

Adaptive Ising Machine Based on Phase Locking of an Auto-Oscillator to a Biharmonic External Driving with Noise

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An auto-oscillator driven by a harmonic signal at about twice its free-running frequency is characterized by a bistable phase dynamics where the two states are separated by π radians. This phase bistability enables an oscillator to emulate a single Ising spin, providing a fundamental building block for the oscillator-based Ising machines (OIM). At the same time, a driving signal close to the oscillator free-running frequency locks the oscillator's phase at a single value, playing the role of a magnetic field bias in ensembles of real spins. We introduce a universal theory of phase auto-oscillators driven by a biharmonic signal (having frequency components close to single and double of the free-running oscillator frequency) with noise; with it, we show how deterministic phase locking and stochastic phase slips can be continuously tuned by varying the relative amplitudes and frequencies of the driving components. Using, as an example, a spin-torque nano-oscillator, we numerically validate this theory by implementing a deterministic Ising machine paradigm, a probabilistic one, and dual-mode operation of the two. This demonstration introduces the concept of adaptive Ising machines (AIM), a unified oscillator-based architecture that dynamically combines both regimes within the same hardware platform by properly tuning the amplitudes of the biharmonic driving relative to the noise strength. Benchmarking on different classes of combinatorial optimization problems, the AIM exhibits complementary performance compared to OIMs and probabilistic Ising machines, with adaptability to the specific problem class. This Letter introduces the first OIM capable of transitioning between deterministic and probabilistic computation taking advantage of a proper design of the trade-off between the strength of phase-locking of an auto-oscillator to a biharmonic external driving and noise, opening a path toward scalable, CMOS-compatible hardware for hybrid optimization and inference.

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Synchronization of nonlinear oscillators is a universal phenomenon that describes collective dynamics in physics, biology, and engineering. The Kuramoto synchronization model [1] is one of the most widely used theoretical frameworks for understanding the phase dynamics of externally driven auto-oscillators. In recent years, the phase dynamics described by [1] has emerged as a powerful framework for unconventional computing, enabling the mapping of oscillator phase states onto Ising spins for the

solution of combinatorial optimization problems (COPs) [2–6]. Oscillator-based Ising machines (OIMs), which exploit second-harmonic injection locking to discretize oscillator phases, have demonstrated efficient deterministic minimization of the Ising Hamiltonian in optical, electronic, and spintronic systems [7–12]. A complementary approach is based on probabilistic Ising machines (PIMs), which leverages Gibbs sampling to generate a discrete dynamics that samples from the Boltzmann distribution, has been also developed recently [13–15]. While OIMs are effective at rapid deterministic convergence with a speed defined by their oscillation frequency, they lack the ability to explore solution spaces statistically. On the other hand, PIMs provide an effective platform for sampling but with a convergence speed fixed by the random number generation and the sequential Gibbs sampling protocol. Recent works have also demonstrated promising performance

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for stochastic sampling in noise-injected opto-electronic Ising machines [16], Kerr parametric oscillator systems [17], and a network of coupled optical parametric oscillators [18]. However, to date, no oscillator-based system has provided programmable interpolation between probabilistic, deterministic, and adaptive Ising operation within the same hardware architecture, using a single control parameter and exploiting only the intrinsic dynamical properties of the oscillators.

In this letter, we introduce the concept of adaptive Ising machine (AIM), an oscillator-based architecture that unifies the working principle of OIMs and PIMs on the same hardware platform. The AIM operates by dynamically tuning the balance between deterministic phase locking and stochastic phase fluctuations through continuous control of the oscillator's input parameters, without introducing external noise sources. Our AIM is different from the self-adaptive Ising machine proposed by Delacour [19], which archives adaptation at the algorithm level to handle constrained optimization problem. Building on a universal model of the phase dynamics in an auto-oscillator under composite and noisy driving, we developed an analytical theory describing how deterministic phase locking and stochastic phase slips can be continuously tuned by external control parameters. Formulated at the level of generic phase dynamics, the proposed protocol is, in principle, applicable to any class of auto-oscillators, including optical systems [20–22]. Using a spin-torque nano-oscillator (STNO), as a representative example, we identify the potential limitations of OIMs for facing COPs having the local bias that is a phase difference between the two state phases of the injection locked oscillator different than π . We also demonstrate that AIMs can work as OIMs, as PIMs, and in a dual-mode operation regime, where robust deterministic binarization and tunable probabilistic sampling are key elements that provide a distinct form of hybridization at the device level. By integrating deterministic dynamics and probabilistic sampling within the same physical architecture—rather than combining heterogeneous systems—AIM can improve both feasibility and scalability compared to previous approaches that, for instance, combine classical Ising machines with postprocessing, integrate annealers with digital solvers [23], apply constraint correction [24], decompose Ising problems into smaller subproblems, or rely on coupling-based methods. We further benchmark AIM on instances of two COPs, maximum satisfiability problem (Max-SAT) and maximum cut problem (Max-Cut). AIM, therefore, establishes the first oscillator-based Ising machine capable of transitioning between deterministic and probabilistic computation, providing a scalable, CMOS-compatible approach with potential applications in hybrid combinatorial optimization and inference.

The dynamics of an auto-oscillatory system can be described by its phase $\varphi(t)$. Under weak periodic driving

signal and noise, the phase dynamics in a general case is governed by the equation:

$$\frac{d\varphi}{dt} = \omega_0 - h(\omega_s t, \varphi) + \xi(t), \quad (1)$$

where ω_0 is free-running frequency of the auto-oscillator, $h(\omega_s t, \varphi)$ accounts for the effect of the periodic driving signal with fundamental frequency ω_s , and $\xi(t)$ is the thermal noise, modeled as white Gaussian noise with zero mean and second-order correlator $\langle \xi(t)\xi(t') \rangle = 2\Delta\omega_0\delta(t-t')$, where $\Delta\omega_0$ is the half-linewidth of the free-running oscillator ($h=0$). For slow phase dynamics, it is convenient to introduce the phase difference between the oscillator and the driving signal $\phi(t) = \varphi(t) - \omega_s t$. In this scenario, Eq. (1) can be written as $(d\phi/dt) = -\Omega_s - H(\phi) + \xi(t)$, where $\Omega_s = \omega_s - \omega_0$ is the frequency mismatch or detuning, and $H(\phi)$ is the averaged driving action. To describe the general case of an auto-oscillator under composite and noisy driving, we introduce the probability distribution function (PDF) $P_D(t, \phi)$ of the oscillator phase.

The standard stochastic analysis leads to the following Fokker-Planck equation:

$$\frac{\partial P_D}{\partial t} - \frac{\partial}{\partial \phi} [(\Omega_s + H(\phi))P_D] - \Delta\omega_0 \frac{\partial^2 P_D}{\partial \phi^2} = 0. \quad (2)$$

This equation describes the diffusion of an effective Brownian particle moving in a potential landscape defined by detuning, driving, and noise. For the driven oscillator ($\Omega_s + H(\phi) \neq 0$), the effective potential takes the form

$$\begin{aligned} U(\phi) &= \int_{\phi_0}^{\phi} [\Omega_s + H(\phi')] d\phi' \\ &= \Omega_s \phi - \sum_{n=1}^{\infty} \frac{\Lambda_n}{n} \cos[n(\phi - \beta_n)], \end{aligned} \quad (3)$$

where the coefficient $\Lambda_n > 0$ is proportional to the amplitude of the n th harmonics of the driving signal, and β_n is the corresponding internal phase shift. The stationary solution of Eq. (2) is given by

$$P_D(\phi) = p_0(1 + \rho G(\phi)) \exp\left(-\frac{U(\phi)}{\Delta\omega_0}\right), \quad (4)$$

with integration constant p_0 and corrections specified by ρ and $G(\phi)$. Full derivations are provided in Sec. I of Supplemental Material [25].

The presence of a second harmonic driving term gives rise to two distinct stable phase states, $\phi_A > \pi$ and $\phi_B < \pi$, separated by $|\phi_A - \phi_B| = \pi$. In the absence of the first harmonic, these states are energetically equivalent, and the oscillator exhibits symmetric phase slips between them with equal probability. Adding a first harmonic driving breaks this symmetry, reshaping the energy landscape and

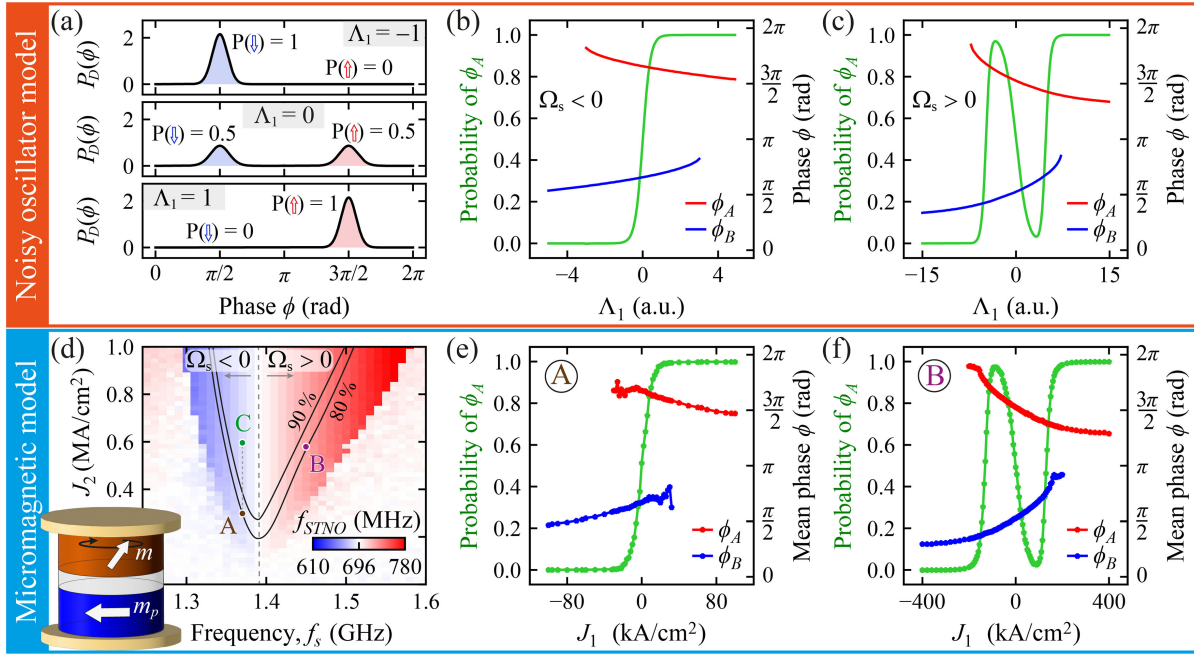


FIG. 1. A comparison of theoretical and numerical results for the oscillator's dynamics. (a) PDF of a noisy oscillator for different values of Λ_1 . (b),(c) Phase values from the analytical model for negative and positive detuning, respectively, and the probability dependence of ϕ_A on Λ_1 (green curve). (d) Arnold tongue region centered at $2f_0$. The point C is used for the simulations shown in Fig. 2. (e),(f) Corresponding phase values and probability of ϕ_A from numerical simulations of STNO for negative [point A in panel (d)] and positive detuning [point B in panel (d)], respectively.

biasing the system toward one of the two phases, as illustrated in Fig. 1(a). Detuning further modifies the dynamics by tilting the effective energy potential [26]. For negative detuning, the occupation probability of state ϕ_A as a function of Λ_1 exhibits a sigmoidal dependence, as shown in Fig. 1(b). In contrast, under the same condition but positive detuning, the potential becomes more complex, resulting in a nonmonotonic dependence of the phase probability with Λ_1 , as illustrated in Fig. 1(c). This nonmonotonicity arises from the emergence of an additional minimum with $|\phi_A - \phi_B| < \pi$ (see Sec. II of Supplemental Material [25]) in the energy landscape. This behavior demonstrates that biharmonic driving enables direct control of the phase landscape by exploiting intrinsic dynamical properties of the oscillator.

Those results are valid for any auto-oscillatory system. Here we focus on STNOs realized with magnetic tunnel junctions (MTJs) and subjected to a biharmonics spin-transfer torque (STT) current and thermal noise. The STNO device considered has an elliptical cross-section, an out-of-plane free layer (FL) magnetization $\mathbf{m}$, and an in-plane polarizer magnetization, $\mathbf{m}_p$, as shown in the sketch in Fig. 1(d) [27]. A detailed description of the MTJ structure is provided in Fig. 2 in Sec. III of Supplemental Material [25]. This geometry allows for the realization of STNOs with ultralow threshold for the excitation of self-oscillations [28]. The FL dynamics is studied through micromagnetic simulations based on the Landau-Lifshitz-

Gilbert-Slonczewski (LLGS) equation within the macrospin approximation [29,30]. The STNO is driven by an ac current with first and second harmonics, $J_{ac} = J_1 \sin(\omega_s t + \beta_1) + J_2 \sin(2\omega_s t + \beta_2)$, where $\omega_s = 2\pi f_s$ is the fundamental frequency. As a first step we analyze the effect of one component of the driving signal, $J_2 \sin(2\omega_s t + \beta_2)$. The STNO exhibits two synchronization regions (Arnold tongues) [26]: the first-harmonic injection locking (FHIL), where $2\omega_s$ is centered around the free-running frequency ω_0 , and the second-harmonic injection locking (SHIL), where $2\omega_s$ is centered around $2\omega_0$. Figure 1(d) shows the SHIL region, where two phase states separated by π are stabilized. The black contour lines define the areas within the Arnold tongue where the injection-locking is stable for more than 90% and 80% of the simulation time. Details of the model and STNO characterization are provided in Sec. III of Supplemental Material [25]. The STNO locked dynamics is affected by the combined effect of the amplitudes of the biharmonic driving, by detuning effect that depends on the working point within the Arnold tongue, and by thermal noise strength.

At high J_2 (e.g., point C), the injection-locked oscillator remains fixed in one of the two stable phase states within the simulation time. As J_2 decreases, phase slip events are observed (e.g., point A), and their occurrence rate increases toward the boundary of the Arnold tongue. At lower values of J_2 , the STNO's phase is no longer synchronized with the external signal. Thermal noise further enhances the

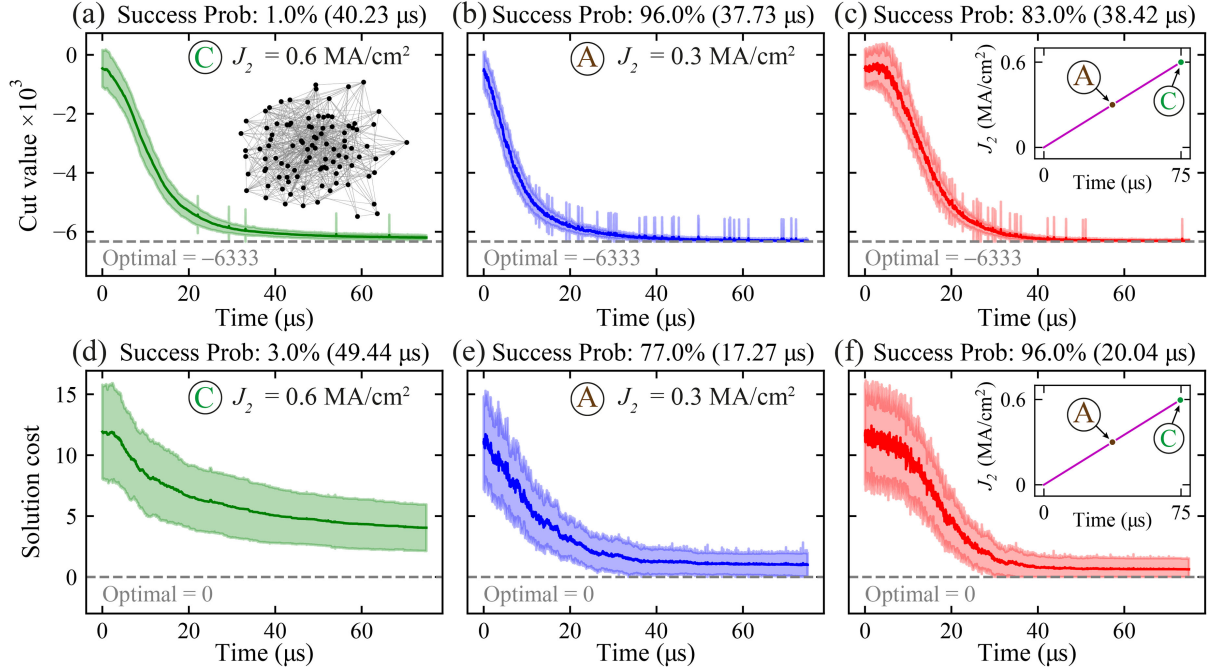


FIG. 2. Performance benchmark of the different types of STNO-based IMs by using simulated annealing as the energy minimization algorithm. (a)–(c) Cut value over time for the Max-Cut problem instance “gkald” on a 101-node graph [inset in (a)], evaluated in three different operating modes: (a) OIM—point C ($J_2 = 0.6$ MA/cm²) (b) PIM—point A ($J_2 = 0.3$ MA/cm²) and (c) AIM: J_2 is linearly ramped from 0 to 0.6 MA/cm², enabling a dynamic transition from unlocked state to OIM. (d)–(f) Time evaluation of solution cost for a Max-SAT problem instance with 112 Ising spins, evaluated under the same three operating modes as in (a)–(c). The solid lines represent the average loss as a function of simulation time, with shaded areas indicating the first standard deviation of the distribution over 100 trials. The dashed gray lines mark the optimal value for each COPs.

occurrence of phase slips, promoting random transitions between the two stable phase states and reducing the overall locking stability [31–33]. As predicted analytically, the phases can be externally tuned by applying the first-harmonic driving component $J_1 \sin(\omega_s t + \beta_1)$, Figs. 1(e) and 1(f) show the probability of the STNO being locked in phase ϕ_A as a function of the J_1 , evaluated at two points chosen for their highest phase slips rates, within the region of negative (point A) and positive (point B) detuning, respectively. The behavior is in quantitative agreement with the theory. This observed trend is robust over a wide parameter range within the Arnold tongue (see Sec. IV of Supplemental Material [25]). Moreover, the locking probability can be further controlled by adjusting the internal phase β_1 . Section V of Supplemental Material [25] summarizes the dependence of the locking probability of ϕ_A at point B as a function of J_1 for different values of β_1 .

An STNOs operating in the SHIL regime with vortex-state static magnetization have already been shown to exploit phase slip dynamics to generate unbiased binary random numbers with 50% probability, proposing them as a building block for stochastic computing [31]. However, to realize PIMs based on STNOs, it is crucial that the probability of the phase slip events can be tuned via an external parameter with the necessary hyperbolic tangent

($\tanh$) dependence [13]. Our findings demonstrate such a tunability, and highlight the importance of the working point position inside the Arnold tongue. Furthermore, by simply tuning J_2 , the STNO can be driven from a deterministic regime with stable phase locking to a stochastic regime characterized by frequent phase slips. This fundamental property enables the design and the implementation of both OIMs and PIMs on a unified hardware platform, controlled by the strength of the relative amplitudes of the biharmonic driving signal. Importantly, in contrast to other implementations, stochasticity here originates from the intrinsic phase noise of the oscillator, and therefore does not require externally injected noise sources. Building on this concept, we introduce the AIM, based on a network of N coupled STNOs, that collectively attempt to minimize the Ising Hamiltonian $H(\sigma) = -(\frac{1}{2} \sum_{i,j} \mathcal{J}_{ij} \sigma_i \sigma_j + \sum_i h_i \sigma_i)$, where σ_i are the binary Ising spins, $\mathcal{J}_{ij}$ is the coupling strength between the spins i and j , and h_i is an external bias applied to the spin i . The time evolution of the normalized magnetization vector $\mathbf{m}_i$ of the FL in the i^{th} oscillator $\forall i \in [1, N]$ is described by a system of N coupled LLGS equations. The coupling among the STNOs is implemented with the binarized phases σ_i , in analogy with the PIMs [13,14]. This approach overcomes the limitation of direct phase coupling in conventional OIMs. In this configuration,

dedicated CMOS circuitry computes the effective coupling force applied to each oscillator according to $J_{i1} = \gamma(\sum_j \mathcal{J}_{ij}\sigma_j + h_i)$, where γ is the inverse pseudotemperature used for implementing the annealing schedule, similarly to PIMs [13,15]. To evaluate the performance of the proposed AIM, we first assess the AIM working in the PIM regime at point A in Fig. 1(d). The phase probability in Fig. 1(e) follows a tanh dependence with slope of 10^{-4} , which we use as a scaling constant for the inverse pseudotemperature γ . Sampling capability is tested by implementing the invertible AND gate, mapped onto a three-node Ising model. The resulting phase-based probability distribution closely matches the expected Boltzmann distribution, confirming that the system correctly samples all four valid logic states with uniform probability (see Sec. VI of Supplemental Material [25]).

We next benchmark the AIM on two representative COPs, with approximately 100 Ising spins—corresponding to a network of 100 STNOs—and compare its performance against the OIM and the PIM. As discussed above, the STNO-based IM supports three modes: (i) OIM, stable deterministic injection locking at large J_2 [point C in Fig. 1(d)]; (ii) PIM, stochastic phase locking at small J_2 [point A in Fig. 1(d)]; and (iii) AIM, where J_2 is linearly ramped from 0 to point C, enabling a continuous transition from unlocked to stochastic and, then, to deterministic dynamics. Figures 2(a)–2(c) summarizes the results for the Max-Cut instance “gka1d” (101 nodes) from the BiqMac library, and Figs. 2(d)–2(f) show the performance on a Max-SAT instance with 20 clauses and 91 variables (112 spins). In both cases, we apply a standard simulated annealing schedule, with γ linearly increased over a total simulation time of 75 μ s. Success probabilities, calculated over 100 trials, are reported in the subfigure titles, with average solution times in parentheses. The results reveal clear performance distinctions across the operating modes. The deterministic OIM struggles to explore the solution space. Both the PIM and the AIM achieve high success probabilities, with the PIM outperforming the AIM on the Max-Cut instance, whereas the AIM exhibits superior performance on the Max-SAT instance (see Sec. VII in Supplemental Material [25] for additional results on large-scale instances). These findings demonstrate that no single mode universally dominates, emphasizing the importance of adaptive strategies that dynamically select or combine computational regimes according to the problem.

We have introduced the idea of AIM, an oscillator-based architecture that unifies deterministic and probabilistic regimes within a single hardware platform, rather than combining heterogeneous approaches. By establishing a universal theory of auto-oscillators under biharmonics noisy driving, and validating it with STNOs, we demonstrated how AIMS exploit transitions between deterministic dynamics and tunable probabilistic sampling by tuning the phase slip probability in facing different classes of COPs.

The tunability of the AIM working regime is directly linked to the strategy for developing Ising solvers that implement hybrid approaches, either digital-analog or quantum-classical, which are expected to be essential for overcoming the scaling limits of any single paradigm [34,35]. These features not only broaden the scope of problems that OIMs can address, but also point toward scalable, CMOS-compatible hardware for hybrid optimization and sampling. Notably, spintronic technology and processes for co-integrating MTJs with CMOS are already well established in several fabs [36]. In this context, AIM complements algorithmic hybrid approaches, and positions oscillator-based systems as promising candidates for the next-generation of Ising machines.

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Data availability—The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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- [1] J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort, and R. Spigler, The Kuramoto model: A simple paradigm for synchronization phenomena, *Rev. Mod. Phys.* **77**, 137 (2005).
 - [2] T. Wang and J. Roychowdhury, OIM: Oscillator-based ising machines for solving combinatorial optimisation problems, in *Unconventional Computation and Natural Computation* (Springer, New York, 2019), pp. 232–256.
 - [3] T. Zhang *et al.*, A review of Ising machines implemented in conventional and emerging technologies, *IEEE Trans. Nanotechnol.* **23**, 704 (2024).
 - [4] D. I. Albertsson and A. Rusu, Highly reconfigurable oscillator-based ising machine through quasiperiodic modulation of coupling strength, *Sci. Rep.* **13**, 4005 (2023).

- [5] S. Chiavazzo, M. Calvanese Strinati, C. Conti, and D. Pierangeli, Ising machine by dimensional collapse of nonlinear polarization oscillators, *Phys. Rev. Lett.* **135**, 063801 (2025).
- [6] Z.-Y. Zhou, C. Gneiting, J. Q. You, and F. Nori, Frustration elimination and excited state search in coherent Ising machines, *Phys. Rev. Lett.* **134**, 090401 (2025).
- [7] T. Honjo *et al.*, 100,000-spin coherent Ising machine, *Sci. Adv.* **7**, 952 (2021).
- [8] H. Yamashita, K. Okubo, S. Shimomura, Y. Ogura, J. Tanida, and H. Suzuki, Low-rank combinatorial optimization and statistical learning by spatial photonic Ising machine, *Phys. Rev. Lett.* **131**, 063801 (2023).
- [9] J. Vaidya, R. S. Surya Kanthi, and N. Shukla, Creating electronic oscillator-based Ising machines without external injection locking, *Sci. Rep.* **12**, 981 (2022).
- [10] M. K. Bashar, Z. Li, V. Narayanan, and N. Shukla, An FPGA-based Max-K-Cut accelerator exploiting oscillator synchronization model, in *Proceedings of the 2024 25th International Symposium on Quality Electronic Design (ISQED)* (IEEE, New York, 2024), pp. 1–8.
- [11] A. Grimaldi, L. Sánchez-Tejerina, N. Anjum Aadit, S. Chiappini, M. Carpentieri, K. Camsari, and G. Finocchio, Spintronics-compatible approach to solving maximum-satisfiability problems with probabilistic computing, invertible logic, and parallel tempering, *Phys. Rev. Appl.* **17**, 024052 (2022).
- [12] Y.-T. Liu, C.-C. Peng, T.-Y. Hung, Y.-H. Huang, and C.-F. Pai, Advancing the problem-solving capabilities of Ising machines based on spin Hall nano-oscillators, *Phys. Rev. Appl.* **22**, 064009 (2024).
- [13] K. Y. Camsari, R. Faria, B. M. Sutton, and S. Datta, Stochastic P-bits for invertible logic, *Phys. Rev. X* **7**, 031014 (2017).
- [14] N. A. Aadit, A. Grimaldi, M. Carpentieri, L. Theogarajan, J. M. Martinis, G. Finocchio, and K. Y. Camsari, Massively parallel probabilistic computing with sparse Ising machines, *Natl. Electron. Rev.* **5**, 460 (2022).
- [15] E. Raimondo *et al.*, High-performance and reliable probabilistic Ising machine based on simulated quantum annealing, *Phys. Rev. X* **15**, 041001 (2025).
- [16] F. Böhm, D. Alonso-Urquijo, G. Verschaffel, and G. Van der Sande, Noise-injected analog Ising machines enable ultrafast statistical sampling and machine learning, *Nat. Commun.* **13**, 5847 (2022).
- [17] G. Margiani, J. del Pino, T. L. Heugel, N. E. Bousse, S. Guerrero, T. W. Kenny, O. Zilberberg, D. Sabonis, and A. Eichler, Deterministic and stochastic sampling of two coupled Kerr parametric oscillators, *Phys. Rev. Res.* **5**, L012029 (2023).
- [18] J. Tosca, M. C. Strinati, C. Conti, and C. Ciuti, Emergent equilibrium in all-optical single quantum-trajectory Ising machines, *Phys. Rev. Lett.* **134**, 230404 (2025).
- [19] C. Delacour, Self-adaptive Ising machines for constrained optimization, in *2025 Design, Automation & Test in Europe Conference* (IEEE, New York, 2025), pp. 1–7.
- [20] C. C. Rodrigues, C. M. Kersul, A. G. Primo, M. Lipson, T. P. M. Alegre, and G. S. Wiederhecker, Optomechanical synchronization across multi-octave frequency spans, *Nat. Commun.* **12**, 5625 (2021).
- [21] G. Beltramo, R. Horváth, G. Modica, and R. Braive, Injection-locking dynamics of two self-pulsing nanocavities optically coupled on photonic integrated circuit, *Commun. Phys.* **9**, 12 (2025).
- [22] D. Zou, Y. Song, O. Gat, M. Hu, and P. Grelu, Synchronization of the internal dynamics of optical soliton molecules, *Optica* **9**, 1307 (2022).
- [23] Y. Atobe, M. Tawada, and N. Togawa, A novel classical-Ising hybrid annealing method with QUBO model cutting, in *Proceedings of the 2024 IEEE 67th International Midwest Symposium on Circuits and Systems (MWSCAS)* (IEEE, New York, 2024), pp. 1154–1157.
- [24] S. Kikuchi, N. Togawa, and S. Tanaka, Hybrid optimization method using simulated-annealing-based Ising machine and quantum annealer, *J. Phys. Soc. Jpn.* **92**, 124002 (2023).
- [25] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/jv9m-4mny> for the full derivation of the general theory of synchronization of a noisy phase oscillator, for spin-torque nano oscillator characterization and phase probability at different working points.
- [26] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, England, 2001).
- [27] B. Fang *et al.*, Giant spin-torque diode sensitivity in the absence of bias magnetic field, *Nat. Commun.* **7**, 11259 (2016).
- [28] Z. Zeng *et al.*, Ultralow-current-density and bias-field-free spin-transfer nano-oscillator, *Sci. Rep.* **3**, 1426 (2013).
- [29] D. C. Ralph and M. D. Stiles, Spin transfer torques, *J. Magn. Magn. Mater.* **320**, 1190 (2008).
- [30] A. Meo, C. E. Cronshaw, S. Jenkins, A. Lees, and R. F. L. Evans, Spin-transfer and spin-orbit torques in the Landau–Lifshitz–Gilbert equation, *J. Phys. Condens. Matter* **35**, 025801 (2023).
- [31] N.-T. Phan *et al.*, Unbiased random bitstream generation using injection-locked spin-torque nano-oscillators, *Phys. Rev. Appl.* **21**, 034063 (2024).
- [32] M. Tortarolo, B. Lacoste, J. Hem, C. Dieudonné, M.-C. Cyrille, J. A. Katine, D. Mauri, A. Zeltser, L. D. Buda-Prejbeanu, and U. Ebels, Injection locking at 2f of spin torque oscillators under influence of thermal noise, *Sci. Rep.* **8**, 1728 (2018).
- [33] J. F. Sierra, M. Quinsat, F. Garcia-Sanchez, U. Ebels, I. Joumar, A. S. Jenkins, B. Dieny, M.-C. Cyrille, A. Zeltser, and J. A. Katine, Influence of thermal fluctuations on the emission linewidth in MgO-based spin transfer oscillators, *Appl. Phys. Lett.* **101**, 062407 (2012).
- [34] N. Mohseni, P. L. McMahon, and T. Byrnes, Ising machines as hardware solvers of combinatorial optimization problems, *Nat. Rev. Phys.* **4**, 363 (2022).
- [35] T. Tran, M. Do, E. Rieffel, J. Frank, Z. Wang, B. O’Gorman, D. Venturelli, and J. Beck, A hybrid quantum-classical approach to solving scheduling problems, *Proc. Int. Symp. Comb. Search* **7**, 98 (2021).
- [36] MRAM-Info MRAM Industry Portal, <https://www.mram-info.com/>.
- [37] www.petaspin.com.