

Perfect Wave Transfer in Continuous Quantum Systems

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The transfer of information from one part of a quantum system to another is fundamental to the understanding and design of quantum information processing devices. In the realm of discrete systems such as spin chains, inhomogeneous networks have been engineered that allow for the perfect transfer of qubits from one end to the other. Here, by contrast, we investigate the perfect transfer of information in continuous systems, phrased in terms of wave propagation. A remarkable difference is found between systems that possess conformal invariance and those that do not. Systems in the first class enjoy perfect wave transfer (PWT), explicitly shown for one-particle excitations and anticipated in general. In the second class, those that exhibit PWT are characterized as solutions to an inverse spectral problem. As a concrete example, we demonstrate how to formulate and solve this problem for a prototypical class of bosonic theories, showing the importance of conformal invariance for these theories to enjoy PWT. Using bosonization, our continuum results extend to theories with interactions, broadening the scope of perfect information transfer to more general quantum systems.

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Introduction—A fundamental task in quantum information processing is to transfer information from one register to another [1,2]. Optical fibers may be used, but efficiency can be improved using a quantum wire as both communication channel and registers [3]. Since this relies on collective phenomena to transfer quantum states, efficiency is optimized by engineering the system's many-body properties. This allows not only for high-fidelity but even perfect transfer of states without dynamical control of individual sites. Such *perfect state transfer* (PST) was first shown theoretically for wires modeled as spin chains with inhomogeneous hopping [4,5], then extended to spins on graphs and other networks [6–15] and demonstrated experimentally [16–21]. However, PST has so far been addressed only in the discrete realm, while powerful continuum frameworks to describe collective quantum phenomena remain untapped.

In this Letter, we harness that power to investigate quantum information transfer in inhomogeneous continuous systems in $1+1$ dimensions ($1+1D$). At low

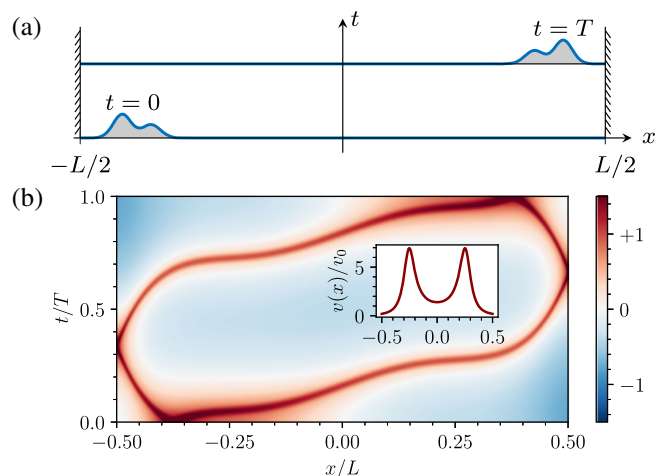


FIG. 1. (a) Illustration of PWT with Neumann BCs showing an initial wave that evolves to its reflection at a certain time T . (b) Its manifestation in an inhomogeneous CFT with a given profile $v(x)/v_0$ (inset), showing a localized initial wave propagating along curved light-cone trajectories that recombine at the reflected point at $t = T \equiv L/v_0$ given by Eq. (7). Specifically, the real part of boson two-point correlations (17) at the free-fermion point is plotted, which by bosonization encodes the same information as $\log F(t)$ in Eq. (5) for deltalike waves ξ_1 centered at $-3L/8$ and ξ_2 at x (setting $k_F = 0$ for simplicity and $t' = 0$).

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energies, such examples are realizable using, e.g., spin ladders [22], Josephson-junction arrays [23], ultracold atoms [24], and nanowires [25–27]. In this framework, rephrasing using waves becomes natural, motivating the term *perfect wave transfer* (PWT): As for states in PST, a system on an interval $[-L/2, L/2]$ of length L with suitable boundary conditions (BCs) exhibits PWT if propagating any initial wave for some time T amounts to a reflection; see Fig. 1. Concretely, *there is PWT* if

$$\langle \mathcal{O}(x, 0) \rangle = \langle \mathcal{O}(-x, T) \rangle \quad (1)$$

for all time-evolved observables $\mathcal{O}(x, t)$ and expectation values $\langle \cdot \rangle$ with respect to any state. The dynamics will be time independent, but the problem is also interesting for driven systems [19–21], with randomness [28], or relaxed to a property for specific classes of observables or states. Ultimately, our motivation for understanding PWT is the efficient high-fidelity transfer of information—a problem that has attracted recent attention, e.g., in the design of distributed networks for quantum computation [29].

Related advances include nonequilibrium studies of inhomogeneous versions of conformal field theory (CFT), i.e., conformally invariant quantum field theories [30] deformed so that the usual constant propagation velocity becomes a profile $v(x)$ [31–36]. Among their applications, such *inhomogeneous CFTs* effectively describe gapless quantum many-body systems with mesoscopic inhomogeneities, e.g., spin chains with spatially varying hopping and trapped ultracold atoms; cf. [37]. Special cases include the profile $v(x) = v\sqrt{1 - (2x/L)^2}$ for some constant $v > 0$ [38,39]—the continuum counterpart of the inhomogeneous spin chain [4] first exhibiting PST—and profiles produced by Möbius transformations [40–47]. Note that for PWT, as for other transport properties in finite systems, the choice of BCs [39,48–50] is important.

Engineering PWT will, at heart, be a spectral problem, amounting to whether the propagator coincides with the parity operator at certain times. This is shown for prototypical quantum field theories describing an inhomogeneous quantum wire—our continuous channel—focusing on universal low-energy descriptions where conformal invariance can be parametrically turned on or off. Notably, these are directly applicable to ultracold atoms [49,51,52], while such experimental systems are even fundamentally continuous [53], in itself warranting a continuum treatment of information transfer. Remarkably, for bosonic theories [54–61], we arrive at an inverse spectral problem for Sturm-Liouville operators. Its solution determines if the channel enjoys PWT, and we show that conformal invariance is important if the system is required to be sufficiently regular. Using bosonization [62–67], our results extend PWT or PST to a large class of theories, including interacting fermions, and we also discuss a proposal to extend beyond low energies. Lastly, in addition to quantum communication in

solid-state devices [68,69], our PWT results relate to relativistic quantum information [70,71], and potentially to communication over long distances [72].

Inhomogeneous CFT—To begin, we study the continuum limit of inhomogeneous spin chains [3,4]. In the gapless regime, their continuous counterpart is an inhomogeneous CFT with a velocity profile $v(x) > 0$ [31–36]. The simplest example is that of inhomogeneous free fermions, given by the Hamiltonian (setting $\hbar = 1$) [73]

$$H_{\text{IFF}} \equiv \int_{-L/2}^{L/2} dx \frac{v(x)}{2} \left[: \psi_+^\dagger (-i) \partial_x \psi_+ : + : \psi_-^\dagger i \partial_x \psi_- : + \text{H.c.} \right] \quad (2)$$

with fields $\psi_\pm(x)$ of right- (+) and left- (−) moving fermions satisfying $\{\psi_\pm(x), \psi_\pm^\dagger(x')\} = \delta(x - x')$ and $\{\psi_\pm(x), \psi_\mp^\dagger(x')\} = 0$. Here, $: \cdots :$ denotes Wick (normal) ordering with respect to the ground state $|\Omega\rangle$ (which, for bilinears in the fields, amounts to an additive renormalization). For simplicity, we focus on Eq. (2), while general CFTs are discussed in the Supplemental Material (SM) [74]. We assume open (Neumann) BCs at $x = \pm L/2$, incorporated by standard unfolding methods; cf. [48,65,66,75–77]: Couple right and left movers by identifying $\psi_-(x) = \psi_+(L - x)$, giving a chiral theory of only right-moving fermions on $[-L/2, 3L/2]$ with periodic BCs and velocity $v(x)$ for $x \in [-L/2, L/2]$ and $v(L - x)$ for $x \in [L/2, 3L/2]$ [78].

PST was probed in [4] using a time-dependent overlap between one-particle states that were mutual reflections. Their continuum counterparts are states created by fields

$$\psi(x) \equiv e^{ik_F x} \psi_+(x) + e^{-ik_F x} \psi_-(x), \quad (3)$$

where k_F denotes the Fermi momentum [79]. As a subset of possible states, consider

$$|\Psi_j\rangle \equiv \int_{-L/2}^{L/2} dx \xi_j(x) \psi^\dagger(x) |\Omega\rangle \quad (j = 1, 2) \quad (4)$$

for smooth enough functions $\xi_j(x)$, interpreted as “waves” of fermionic excitations in each state, and study the time-dependent overlap

$$F(t) \equiv \langle \Psi_2 | e^{-iH_{\text{IFF}} t} | \Psi_1 \rangle. \quad (5)$$

The states in Eq. (4) form a reduced class, as only one-particle excitations are possible, and right and left movers are equally distributed. (The latter restriction is lifted in the SM [74].)

To probe PWT for one-particle excitations as in [4], we restrict to specular waves, $\xi_2(x) = \xi_1(-x)$, and compare $F(t)$ in Eq. (5) with $\langle \Psi_1 | \Psi_1 \rangle$. Specifically, *there is one-particle PWT* if there exists $T > 0$ so that

$$|F(T)| = |\langle \Psi_1 | \Psi_1 \rangle| \quad (6)$$

for all initial waves $\xi_1(x)$ in Eq. (4) [80].

Equation (6) amounts to a comparison between two-point correlation functions at equal and nonequal times. Such correlations can be computed in any inhomogeneous CFT by mapping them to the corresponding homogeneous correlations [36]. The details given in the SM [74] distill to comparing spacetime points after conformal transformations to coordinates

$$y = y(x) \equiv \int_0^x ds \frac{v_0}{v(s)}, \quad \frac{1}{v_0} \equiv \frac{1}{L} \int_{-L/2}^{L/2} \frac{ds}{v(s)}. \quad (7)$$

The comparison's outcome shows that reflection symmetry is essential: *Inhomogeneous CFTs exhibit one-particle PWT if and only if $v(x)$ is even.* Moreover, this transfer occurs at odd multiples $(2m+1)T$ (for $m = 0, 1, \dots$) of the time $T = L/v_0$ for excitations to cross the system's entire length. Note that $y \in [-L/2, L/2]$ if $v(x)$ is even.

As an example, consider the square-root profile $v(x) = v\sqrt{1 - (2x/L)^2}$, effectively describing the spin chain in [4]. It exhibits PWT, just like for any even profile $v(x)$, in contrast to [4], where that profile was exceptional in this respect. Notably, the square-root profile is positive everywhere except at $x = \pm L/2$. Thus, strictly speaking, $v(x)$ should be regularized before using the tools in [36], but since the integral defining $1/v_0$ in Eq. (7) converges, it has no consequence for PWT [81].

Bosonic theories—The above result and observations should not come as a surprise. Indeed, if one compares discrete and continuous theories, then an inhomogeneous CFT can describe only low-energy excitations of a given lattice model. Moreover, we show [74] that any Hamiltonian (with pure point spectrum) must commute with the parity operator for PWT to be possible, i.e., $v(x)$ must be even, as should be expected. Furthermore, since excitations propagate ballistically in CFT, one might anticipate that this is not only necessary but also sufficient to ensure PWT with respect to all states. E.g., the model (2) has a linear spectrum and is free, suggesting that PWT for one-particle excitations extends to many particles. The rest of this Letter concerns to what extent this is true more generally: What are *necessary* and *sufficient* conditions for PWT as defined in Eq. (1)?

To answer this question and understand the role of conformal invariance, we should *not* assume the latter. Specific nonconformal examples include 1 + 1D fermions with nonlocal interactions [82–84] or the Lieb-Liniger model [85]. While interesting in their own right, we address this question using a paradigmatic low-energy theory, namely, inhomogeneous Tomonaga-Luttinger liquids (TLLs) [54–61]. By bosonization, this covers fermionic models generalizing the one in Eq. (2), while manipulating the inhomogeneities allows one to controllably turn

conformal invariance on or off. The theory is formulated using a compactified bosonic field $\varphi(x)$ (modulo 2π) with conjugate $\Pi(x)$ satisfying $[\partial_x \varphi(x), \Pi(x')] = i\partial_x \delta(x - x')$. The Hamiltonian is [73]

$$H_{\text{TLL}} \equiv \int_{-L/2}^{L/2} \frac{dx}{2\pi} v(x) : \left(\frac{[\pi \Pi(x)]^2}{K(x)} + K(x) [\partial_x \varphi(x)]^2 \right) :, \quad (8)$$

where, in addition to $v(x) > 0$, the Luttinger parameter $K(x) > 0$ also depends on x . For simplicity, we assume that both functions are smooth. For $K(x) = K$ constant, Eq. (8) is an inhomogeneous CFT, and if $K = 1$, it is the bosonized version of the free-fermion model (2) [62–67]. However, $K'(x) \neq 0$ locally couples right and left movers nontrivially; thus, they do not commute, and the theory is not conformal [61]. Again, we impose open BCs:

$$J(x)|_{x=\pm L/2} = 0, \quad \text{where } J(x) \equiv -\frac{v(x)K(x)}{\pi} \partial_x \varphi(x) \quad (9)$$

is the particle current. As a rule, we further assume that $K'(x)$ is zero at $x = \pm L/2$ so that right and left movers are coupled only via the BCs (9) at those points [86].

Sturm-Liouville (SL) theory—Following [37,87–92], we use SL theory to find eigenfunctions, using which the fields can be expanded and the Hamiltonian (8) diagonalized [93]. Indeed, using integration by parts and Eq. (9),

$$H_{\text{TLL}} = \int_{-L/2}^{L/2} \frac{dx}{2\pi} w(x) : \left(\left[\frac{\pi \Pi(x)}{w(x)} \right]^2 + \varphi(x) \mathcal{A} \varphi(x) \right) : \quad (10)$$

with the SL operator

$$\mathcal{A} \equiv -\frac{1}{w(x)} [\partial_x p(x) \partial_x + q(x)], \quad (11)$$

$$w(x) \equiv \frac{K(x)}{v(x)}, \quad p(x) \equiv v(x)K(x), \quad q(x) = 0, \quad (12)$$

where $\mathcal{A}$ acts on the space $\mathcal{D}$ of smooth [94] functions $u(x)$ on $[-L/2, L/2]$ that satisfy $v(x)K(x)\partial_x u(x)|_{x=\pm L/2} = 0$, i.e., Neumann BCs when $v(x), K(x) > 0$.

We want to solve the SL problem

$$\mathcal{A}u = \lambda u, \quad u \in \mathcal{D}. \quad (13)$$

It is regular if certain conditions are satisfied [95], and irregular otherwise. This matters due to properties that hold for regular problems [96]: (i) The spectrum consists of eigenvalues λ_n that are real and ordered so that $0 \leq \lambda_0 < \lambda_1 < \dots$ (ii) Each λ_n has a unique square-integrable eigenfunction $u_n(x)$ with n isolated zeros in $[-L/2, L/2]$. (iii) The $u_n(x)$ form a complete orthonormal basis for square-integrable functions on $[-L/2, L/2]$ with weight $w(x)$, i.e., $\int_{-L/2}^{L/2} dx w(x) u_n(x) u_{n'}(x) = \delta_{n,n'}$. Furthermore, assuming that the integral defining $1/v_0$ in

Eq. (7) converges, Weyl's law implies the following: (iv) λ_n grows at most as n^2 for $n \gg 1$, i.e., $\lambda_n = O(n^2)$; cf. the SM [74] and [97,98].

Unless otherwise stated, our SL problem will be regular. If irregular, the above properties are not guaranteed, while in certain cases they survive. Note also that, since $q(x) = 0$ here, $u_0(x) = \text{const}$ solves Eq. (13) with $\lambda_0 = 0$.

The stated properties allow us to expand

$$\varphi(x) = \sum_{n=0}^{\infty} \varphi_n u_n(x), \quad \Pi(x) = \sum_{n=0}^{\infty} \Pi_n w(x) u_n(x) \quad (14)$$

with $[\varphi_n, \Pi_{n'}] = i\delta_{n,n'}$ and thereby diagonalize H_{ITLL} in Eq. (8). Indeed, the latter becomes

$$H_{\text{ITLL}} = \frac{1}{2\pi} [\pi \Pi_0]^2 + \sum_{n=1}^{\infty} E_n a_n^\dagger a_n \quad (15)$$

with eigenenergies

$$E_n = \sqrt{\lambda_n}, \quad (16)$$

where we expressed $\varphi_n = \sqrt{\pi/2E_n}(a_n + a_n^\dagger)$ and $\Pi_n = -i\sqrt{E_n/2\pi}(a_n - a_n^\dagger)$ for $n \geq 1$ in bosonic operators $a_n^\dagger, a_n$ satisfying $[a_n, a_{n'}^\dagger] = \delta_{n,n'}$ [99]. The SM [74] also examines operators of the form (11) with $q(x) \neq 0$, corresponding to an inhomogeneous (possibly imaginary) mass $M(x)$ by adding $v_0^2 M(x)^2 \varphi(x)^2$ inside the big parentheses in Eq. (8).

The time evolution of any observable, such as $\varphi(x)$, contains terms like $a_n^\dagger e^{iE_n t} u_n(x)$ and $a_n e^{-iE_n t} u_n(x)$. Thus, for Eq. (1) to hold for all states, each $u_n(x)$ must have definite parity, i.e., be invariant under reflections $x \mapsto -x$ up to a sign, which must equal each phase factor $e^{\pm iE_n T}$ for a common shortest possible $T > 0$. Moreover, the $u_n(x)$ have definite parity if and only if $v(x)$, $K(x)$ [and $q(x)$ if nonzero] are even, i.e., H_{ITLL} is reflection symmetric. This leads to a statement in terms of spectral requirements [74]: *Inhomogeneous TLLs exhibit PWT as defined in Eq. (1) if and only if $v(x)$ and $K(x)$ are even and the eigenenergies are $E_n = \pi m_n/T$ for integers m_n so that* [100] (a) $0 = m_0 < m_1 < \dots$ with each m_n even or odd if n is even or odd, (b) $m_n \sim n$ for large n , i.e., $m_n = n + o(n)$.

To probe PWT based on Eq. (5), note that the results above allow one to compute correlations for inhomogeneous TLLs [74]. In particular, the ground-state $\varphi\varphi$ correlation function is [99]

$$\langle \Omega | \varphi(x, t) \varphi(x', t') | \Omega \rangle = \sum_{n=1}^{\infty} \frac{\pi}{2E_n} u_n(x) u_n(x') e^{-iE_n(t-t')}. \quad (17)$$

Thus, for a spectrum satisfying the PWT requirements, $\langle \Omega | \varphi(x, T) \varphi(x', 0) | \Omega \rangle = \langle \Omega | \varphi(-x, 0) \varphi(x', 0) | \Omega \rangle$, meaning that, for fixed x' , the correlations at $t = T$ are reflections of

those at $t = 0$. In Fig. 1, we plot Eq. (17) for $K(x) = 1$, which, by bosonization, is related to $\log F(t)$ in Eq. (5), in that it encodes the same information [74].

Inverse Sturm-Liouville problem—Reversing the logic, the spectral requirements for PWT yield an inverse problem for SL operators with Neumann BCs [101]: *Given a spectrum $E_n = \pi m_n/T$ for integers m_n as above, determine the even functions $v(x)$, $K(x)$ [and $q(x)$ if nonzero].*

Before a general discussion, we consider two examples.

First, the solution is known if the spectrum is exactly linear: For $K(x) = K$ constant but $v(x)$ general, then $\mathcal{A} = -v(x)\partial_x v(x)\partial_x = -v_0^2 \partial_y^2$ using Eq. (7) with $E_n = \pi n v_0/L$ and $u_{n>0}(x) = \sqrt{2v_0/L} \cos(n\pi[L + 2y]/2L)$, $u_0(x) = \sqrt{v_0/L}$. This is the conformal case of uncoupled right and left movers, enjoying PWT for any even $v(x) > 0$.

Second, for $v(x) = v\sqrt{1 - (2x/L)^2}$ and $K(x) = K[1 - (2x/L)^2]^\alpha$ with $\alpha > -1/2$ and $v, K > 0$, we get an irregular problem possessing Properties (i)–(iv) [102]. The eigenfunctions are Gegenbauer polynomials $C_n^{(\alpha)}(2x/L)$ (up to normalizations), which have definite parity and satisfy Eq. (13) with $\lambda_n = (\pi v_0/L)^2 n(n + 2\alpha)$, $v_0 = 2v/\pi$; cf. Sec. 18.3 in [103]. Clearly, only $\alpha = 0$ (Chebyshev polynomials of the first kind) exhibits PWT, while all $\alpha \neq 0$ do not, including $\alpha = 1/2$ (Legendre polynomials) and $\alpha = 1$ (Chebyshev polynomials of the second kind); cf. Sec. 18.7 in [103]. The respective $\varphi\varphi$ correlations are plotted in Fig. 2.

To solve the inverse SL problem in general, we write the operator (11) in Liouville normal form, as is always possible for regular problems; cf. Sec. 1.13(viii) in [103]: Let $\mathbf{u}(y) \equiv u(x)\sqrt{K(x)}$ for $y = y(x)$ in Eq. (7) and define

$$\hat{q}(y) \equiv -\frac{q(x)}{w(x)} + \frac{1}{\sqrt{K(x)}} v_0^2 \partial_y^2 \sqrt{K(x)}. \quad (18)$$

The eigenvalue problem (13) then becomes

$$v_0^2 \partial_y^2 \mathbf{u}(y) + [\lambda - \hat{q}(y)] \mathbf{u}(y) = 0 \quad (19)$$

with unchanged BCs [104], and if reflection symmetric in x , then so in y . Clearly, $\mathbf{u}_n(y)$ have the same eigenvalues as the respective $u_n(x)$, and they similarly form a complete orthonormal basis but with identity weight.

Two sets of eigenvalues for different BCs are usually required to uniquely solve an inverse SL problem. However, one set suffices for reflection-symmetric operators with certain BCs. Indeed, an even $\hat{q}(y)$ is uniquely determined by the spectrum for Neumann BCs [105,106]. Thus, taking $\hat{q}(y)$ as given, Eq. (18) [with $q(x) = 0$] becomes a second-order differential equation for $\sqrt{K(x)}$. By uniqueness, this determines an even $K(x)$ up to changing coordinates as in Eq. (7) and an overall

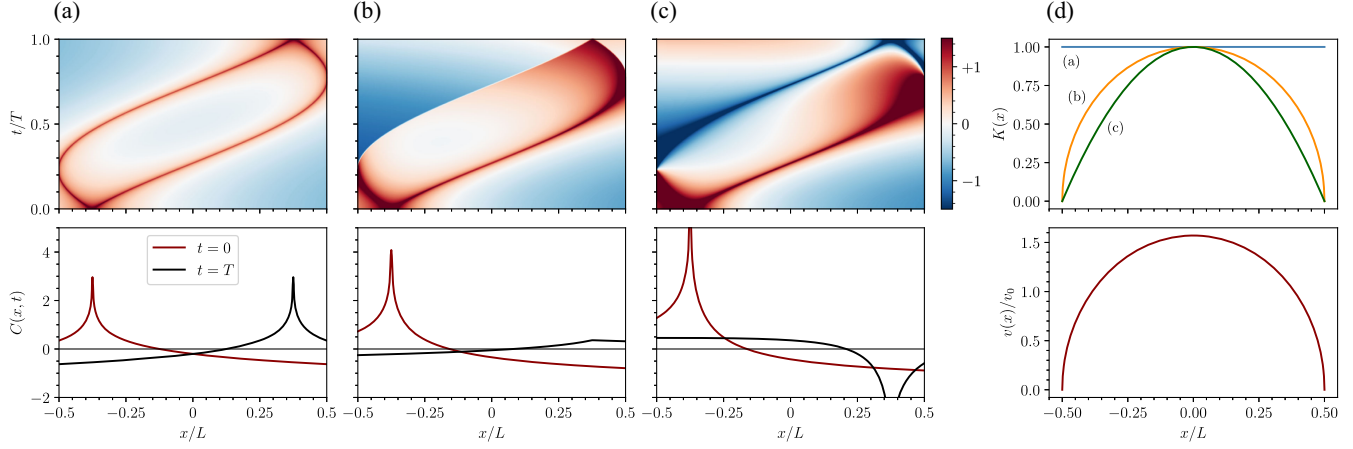


FIG. 2. Plot of $C(x, t) \equiv \text{Re}\langle \Omega | \varphi(x, t) \varphi(-3L/8, 0) | \Omega \rangle$ given by Eq. (17) for eigenfunctions in the form of (a) Chebyshev polynomials of the first kind, which exhibit PWT, contrasted with (b) Legendre polynomials and (c) Chebyshev polynomials of the second kind, which do not. The upper row contains intensity plots of the correlations as functions of $x/L \in [-1/2, 1/2]$ and $t/T \in [0, 1]$, while the lower row compares their profiles at $t = 0$ and T . Note the asymmetry in (b) and (c). The velocity $v(x)$ (same in all cases) and the respective Luttinger parameter $K(x)$ are shown in (d).

rescaling [assuming that $\hat{q}(y)$ is continuous and the solution $\sqrt{K(x)}$ is positive].

The results [105,106] do not reconstruct $\hat{q}(y)$, although there are constructive approaches [107–109]. However, as $K(x) = K$ gives an SL problem with $m_n = n$ and $\hat{q}(y) = 0$, a constant Luttinger parameter is the unique solution for the inverse problem given an exactly linear spectrum. Beyond this case, one may imagine that the spectrum could deviate from the leading n behavior as long as the requirements on the m_n are satisfied. By an exact Bohr-Sommerfeld/WKB quantization condition [110–113], we show in the SM [74] that this depends on the regularity: If the system is regular also after unfolding the interval $[-L/2, L/2]$ to the circle of twice the length [114], then PWT requires $\hat{q}(y) = 0$. Thus, the conformal case of an exactly linear spectrum is the only possibility for such regular inhomogeneous TLLs to exhibit PWT.

Discussion—We investigated the collective transfer of quantum information in continuous communication channels, introducing the notion of perfect wave transfer (PWT). Assuming conformal invariance, we showed that PWT is exhibited for one-particle excitations whenever the system is reflection symmetric. More generally, PWT with respect to arbitrary states amounts to an inverse spectral problem, studied in detail for inhomogeneous TLL theory—a prototypical model of $1 + 1$ D bosons with coupled right and left movers. Its solution shows that conformal invariance is essential for PWT if sufficient regularity is required. This approach is similar to results for spin chains with nearest-neighbor hopping: Their Hamiltonians can be represented as Jacobi matrices, yielding an inverse eigenvalue problem [9,115] whose solution is unique [109,116] given one set of eigenvalues for matrices symmetric with respect to the antidiagonal

(reflection symmetry) [117]. Remarkably, however, the continuum problem is more constrained due to asymptotic requirements, which imply that the spectrum must be exactly linear for inhomogeneous bosons to exhibit PWT under strong enough regularity conditions. This is not the case for discrete systems due to their intrinsic ultraviolet cutoff. Such a cutoff is also relevant for applications of TLL theory [49], and if present, there are no asymptotic requirements and more possibilities for PWT even for regular inhomogeneous TLLs. Moreover, as shown in the SM [74], PWT is also possible in inhomogeneous TLLs with fine-tuned mass terms, governed by a similar inverse spectral problem but with $q(x) \neq 0$.

By bosonization, our results for inhomogeneous TLLs cover fermionic theories with position-dependent interactions [61]. The essential property for these systems to exhibit PWT as in Eq. (1) was shown to be conformal invariance, assuming that the inhomogeneities are sufficiently regular, while it remains unclear what is possible if the regularity is relaxed. Consequently, PWT for all (not only one-particle) states holds for particular CFTs, such as inhomogeneous CFTs of free fermions and compactified bosons. It would be interesting to prove if PWT holds for any number of excitations in general CFTs and to extend to systems on the half or full infinite line.

Our Letter can be seen as a first investigation of PWT for continuous quantum systems, focusing on universal low-energy descriptions within CFT and TLL theory. A second step would be to study PWT for specific gapless $1 + 1$ D models, including Lieb-Liniger [85]. For those, our results apply at low energies, while full treatments require tailored solution methods, such as Bethe ansatz, but typically extended to inhomogeneous systems for PWT to be possible—a very difficult problem in itself. Generic

interaction or curvature effects may also spoil PWT beyond the low-energy regime. Refinements may then be needed, either relaxing PWT to specific classes of observables or states, or reformulating it as *almost* PWT with fidelity $1 - \epsilon$ ($\epsilon > 0$) for the transfer of information. A concrete proposal for a general approach that builds on our present results would be to use nonlinear TLL theory [118,119] to study such questions.

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- [101] A related prominent problem is the inverse scattering for integrable differential equations, solved by the Gelfand-Levitan-Marchenko equation. That case is on the infinite line, while we have a finite interval.
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