. **75**, 219 (2016).
[36] R. M. Howe and S. T. Lee, Why should the Littlewood–Richardson rule be true?, Bull. Am. Math. Soc. **49**, 187 (2012).
[37] G. D. James, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its Applications (Cambridge University Press, Cambridge, England, 1984).
[38] B. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions* (Springer Science & Business Media, New York, 2001), Vol. 203.
[39] G. Panova, Polynomial time classical versus quantum algorithms for representation theoretic multiplicities, arXiv:2502.20253.
[40] We made a distinction between a hypothesis and a conjecture. By a hypothesis, we mean claim that we strongly believe in but cannot prove. A conjecture labels a claim that we cannot prove and have no strong beliefs about.
[41] R. Stanley, Positivity problems and conjectures in algebraic combinatorics, in *Mathematics: Frontiers and Perspectives* (Amer. Math. Soc., 1999), Vol. 295, pp. 295–319, ISBN 978-1-4704-4661-1.
[42] C. Ikenmeyer, K. D. Mulmuley, and M. Walter, On vanishing of Kronecker coefficients, Comput. Complex. **26**, 949 (2017).
[43] I. Pak and G. Panova, On the complexity of computing Kronecker coefficients, Comput. Complex. **26**, 1 (2017).
[44] C. Ikenmeyer and S. Subramanian, A remark on the quantum complexity of the Kronecker coefficients, arXiv:2307.02389.
[45] S. Bravyi, A. Chowdhury, D. Gosset, and P. Wocjan, Quantum Hamiltonian complexity in thermal equilibrium, Nat. Phys. **18**, 1367 (2022).
[46] M. Christandl, B. Doran, and M. Walter, Computing multiplicities of lie group representations, in *2012 IEEE 53rd Annual Symposium on Foundations of Computer Science* (IEEE, Los Alamitos, CA, 2012), pp. 639–648.
[47] V. Baldoni, M. Vergne, and M. Walter, Computation of dilated Kronecker coefficients, J. Symb. Comput. **84**, 113 (2018).
[48] M. Mishna and S. Trandafir, Estimating and computing Kronecker coefficients: A vector partition function approach, arXiv:2210.12128.
[49] J. A. D. Loera, R. Hemmecke, J. Tauzer, and R. Yoshida, Effective lattice point counting in rational convex polytopes, J. Symb. Comput. **38**, 1273 (2004).
[50] J. A. D. Loera, R. Hemmecke, J. Tauzer, and R. Yoshida, Effective lattice point counting in rational convex polytopes, J. Symb. Comput. **38**, 1273 (2004).
[51] J. A. D. Loera and T. B. McAllister, On the computation of Clebsch-Gordan coefficients and the dilation effect, arXiv:math/0501446.
[52] J. A. De Loera, The many aspects of counting lattice points in polytopes, Mathematische Semesterberichte **52**, 175 (2005).
[53] A. Barvinok, A polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed, in *Proceedings of 1993 IEEE 34th Annual Foundations of Computer Science* (IEEE Computer Society, Los Alamitos, CA, 1993), pp. 566–572.
[54] A. I. Barvinok and J. E. Pommersheim, An algorithmic theory of lattice points in polyhedra, in *New Perspectives in Algebraic Combinatorics*, Math. Sci. Res. Inst. Publ.

Vol. 38 (Cambridge University Press, Cambridge, England, 1999), pp. 91–147.

[55] M. Dyer and R. Kannan, On Barvinok's algorithm for counting lattice points in fixed dimension, Math. Oper. Res. **22**, 545 (1997).

[56] There is no polynomial time randomized algorithm that (with high probability) computes $g_{\lambda\mu\nu}$ for $\mu, \nu, \lambda \vdash n$ and $d_\nu/(d_\lambda d_\mu) \geq 1/\mathrm{poly}(n)$. The key challenge then becomes to find a nicely parametrized set of inputs such that d_ν, d_λ, d_μ are all superpolynomial, their ratio is bounded by a polynomial, and the partitions do not fall into one of the special cases that can be simulated easily (such as hooks). As long as λ, μ have number of parts that scale with n, the quantum algorithm can do better than the bound in Eq. (15) as well as the state-of-the-art classical algorithms, but the new result in [39] supersedes this obstruction.

[57] S. P. Jordan, Fast quantum algorithms for approximating some irreducible representations of groups, arXiv:0811.0562.

[58] Y. Roichman, Characters of the symmetric groups: Formulas, estimates and applications, in *Emerging Applications of Number Theory*, edited by D. A. Hejhal, J. Friedman, M. C. Gutzwiller, and A. M. Odlyzko (Springer, New York, 1999), pp. 525–545.