

# Unconventional Fractional Phases in Multiband Vortexable Systems

Siddhartha Sarkar<sup>1,2,\*</sup>, Xiaohan Wan<sup>1,\*</sup>, Ang-Kun Wu<sup>3</sup>, Shi-Zeng Lin<sup>4,5,†</sup> and Kai Sun<sup>1,‡</sup>

<sup>1</sup>*Department of Physics, University of Michigan, Ann Arbor, Michigan 48109, USA*

<sup>2</sup>*Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Straße 38, 01187 Dresden, Germany*

<sup>3</sup>*Theoretical Division, T-4, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*

<sup>4</sup>*Theoretical Division, T-4 and CNLS, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*

<sup>5</sup>*Center for Integrated Nanotechnologies (CINT), Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*



(Received 20 May 2025; revised 2 September 2025; accepted 28 October 2025; published 20 November 2025)

We study topological flat bands with distinct features that deviate from conventional Landau level behavior. We show that even in the ideal quantum geometry limit, moiré flat band systems can exhibit physical phenomena fundamentally different from Landau levels without lattices. In particular, we find new fractional quantum Hall states emerging from multiband vortexable systems, where multiple exactly flat bands appear at the Fermi energy. While the set of bands as a whole exhibits ideal quantum geometry, individual bands separately lose vortexability, and thus making them very different from a stack of Landau levels. At certain filling fractions, we find fractional states whose Hall conductivity deviates from the filling factor. Through careful numerical and analytical studies, we rule out all known mechanisms—such as fractional quantum Hall crystals or separate filling of trivial and topological bands—as possible explanations. Leveraging the exact solvability of vortexable systems, we use analytic Bloch wave functions to uncover the origin of these new fractional states, which arises from the commensurability between the moiré unit cell and the magnetic unit cell of an emergent effective magnetic field.

DOI: [10.1103/jmk4-46tj](https://doi.org/10.1103/jmk4-46tj)

**Introduction**—The interplay between band topology and strong interactions in Landau levels (LLs) gives rise to exotic quantum states, known as the fractional quantum Hall (FQH) effect [1,2]. Inspired by Haldane’s seminal work on the quantum anomalous Hall effect [3], it was predicted that fractionalized states could also arise without magnetic fields, by utilizing topological flat bands—leading to the concepts of fractional quantum anomalous Hall effects and fractional Chern insulators (FCIs) [4–9]. Recent proposals suggested moiré systems as ideal platforms [10–14], and FCI states have now been observed in twisted bilayer MoTe<sub>2</sub> [15–18] and rhombohedral graphene plus hBN superlattices [19,20].

Although FCIs do not require external magnetic fields, most known examples are adiabatically connected to FQH states in LLs or multilayered LL systems [8,9,21–24]. It has

also been shown that topological flat bands with quantum geometry more analogous to Landau levels—namely, with more uniform Berry curvature distribution and ideal quantum geometry (meaning that the trace of the quantum metric tensor, which measures the spread of the wave functions [25], equals the Berry curvature)—host more robust fractional states with larger energy gaps [11,21,24,26–31]. In this context, the concept of “vortexable” bands [31] have recently been put forward; these bands are defined as bands with wave functions  $\psi$  such that after multiplying the wave function by a holomorphic function  $f(z)$  of complexified real space coordinate  $z = x + iy$ , the new function  $f(z)\psi$  is fully contained in the band. These bands share the same exotic properties as the remarkable lowest Landau level (LLL)—such as ideal quantum geometry and nontrivial algebra—and exhibit corresponding experimental signatures: for example, vortex insertion costs no energy, fractional states can emerge from short-range repulsion [11,28–32].

In this Letter, we explore the opposite question: Can topological flat bands support fractionalized physics that fundamentally deviates from LL behavior? To address this, we focus on exactly flat bands with ideal quantum geometry as a controlled platform, while emphasizing that our conclusions extend beyond the ideal limit due to many-body gap protection. Recent studies [32–40] reveal that vortexable systems encompass a rich family

\*These authors contributed equally to this work.

†Contact author: szl@lanl.gov

‡Contact author: sunkai@umich.edu

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI. Open access publication funded by the Max Planck Society.

of models: while some closely resemble LLs, others exhibit fundamentally new behaviors. These systems preserve advantages such as flatness and exact solvability, yet display features absent in conventional LLs.

A striking example is the emergence of multiband vortexable systems [35,37–40], where multiple exactly flat bands appear at the Fermi energy. As a whole, they remain vortexable with ideal quantum geometry, but individually lose vortexability—unlike stacks of LLs, where each band retains this property. Moreover, instead of carrying total Chern number  $C = N$  as in  $N$  LLs, these systems universally exhibit  $C = 1$ , independent of degeneracy or symmetry [40].

Using exact numerics, we identify fractional states across various multiband vortexable models. Depending on filling, the behavior varies: some mimic  $N - 1$  trivial bands plus a  $C = 1$  Chern band, while others exhibit fundamentally new fractionalization. For instance, at  $\nu = 2/3$ , we observe a fractional state with many-body Chern number  $C_{\text{mb}} = 1/3$ , violating the conventional relation  $C_{\text{mb}} = \nu$ . Through detailed numerical and theoretical analyses, we rule out known explanations such as fractional crystals or simple band fillings. Crucially, the many-body ground states show strong interband entanglement, defying descriptions based on individual bands.

Leveraging the exact solvability of vortexable systems, we analytically solve for the Bloch wave functions and identify the underlying mechanism: the commensurability between the moiré unit cell and the effective magnetic unit cell of an emergent magnetic field. In contrast to LLs (without lattices), where a single magnetic length governs physics, topological flat bands feature two length scales—the moiré unit cell and an emergent magnetic length. Spatial periodicity requires that the unit cell encloses an integer ( $N$ ) number of flux quanta. When  $N = 1$ , the system mimics a LL; when  $N > 1$ , new types of fractional states emerge, as schematically shown in Fig. 1. Our Letter focuses on this  $N > 1$  regime, revealing novel fractionalization fundamentally distinct from conventional Landau levels.

**Model**—Although our results are generic to any continuum model with multiple flat bands per sublattice, spin, or valley with ideal quantum geometry, as a concrete example, we consider a continuum Hamiltonian which describes a quadratic band crossing point (QBCP) at a time reversal invariant momentum coupled to a moiré periodic strain field [35],

$$\mathcal{H}(\mathbf{r}) = \begin{pmatrix} 0 & \text{H.c.} \\ -4\partial_z^2 + \tilde{A}(\mathbf{r}) & 0 \end{pmatrix}. \quad (1)$$

Here  $z = x + iy$  is the complex coordinate,  $\tilde{A}(\mathbf{r}) = A_x(\mathbf{r}) + iA_y(\mathbf{r})$  where  $A_x = u_{xx} - u_{yy}$  and  $A_y = u_{xy}$  are moiré periodic shear strain fields. Note that this Hamiltonian is chiral or sublattice symmetric  $\{\sigma_z, \mathcal{H}(\mathbf{r})\} = 0$ . As shown in

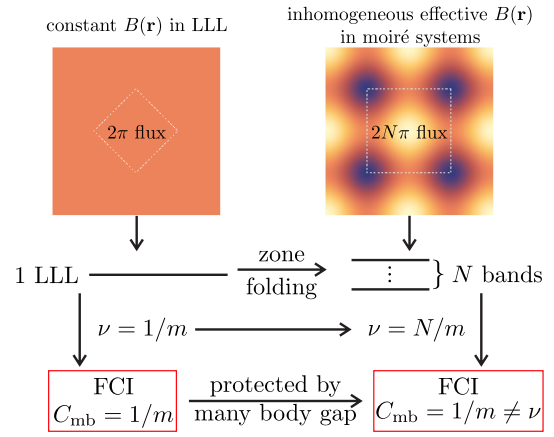


FIG. 1. Schematic showing how a FCI state with unequal filling fraction and many-body Chern number arises in multiband vortexable systems due to inhomogeneous effective magnetic field distribution  $B(\mathbf{r})$  and commensurability between moiré unit cell and magnetic length. The white dashed lines show the smallest unit cells.

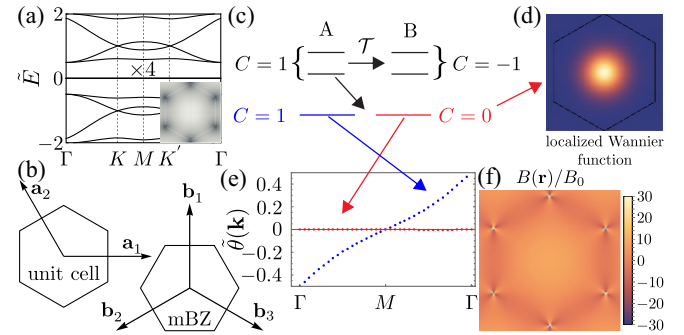


FIG. 2. Four exact flat bands in single layer system with QBCP under periodic strain field. (a) Band structure for the Hamiltonian in Eq. (1) with  $\tilde{A}(\mathbf{r}) = -(\alpha/2) \sum_{n=1}^3 e^{i(4-n)\phi} \cos(\mathbf{b}_n \cdot \mathbf{r})$  showing 4 exact flat bands at the “magic” value  $\tilde{\alpha} = (\alpha/|\mathbf{b}|^2) = -2.88$  where  $\mathbf{b}_i$  ( $i = 1, 2, 3$ ) are the moiré reciprocal lattice vectors shown in (b). Panel (b) also shows the moiré unit cell, lattice vectors  $\mathbf{a}_1$  and  $\mathbf{a}_2$ , moiré Brillouin zone (mBZ). Density plot of  $|\psi_{\Gamma}(\mathbf{r})|$  (normalized by its maximum) is shown at the bottom right corner of (a). The dark spots represent zeros of  $|\psi_{\Gamma}(\mathbf{r})|$ . There are two zeros at the corners of the unit cell. (c) Decomposition of the four flat bands in sublattice space. Two bands that are polarized on A sublattice together have  $C = 1$ , while the other two bands polarized on B sublattice together have  $C = -1$ . The two sublattices are related by time reversal symmetry  $\mathcal{T}$ . The two bands polarized on A sublattice can be decomposed into one  $C = 1$  band and one  $C = 0$  band. The  $C = 0$  band corresponds to a very localized Wannier function  $w(\mathbf{r})$ . (d) Density plot of  $|w(\mathbf{r})|^2$ , where brighter color represent larger value of the norm of the wave function. (e) Wilson loop spectra  $\tilde{\theta}(\mathbf{k}) = [\theta(\mathbf{k})/2\pi]$  of the  $C = 1$  and  $C = 0$  band plotted in blue and red, respectively. (f) Effective magnetic field  $B(\mathbf{r})/B_0$  in the moiré unit cell for the flatbands obtained by comparing the flatband wave functions to lowest Landau Level (LLL) wave functions.

Fig. 2(a), it was shown in [35] that for a simple first harmonic moiré strain field with  $p6mm$  space group symmetry there are four exact flat bands at some magic value of strain strength. Two of the four bands are polarized on the  $A$  sublattice while the other two are polarized on  $B$  sublattice. Notably, one of the two sublattice-polarized  $\Gamma$  point wave functions  $\psi_{\Gamma}(\mathbf{r})$  has two zeros [as shown in Fig. 2(a)] at the corners  $\mathbf{r}_0^{(1)}$  and  $\mathbf{r}_0^{(2)}$  of the unit cell. Using these two zeros two holomorphic Bloch-periodic functions  $f_{\mathbf{k}}^{(1)}(z)$  and  $f_{\mathbf{k}}^{(2)}(z)$  can be constructed (Supplemental Material [41]) such that the two Bloch-periodic wave functions  $\psi_{\mathbf{k}}^{(i)}(\mathbf{r}) = f_{\mathbf{k}}^{(i)}(z)\psi_{\Gamma}(\mathbf{r})$  ( $i = 1, 2$ ) are zero modes of the Hamiltonian, and hence they are the exact wave functions of the two sublattice-polarized flatbands. The two flat bands polarized to one sublattice together have Chern number  $C = 1$ . Furthermore, due to the form of the Hamiltonian, for any flatband wave function  $\psi_{\mathbf{k}}^{(i)}(\mathbf{r})$  defined above,  $f(z)\psi_{\mathbf{k}}^{(i)}(\mathbf{r})$  is also a zero mode for any holomorphic function  $f(z)$  of  $z$ , and hence contained in the two flatbands; this implies that these bands are vortexable [31]. All of these properties mentioned until now are generic to all examples of multiple sublattice-polarized flatbands found in different models [37–40]. Importantly, the two sublattice-polarized bands can be decomposed (we perform this decomposition using the software Wannier90 [55–57], see Supplemental Material [41]) into a topologically trivial band with very localized Wannier function and a topological band with Chern number  $C = 1$  [see Figs. 2(c)–2(e)]. After this decomposition, the Chern band has ideal quantum geometry, but the trivial band does not have ideal quantum geometry as we show in Supplemental Material [41]. However, it is worth reiterating that the two bands together are vortexable [40].

To study interacting states in these four flat bands, we note that if the energy scale of interaction is much smaller than the single particle gap between the flatbands and remote bands, we can project the interacting Hamiltonian onto the flatbands. When filling fraction is  $\nu = 2$ , exchange interaction drives all electrons to the two bands polarized on one of the sublattices (Hund’s rule), and the resulting ground state is a spontaneous time reversal symmetry breaking Chern insulator. We assume that for  $\nu < 2$ , electrons are still polarized on one of the sublattices, and write down the two-band (polarized on the same sublattice) projected interacting Hamiltonian

$$H_{\text{int}} = \frac{1}{2A} \sum_{\mathbf{q}} V(\mathbf{q}) \rho(\mathbf{q}) \rho(-\mathbf{q}), \quad (2)$$

where  $A$  is the system area,  $V(\mathbf{q})/A = 4\pi U \tanh(dq)/(N_s \sqrt{3}qa)$  is the screened Coulomb interaction,  $d$  is the separation between the two electrodes,  $a$  is the moiré lattice constant,  $q = |\mathbf{q}|$ ,  $N_s$  is the number of moiré unit cells,  $U$  is the bare Coulomb energy between two particles at a

distance  $a$ , and we introduced the projected density operator  $\rho(\mathbf{q}) = \sum_{\mathbf{k}} \lambda_{i,j,\mathbf{q}}(\mathbf{k}) c_{\mathbf{k},i}^{\dagger} c_{\mathbf{k}+\mathbf{q},j}$ , where  $\lambda_{i,j,\mathbf{q}}(\mathbf{k}) = \langle \psi_{\mathbf{k}}^{(i)}(\mathbf{r}) | e^{-i\mathbf{q}\cdot\mathbf{r}} | \psi_{\mathbf{k}+\mathbf{q}}^{(j)}(\mathbf{r}) \rangle$  is the form factor, and  $i, j$  run over the Chern ( $C$ ) and Wannier ( $W$ ) bands. Note that  $\rho(\mathbf{q})$  can be split into three parts  $\rho(\mathbf{q}) = \rho_{CC}(\mathbf{q}) + \rho_{WW}(\mathbf{q}) + \rho_{CW}(\mathbf{q})$ , where the first two terms are the densities in the Chern band and the trivial band, respectively, and the third term is a tunneling term between the two bands [this term breaks the  $U(1) \times U(1)$  symmetry]. Below we present exact diagonalization results at different filling fractions.

*Filling fractions  $\nu = 1$ ,  $\nu = 4/3$ ,  $\nu = 5/3$ , and  $\nu = 1/3$* — At filling fraction  $\nu = 1$ , we find an insulating ground state [Fig. 3(a)] with the trivial band fully occupied and the Chern band empty as can be seen from the occupation number plot of the ground state shown in Fig. 3(b). This can be understood from the fact that the Wannier functions of

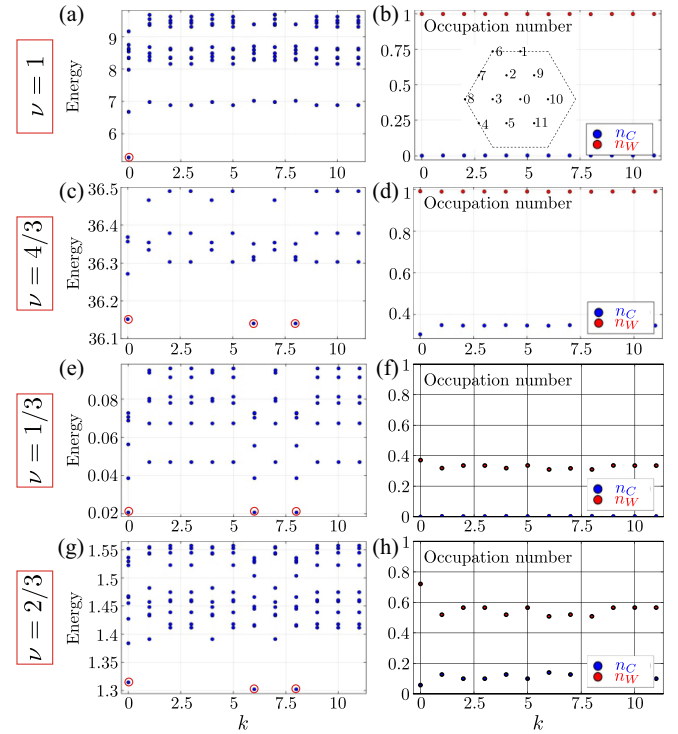


FIG. 3. Many-body energy spectra (a),(c),(e),(g) and occupation number in the Chern [ $n_C(\mathbf{k}) = \langle c_C^\dagger(\mathbf{k}) c_C(\mathbf{k}) \rangle$ ] and Wannier [ $n_W(\mathbf{k}) = \langle c_W^\dagger(\mathbf{k}) c_W(\mathbf{k}) \rangle$ ] bands of the ground states (averaged over the three ground states) (b),(d),(f),(h) at filling  $\nu = 1, 4/3, 1/3$  and  $2/3$ , respectively. The momentum numbers are shown in the inset in (b). The ground states in (a),(c),(e),(g) are marked by red circles. At  $\nu = 1$ , there is a single insulating ground state, whereas for each of the other filling fractions, there are three quasidegenerate ground states. The many-body Chern numbers of the ground states are  $C_{\text{mb}} = 0$  in (a) and (e), and are  $C_{\text{mb}} = 1/3$  for (c) and (g). For  $\nu = 1$  and  $\nu = 1/3$  ground states, (b) and (f) show that the Chern band is fully empty, all electrons are in the Wannier band. For  $\nu = 4/3$  ground state, (d) shows the Wannier band is fully filled, and the Chern band is  $1/3$  filled. For  $\nu = 2/3$  ground state, (h) shows that both bands are partially occupied.

the trivial band are very localized [Fig. 2(d)] meaning that the trace of the quantum metric  $\text{tr}[g(\mathbf{k})]$  of this band is small (see Supplemental Material Fig. S2 [41] for the plot of  $\text{tr}[g(\mathbf{k})]$ ), whereas  $\text{tr}[g(\mathbf{k})]$  of the Chern band is much larger since it is bounded by the Berry curvature. Since Fock contribution of the interaction energy is roughly proportional to  $\text{tr}[g(\mathbf{k})]$  [14], by occupying the trivial band the electrons minimize the Fock contribution. We numerically verified that the ground state has many-body Chern number  $C_{\text{mb}} = 0$  via flux insertion (we follow [14] for  $C_{\text{mb}}$  calculation). At filling  $\nu = 4/3$ , we find three quasidegenerate ground states at momenta consistent with both a CDW and FCI state shown in Fig. 3(c). We numerically find that the trivial band is fully occupied, whereas the Chern band is  $1/3$  filled [Fig. 3(d)], which is consistent with our numerical finding of  $C_{\text{mb}} = 1/3$ . We show in Supplemental Material [41] that the particle entanglement spectrum (PES) [58,59] of the ground states is also consistent with a FCI state. Filling fraction  $\nu = 5/3$  is similar to  $\nu = 4/3$ , the Chern band is two-thirds filled in this case, giving rise to a FCI with  $C_{\text{mb}} = 2/3$  (see Supplemental Material [41]). At  $\nu = 1/3$ , there are three quasidegenerate ground states with center of mass momenta at zone center ( $\Gamma$ ) and zone corners ( $K$  and  $K'$ ) [Fig. 3(e)], with all electrons in the Wannier band. This is a  $\sqrt{3} \times \sqrt{3}$  CDW state; we show that density-density correlation function of this ground state has pronounced peaks at the corner of the BZ (see Supplemental Material [41]).

**Filling fraction  $\nu = 2/3$** —From the discussion above, naïvely one may assume that for  $\nu \leq 1$ , all electrons occupy the trivial band, and for  $\nu > 1$ , the trivial band is fully filled and the rest of the electrons occupy the Chern band. However, remarkably, this simple picture does not hold for  $\nu = 2/3$ . At this filling, we find three quasidegenerate ground states shown in Fig. 3(g), which is consistent with both CDW and FCI. However, there are a few puzzling properties of these ground states: (i) we numerically find the many-body Chern number  $C_{\text{mb}} = 1/3$ , (ii) the occupation number is nonzero in both bands as shown in Fig. 3(h), (iii) we also find (see End Matter Appendix A Fig. 5) that the off-diagonal terms in the occupation number matrix  $n_{ij}(\mathbf{k}) = \langle c_i^\dagger(\mathbf{k})c_j(\mathbf{k}) \rangle$  ( $i, j \in \{C, W\}$ ) as well as its determinant are nonzero at generic momenta which implies this FCI cannot be single-band-like in any basis (see Supplemental Material Sec. XI [41]), (iv) in the PES of the ground states [see End Matter Appendix Fig. 7(a)] the number of states below the gap is consistent with  $\nu = 1/3$  FCI state in a single Chern band with twice the number of single particle states as each band of this system has, (v) we show in Supplemental Material [41] that the tunneling term  $\rho_{CW}(\mathbf{q})$  in the density operator is crucial for this state, as all electrons occupy the trivial band and form a topologically trivial  $\sqrt{3} \times \sqrt{3}$  CDW state if we artificially eliminate this term from  $H_{\text{int}}$  to restore  $U(1) \times U(1)$  symmetry. Below we explain this apparent

discrepancy between the filling fraction and many-body Chern number.

*The two flatbands per sublattice are equivalent to LLL with  $4\pi$  flux per magnetic unit cell*—Recall that one of the flatband wave functions at  $\Gamma$  point  $\psi_\Gamma(\mathbf{r})$  has two zeros in the moiré unit cell. It is known that LLL wave functions on a torus have one zero per magnetic unit cell with  $2\pi$  flux [60]. This gives the first clue that the two flatbands may be one LLL band in a spatially varying magnetic field with  $4\pi$  flux per unit cell, which makes the LLL band fold into two bands. To make it quantitative, we first note that LLL wave functions in a spatially varying magnetic field can be written as  $\psi^{\text{LLL}}(\mathbf{r}) = h(z)e^{-K(\mathbf{r})/4\ell_B^2}$ , where  $h(z)$  is a holomorphic function, and  $B(\mathbf{r}) = B_0\partial_z\bar{\partial}_{\bar{z}}K(\mathbf{r})$  [11] ( $B_0 = 2\pi\hbar/eA_{\text{u.c.}}$ , where  $A_{\text{u.c.}}$  is the area of the magnetic unit cell, and  $\ell_B^2 = \hbar/eB_0$ ). Similarly, writing our flatband wave functions  $\psi_{\mathbf{k}}^{(i)}(\mathbf{r})$  in the form  $\psi_{\mathbf{k}}^{(i)}(\mathbf{r}) = h_{\mathbf{k}}^{(i)}(z)e^{-K(\mathbf{r})/4\ell_B^2}\zeta(\mathbf{r})$  [61], we extracted the effective magnetic field per unit cell  $B(\mathbf{r}) = B_0\partial_z\bar{\partial}_{\bar{z}}K(\mathbf{r})$  as shown in Fig. 2(f) (the magnetic field distribution has interesting singularities, see Supplemental Material [41] for details). The total flux per moiré unit cell of this magnetic field is indeed  $4\pi$ . Hence, the two flatbands together are similar to one LLL band in a spatially varying magnetic field, and the filling fraction  $\nu = 2/3$  is really filling fraction  $\nu_{\text{unfolded}} = 1/3$  of this LLL band, which gives in FQH state with many-body Chern number  $C_{\text{mb}} = 1/3$ . It is worth mentioning that even though the flux of per moiré unit cell is  $4\pi$ , smaller moiré unit cell with  $2\pi$  cannot be chosen without breaking threefold rotation symmetry  $C_{3z}$  due to the spatial variation of the magnetic field; however, it is possible in a  $C_{4z}$  symmetric system without breaking any rotation symmetries. In End Matter Appendix B, we show explicitly in a  $C_{4z}$  symmetric moiré system that this  $\nu = 2/3$  but  $C_{\text{mb}} = 1/3$  state is adiabatically connected to a  $\nu_{\text{unfolded}} = 1/3$  state of a  $C = 1$  Chern band that is folded into a smaller mBZ.

**Competing states at filling  $\nu = 4/3$** —We showed above that for the two flatbands per sublattice with total Chern number  $C = 1$  and ideal quantum geometry, the ground states at different filling fractions can be understood using one of the two interpretations of the bands: (i) one Chern band, and one trivial band, (ii) one  $C = 1$  LLL in a spatially varying magnetic field with total  $4\pi$  flux per unit cell. The fillings  $\nu = 1/3, 1, 5/3$  correspond to  $\nu_{\text{unfolded}} = 1/6, 1/2, 5/6$  having even denominator, hence in the single band interpretation there is no FCI state for these fillings, so in this interpretation the ground state would generically be metallic, which lose energetically to the insulating ground state from the first interpretation. On the other hand, the filling fractions  $\nu = 2/3, 4/3$  correspond to  $\nu_{\text{unfolded}} = 1/3, 2/3$ , and both interpretations lead to gapped ground states with different topological properties, and numerics is needed to know which ground state is realized. At  $\nu = 4/3$ , where the first interpretation predicts

an FCI state with  $C_{\text{mb}} = 1/3$  as discussed earlier [see Figs. 3(c) and 3(d)], whereas within the second interpretation, we have  $\nu_{\text{unfolded}} = 2/3$ , which predicts a FCI state with  $C_{\text{mb}} = 2/3$ . To show a competition between these two states we take the same Hamiltonian in Eq. (1), but choose a  $C_{4z}$  symmetric strain field

$$\tilde{A}(\mathbf{r}) = -2i\alpha(\cos(\tilde{\mathbf{b}}_1 \cdot \mathbf{r}) - \cos(\tilde{\mathbf{b}}_2 \cdot \mathbf{r})) + 2\beta(\cos(\mathbf{b}_1 \cdot \mathbf{r}) - \cos(\mathbf{b}_2 \cdot \mathbf{r})), \quad (3)$$

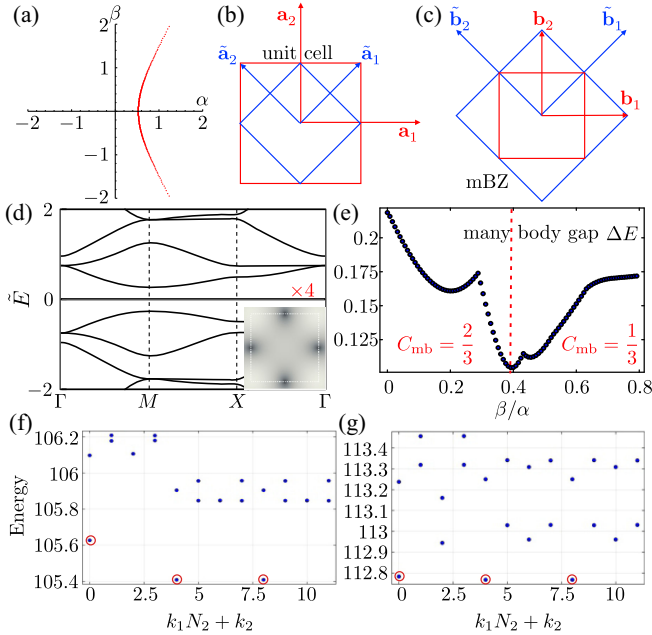


FIG. 4. Phase transition between FCIs with  $C_{\text{mb}} = 2/3$  and  $C_{\text{mb}} = 1/3$  at filling fraction  $\nu = 4/3$  in a  $p4mm$  symmetric system. The single particle Hamiltonian is given in Eqs. (1) and (3). The unit cell, mBZ, (reciprocal) lattice vectors are shown in red in (b) and (c). However, when  $\beta = 0$ , the primitive unit cell and corresponding mBZ are shown in blue in (b) and (c). (a) Line (in red) of “magic” parameters  $(\alpha, \beta)$ , at which four exact flatbands (two per sublattice) appear. The two flatbands per sublattice have total Chern number  $|C| = 1$ . At the point where the red line crosses  $\alpha$  axis ( $\alpha = 0.5279$  and  $\beta = 0$ ), the four flatbands (two per sublattice) corresponding to the red mBZ in (c) can be unfolded into two flatbands (one per sublattice) corresponding to the blue mBZ in (c). (d) Band structure for  $\alpha = 0.6651$ ,  $\beta = 0.7183$  containing 4 exact flatbands at  $\tilde{E} = 0$ . At the bottom right corner, the density plot of wave function  $|\psi_{\Gamma}(\mathbf{r})|$  is shown. The dark spots in the wave functions are the zeros. (f),(g) Many-body energy spectra at filling  $\nu = 4/3$  for  $\alpha = 0.5279$ ,  $\beta = 0$  and  $\alpha = 0.6651$ ,  $\beta = 0.7183$ , respectively. In both cases, we find three quasi degenerate ground states (encircled with red circles). We numerically find the many-body Chern numbers of these quasidegenerate ground states to be  $C_{\text{mb}} = 2/3$  and  $C_{\text{mb}} = 1/3$  for (f) and (g), respectively. (e) Many-body gap  $\Delta E$  between the lowest energy excited state and highest energy ground state (among the three quasidegenerate ground state) as the ratio  $\beta/\alpha$  is varied along red line in (a).

where  $\mathbf{b}_i$  and  $\tilde{\mathbf{b}}_i$  are shown in Fig. 4(c). When  $\beta = 0$ , the primitive unit cell (mBZ) is half (twice) the size of the unit cell for  $\beta \neq 0$ , as shown in Figs. 4(b) and 4(c). Now, by varying  $\alpha$  and  $\beta$ , we find a line in the  $\alpha - \beta$  plane [shown in red in Fig. 4(a)] where there are two flatbands per sublattice at  $\tilde{E} = 0$  as shown in Fig. 4(d) (see Supplemental Material [41] for details of how we find this line). For any  $(\alpha, \beta)$  on the red line, the wave function  $\psi_{\Gamma}(\mathbf{r})$  has two zeros in the unit cell [Fig. 4(d)]. Using this two zeros two flatband wave functions can be constructed just like the  $p6mm$  case in Fig. 2. The two flatbands per sublattice together have Chern number  $|C| = 1$ . We show the effective magnetic field for these flatbands in Supplemental Material [41], and verify that the magnetic flux per moiré unit cell is  $4\pi$ . Hence, these flatbands are really like the flatbands in Fig. 2, just with a different symmetry. Crucially, at the point where  $\beta = 0$  in Fig. 4(a), the two flatbands per sublattice in the red mBZ in Fig. 4(c) can be unfolded to one flatband with  $|C| = 1$  per sublattice in the blue mBZ in Fig. 4(c). The many-body spectra in Figs. 4(f) and 4(g) for  $\nu = 4/3$  show triply degenerate ground states at  $(\alpha = 0.5279, \beta = 0.0)$  and  $(\alpha = 0.6651, \beta = 0.7183)$ , respectively; however, in the former case  $C_{\text{mb}} = 2/3$  and in the latter case  $C_{\text{mb}} = 1/3$ . The former one can be understood from the second picture (single unfolded  $|C| = 1$  with  $\nu_{\text{unfolded}} = 2/3$ ), whereas the latter can be understood from the first picture (one Chern band  $1/3$  filled, and one trivial band fully filled). The PES of these ground states agree with these interpretations as we show in End Matter Appendix D Fig. 8. Note that along the whole red line in Fig. 4(a) the single particle gap does not close; however, as  $\beta/\alpha$  is varied, we numerically find that for  $\beta/\alpha < 0.39$ , the many-body Chern number of ground states is  $C_{\text{mb}} = 2/3$ , whereas for  $\beta/\alpha > 0.39$ , the many-body Chern number of ground states is  $C_{\text{mb}} = 1/3$ . We show the many-body gap as a function of  $\beta/\alpha$  in Fig. 4(e) along the red line in Fig. 4(a). The many-body gap has a global minimum around  $\beta/\alpha \approx 0.39$ ; we conjecture that the many-body gap does not fully close due to finite size effect.

**Discussion**—One interesting platform where the physics described can be realized is twisted bilayer graphene with spatially alternating magnetic field as described in [37]. This system has four flatbands (two per sublattice) per valley per spin, and in the chiral limit the high symmetry momentum wave function  $\psi_{\Gamma}(\mathbf{r})$  has two zeros [40] at the corners of the moiré unit cell indicating that the magnetic flux per moiré unit cell is  $4\pi$ . Also, promising 2D materials with quadratic band crossing at the Fermi level such as PO [62],  $\text{Mg}_2\text{C}$  [63],  $\text{Pd}_3\text{P}_2\text{S}_8$  [64],  $\text{Cu}_2\text{N}$  [65], metal organic framework material  $\text{Cr}_3(\text{HAB})_2$  [66] along with rapidly improving experimental techniques for applying periodic strain field [67–73] means that experimental realization of the Hamiltonian in Eq. (1) will soon be possible. Lastly, although we focus mostly on moiré structure, it is worth noting that the key conclusions hold

more generally for other topological flatbands with similar flatness and wave function characteristics.

**Acknowledgments**—The authors thank Patrick Ledwith for pointing out the similarity between multiple flatbands in moiré systems and lowest Landau level with  $2n\pi$  flux per unit cell. S. S. thanks Debanjan Chowdhury for insightful discussions. This work was supported in part by Air Force Office of Scientific Research No. MURI FA9550-23-1-0334 and the Office of Naval Research No. MURI N00014-20-1-2479 (X. W., S. S., and K. S.) and Grant No. N00014-21-1-2770 (X. W. and K. S.), and by the Gordon and Betty Moore Foundation Grant No. N031710 (K. S.). The work at LANL (S.-Z. L.) was carried out under the auspices of the U.S. DOE NNSA under Contract No. 89233218CNA000001 through the LDRD Program, and was supported by the Center for Nonlinear Studies at LANL, and was performed, in part, at the Center for Integrated Nanotechnologies, an Office of Science User Facility operated for the U.S. DOE Office of Science, under user Proposals No. 2018BU0010 and No. 2018BU0083.

**Data availability**—The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

- 
- [1] H. L. Stormer, D. C. Tsui, and A. C. Gossard, *Rev. Mod. Phys.* **71**, S298 (1999).
  - [2] R. B. Laughlin, *Rev. Mod. Phys.* **71**, 863 (1999).
  - [3] F. D. M. Haldane, *Phys. Rev. Lett.* **61**, 2015 (1988).
  - [4] E. Tang, J.-W. Mei, and X.-G. Wen, *Phys. Rev. Lett.* **106**, 236802 (2011).
  - [5] K. Sun, Z. Gu, H. Katsura, and S. Das Sarma, *Phys. Rev. Lett.* **106**, 236803 (2011).
  - [6] T. Neupert, L. Santos, C. Chamon, and C. Mudry, *Phys. Rev. Lett.* **106**, 236804 (2011).
  - [7] D. Sheng, Z.-C. Gu, K. Sun, and L. Sheng, *Nat. Commun.* **2**, 389 (2011).
  - [8] N. Regnault and B. A. Bernevig, *Phys. Rev. X* **1**, 021014 (2011).
  - [9] B. A. Bernevig and N. Regnault, *Phys. Rev. B* **85**, 075128 (2012).
  - [10] C. Repellin and T. Senthil, *Phys. Rev. Res.* **2**, 023238 (2020).
  - [11] P. J. Ledwith, G. Tarnopolsky, E. Khalaf, and A. Vishwanath, *Phys. Rev. Res.* **2**, 023237 (2020).
  - [12] Z. Liu, A. Abouelkomsan, and E. J. Bergholtz, *Phys. Rev. Lett.* **126**, 026801 (2021).
  - [13] H. Li, U. Kumar, K. Sun, and S.-Z. Lin, *Phys. Rev. Res.* **3**, L032070 (2021).
  - [14] A.-K. Wu, S. Sarkar, X. Wan, K. Sun, and S.-Z. Lin, *Phys. Rev. Res.* **6**, L032063 (2024).
  - [15] J. Cai, E. Anderson, C. Wang, X. Zhang, X. Liu, W. Holtzmann, Y. Zhang, F. Fan, T. Taniguchi, K. Watanabe *et al.*, *Nature (London)* **622**, 63 (2023).
  - [16] Y. Zeng, Z. Xia, K. Kang, J. Zhu, P. Knüppel, C. Vaswani, K. Watanabe, T. Taniguchi, K. F. Mak, and J. Shan, *Nature (London)* **622**, 69 (2023).
  - [17] H. Park, J. Cai, E. Anderson, Y. Zhang, J. Zhu, X. Liu, C. Wang, W. Holtzmann, C. Hu, Z. Liu *et al.*, *Nature (London)* **622**, 74 (2023).
  - [18] F. Xu, Z. Sun, T. Jia, C. Liu, C. Xu, C. Li, Y. Gu, K. Watanabe, T. Taniguchi, B. Tong *et al.*, *Phys. Rev. X* **13**, 031037 (2023).
  - [19] Z. Lu, T. Han, Y. Yao, A. P. Reddy, J. Yang, J. Seo, K. Watanabe, T. Taniguchi, L. Fu, and L. Ju, *Nature (London)* **626**, 759 (2024).
  - [20] J. Xie, Z. Huo, X. Lu, Z. Feng, Z. Zhang, W. Wang, Q. Yang, K. Watanabe, T. Taniguchi, K. Liu *et al.*, *arXiv*: 2405.16944.
  - [21] Y.-H. Wu, J. K. Jain, and K. Sun, *Phys. Rev. B* **86**, 165129 (2012).
  - [22] Y.-L. Wu, B. A. Bernevig, and N. Regnault, *Phys. Rev. B* **85**, 075116 (2012).
  - [23] A. Sterdyniak, C. Repellin, B. A. Bernevig, and N. Regnault, *Phys. Rev. B* **87**, 205137 (2013).
  - [24] Y.-H. Wu, J. K. Jain, and K. Sun, *Phys. Rev. B* **91**, 041119(R) (2015).
  - [25] D. Vanderbilt, *Berry Phases in Electronic Structure Theory: Electric Polarization, Orbital Magnetization and Topological Insulators* (Cambridge University Press, Cambridge, England, 2018).
  - [26] S. A. Parameswaran, R. Roy, and S. L. Sondhi, *C.R. Phys.* **14**, 816 (2013).
  - [27] R. Roy, *Phys. Rev. B* **90**, 165139 (2014).
  - [28] P. J. Ledwith, E. Khalaf, and A. Vishwanath, *Ann. Phys. (Amsterdam)* **435**, 168646 (2021).
  - [29] J. Wang, J. Cano, A. J. Millis, Z. Liu, and B. Yang, *Phys. Rev. Lett.* **127**, 246403 (2021).
  - [30] B. Mera and T. Ozawa, *Phys. Rev. B* **104**, 115160 (2021).
  - [31] P. J. Ledwith, A. Vishwanath, and D. E. Parker, *Phys. Rev. B* **108**, 205144 (2023).
  - [32] P. J. Ledwith, A. Vishwanath, and E. Khalaf, *Phys. Rev. Lett.* **128**, 176404 (2022).
  - [33] G. Tarnopolsky, A. J. Kruchkov, and A. Vishwanath, *Phys. Rev. Lett.* **122**, 106405 (2019).
  - [34] J. Wang and Z. Liu, *Phys. Rev. Lett.* **128**, 176403 (2022).
  - [35] X. Wan, S. Sarkar, S.-Z. Lin, and K. Sun, *Phys. Rev. Lett.* **130**, 216401 (2023).
  - [36] P. M. Eugenio and O. Vafek, *SciPost Phys.* **15**, 081 (2023).
  - [37] C. Le, Q. Zhang, F. Cui, X. Wu, and C.-K. Chiu, *Phys. Rev. Lett.* **132**, 246401 (2024).
  - [38] S. Becker, T. Humbert, and M. Zworski, *arXiv*:2208.01628.
  - [39] S. Becker, T. Humbert, and M. Zworski, *arXiv*:2306.02909.
  - [40] S. Sarkar, X. Wan, S.-Z. Lin, and K. Sun, *Phys. Rev. Lett.* **135**, 016501 (2025).
  - [41] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/jmk4-46tj> for details of analytic and numerical calculations including the derivation of the single particle wave functions of the degenerate flatbands and their similarities with lowest Landau level wave functions with multiple flux quanta per magnetic unit cell, expression and properties of the effective magnetic field in the moiré flat bands, construction of maximally localized Wannier band, further results from exact diagonalization,

- and details of particle entanglement spectrum, which includes Refs. [42–54].
- [42] S. Kharchev and A. Zabrodin, *J. Geom. Phys.* **94**, 19 (2015).
  - [43] D. Guerci, J. Wang, and C. Mora, *Phys. Rev. B* **112**, L041108 (2025).
  - [44] C. Fang, M. J. Gilbert, and B. A. Bernevig, *Phys. Rev. B* **86**, 115112 (2012).
  - [45] M. I. Aroyo, J. M. Perez-Mato, D. Orobengoa, E. Tasci, G. de la Flor, and A. Kirov, *Bulg. Chem. Commun.* **43**, 183 (2011), [http://www.bcc.bas.bg/BCC\\_Volumes/Volume\\_43\\_Number\\_2\\_2011/Volume\\_43\\_Number\\_2\\_2011\\_PDF/2011\\_43\\_2\\_1.pdf](http://www.bcc.bas.bg/BCC_Volumes/Volume_43_Number_2_2011/Volume_43_Number_2_2011_PDF/2011_43_2_1.pdf).
  - [46] M. I. Aroyo, J. M. Perez-Mato, C. Capillas, E. Kroumova, S. Ivantchev, G. Madariaga, A. Kirov, and H. Wondratschek, *Z. Kristallogr.* **221**, 15 (2006).
  - [47] M. I. Aroyo, A. Kirov, C. Capillas, J. Perez-Mato, and H. Wondratschek, *Acta Crystallogr. Sect. A* **62**, 115 (2006).
  - [48] Z.-D. Song and B. A. Bernevig, *Phys. Rev. Lett.* **129**, 047601 (2022).
  - [49] S. Becker, M. Embree, J. Wittsten, and M. Zworski, *Probab. Math. Phys.* **3**, 69 (2022).
  - [50] S. Becker, M. Embree, J. Wittsten, and M. Zworski, *Phys. Rev. B* **103**, 165113 (2021).
  - [51] N. Morales-Durán, N. Wei, J. Shi, and A. H. MacDonald, *Phys. Rev. Lett.* **132**, 096602 (2024).
  - [52] J. Yu, J. Herzog-Arbeitman, Y. H. Kwan, N. Regnault, and B. A. Bernevig, *Phys. Rev. B* **112**, 075110 (2025).
  - [53] J. Jung and A. H. MacDonald, *Phys. Rev. B* **89**, 035405 (2014).
  - [54] X. Wan, S. Sarkar, K. Sun, and S.-Z. Lin, *Phys. Rev. B* **108**, 125129 (2023).
  - [55] N. Marzari and D. Vanderbilt, *Phys. Rev. B* **56**, 12847 (1997).
  - [56] I. Souza, N. Marzari, and D. Vanderbilt, *Phys. Rev. B* **65**, 035109 (2001).
  - [57] G. Pizzi, V. Vitale, R. Arita, S. Blügel, F. Freimuth, G. Géranton, M. Gibertini, D. Gresch, C. Johnson, T. Koretsune *et al.*, *J. Phys. Condens. Matter* **32**, 165902 (2020).
  - [58] H. Li and F. D. M. Haldane, *Phys. Rev. Lett.* **101**, 010504 (2008).
  - [59] A. Sterdyniak, N. Regnault, and B. A. Bernevig, *Phys. Rev. Lett.* **106**, 100405 (2011).
  - [60] F. D. M. Haldane and E. H. Rezayi, *Phys. Rev. B* **31**, 2529 (1985).
  - [61] J. Dong, J. Wang, P. J. Ledwith, A. Vishwanath, and D. E. Parker, *Phys. Rev. Lett.* **131**, 136502 (2023).
  - [62] L. Zhu, S.-S. Wang, S. Guan, Y. Liu, T. Zhang, G. Chen, and S. A. Yang, *Nano Lett.* **16**, 6548 (2016).
  - [63] S.-S. Wang, Y. Liu, Z.-M. Yu, X.-L. Sheng, L. Zhu, S. Guan, and S. A. Yang, *Phys. Rev. Mater.* **2**, 104003 (2018).
  - [64] S. Park, S. Kang, H. Kim, K. H. Lee, P. Kim, S. Sim, N. Lee, B. Karuppannan, J. Kim, J. Kim *et al.*, *Sci. Rep.* **10**, 20998 (2020).
  - [65] X. Hu, R.-W. Zhang, D.-S. Ma, Z. Cai, D. Geng, Z. Sun, Q. Zhao, J. Gao, P. Cheng, L. Chen *et al.*, *Nano Lett.* **23**, 5610 (2023).
  - [66] X. Feng, Z. Zhang, W. Wu, X.-L. Sheng, and S. A. Yang, *Phys. Rev. B* **109**, 045423 (2024).
  - [67] Y. Jiang, J. Mao, J. Duan, X. Lai, K. Watanabe, T. Taniguchi, and E. Y. Andrei, *Nano Lett.* **17**, 2839 (2017).
  - [68] Y. Zhang, Y. Kim, M. J. Gilbert, and N. Mason, *Appl. Phys. Lett.* **115**, 143508 (2019).
  - [69] J. Mao, S. P. Milovanović, M. Anđelković, X. Lai, Y. Cao, K. Watanabe, T. Taniguchi, L. Covaci, F. M. Peeters, A. K. Geim *et al.*, *Nature (London)* **584**, 215 (2020).
  - [70] C. Cho, J. Wong, A. Taqieddin, S. Biswas, N. R. Aluru, S. Nam, and H. A. Atwater, *Nano Lett.* **21**, 3956 (2021).
  - [71] Y. Zhang, M. A. Hossain, K. J. Hwang, P. F. Ferrari, J. Maduzia, T. Peña, S. M. Wu, E. Ertekin, and A. M. van der Zande, *ACS Nano* **18**, 4205 (2024).
  - [72] Y. Zhang, H. L. Zhao, S. Huang, M. A. Hossain, and A. M. van der Zande, *ACS Nano* **18**, 12377 (2024).
  - [73] J. M. Kim, M. F. Haque, E. Y. Hsieh, S. M. Nahid, I. Zarin, K.-Y. Jeong, J.-P. So, H.-G. Park, and S. Nam, *Adv. Mater.* **35**, 2107362 (2023).

## End Matter

*Appendix A: Details of occupation number matrix  $n_{ij}(\mathbf{k})$  for  $(\nu = 2/3, C_{\text{mb}} = 1/3)$  FCI ground states in Fig. 3(g)*

*Appendix B: The  $(\nu = 2/3, C_{\text{mb}} = 1/3)$  state is adiabatically connected to a  $\nu_{\text{unfolded}} = 1/3$  state of a  $C = 1$  Chern band that is folded into a smaller mBZ—* We noted in the main text that a  $C_{3z}$  symmetric moiré unit cell with  $4\pi$  flux cannot be halved to obtain a unit cell with  $2\pi$  flux without breaking  $C_{3z}$  symmetry. However, a  $C_{4z}$  symmetric moiré unit cell with  $4\pi$  can be halved in some limits. To this end, we take the same setup as in Figs. 4(a)–4(c). We find threefold quasidegenerate ground states at filling  $\nu = 2/3$  everywhere along the red line [Figs. 6(b) and 6(c)], and the many-body gap remains open on the entire line as

shown in Fig. 6(d) and the many-body Chern number remains  $C_{\text{mb}} = 1/3$ , hence these ground states along the red line are adiabatically connected. However, the FCI state at  $\nu = 2/3$  filling corresponding to the red mBZ for the  $\beta = 0$  point [see Fig. 6(b)] is really a FCI state at  $\nu_{\text{unfolded}} = 1/3$  filling corresponding to the larger blue mBZ. Hence, the fact that it has  $C_{\text{mb}} = 1/3$  is expected. Since the FCI states at other points on the red line are adiabatically connected to this FCI at the  $\beta = 0$  point, their many-body Chern number must be  $C_{\text{mb}} = 1/3$  even though for these other red points the mBZ cannot be unfolded. Furthermore, in End Matter Appendix C, we show that the PES of these ground states have the same counting of low-lying spectra as the  $\nu = 2/3$  FCI state of the  $p6mm$  system in Figs. 3(g) and 3(h), supporting our claim that all these  $\nu = 2/3$  FCI states are the same.

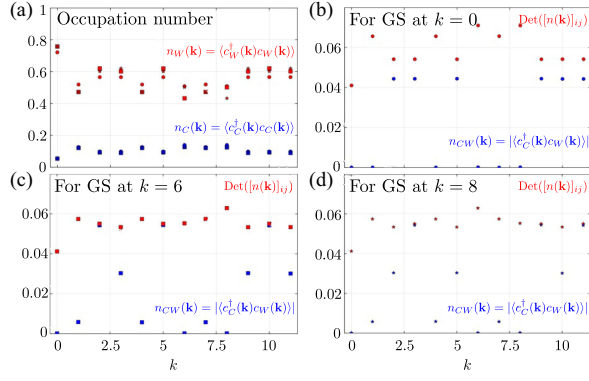


FIG. 5. (a) The diagonal terms  $n_C(\mathbf{k}) = \langle c_C^\dagger(\mathbf{k}) c_C(\mathbf{k}) \rangle$  (in blue) and  $n_W(\mathbf{k}) = \langle c_W^\dagger(\mathbf{k}) c_W(\mathbf{k}) \rangle$  (in red) of the occupation number matrix  $n_{ij}(\mathbf{k})$  for the three quasidegenerate ground states in Fig. 3(g). The circle, square, and star-shaped markers correspond to the ground states at  $k = 0$  (zone center),  $k = 6$ , and  $k = 8$  (zone corners), respectively. (b)–(d) The absolute value of off-diagonal component  $n_{CW}(\mathbf{k}) = |\langle c_C^\dagger(\mathbf{k}) c_W(\mathbf{k}) \rangle|$  (in blue) and the determinant (in red) of the occupation number matrix  $n_{ij}(\mathbf{k})$  for the ground states at momenta  $k = 0$ ,  $k = 6$ , and  $k = 8$  in Fig. 3(g), respectively.

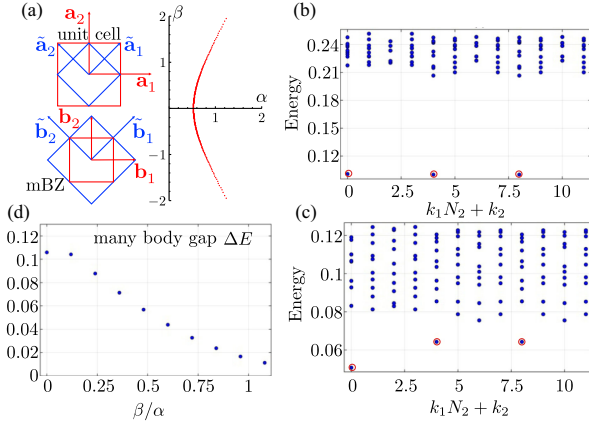


FIG. 6. Evidence that the  $(\nu = 2/3, C_{\text{mb}} = 1/3)$  FCI state is adiabatically connected to a  $(\nu_{\text{unfolded}} = 1/3, C_{\text{mb}} = 1/3)$  FCI state in a  $p4mm$  symmetric system. The single particle Hamiltonian is given in Eqs. (1) and (3). When  $\beta = 0$  in Eq. (3), the primitive unit cell and corresponding mBZ are shown in blue in (a). (a) Line (in red) of magic parameters  $(\alpha, \beta)$ , at which four exact flatbands (two per sublattice) appear. The two flatbands per sublattice have total Chern number  $|C| = 1$ . At the point where the red line crosses  $\alpha$  axis ( $\alpha = 0.5279$  and  $\beta = 0$ ), the four flatbands (two per sublattice) corresponding to the red mBZ can be unfolded into two flatbands (one per sublattice) corresponding to the blue mBZ. (b), (c) Many-body energy spectra at filling  $\nu = 2/3$  for  $\alpha = 0.5279, \beta = 0$  and  $\alpha = 0.6651, \beta = 0.7183$ , respectively. Both show three quasidegenerate ground states (marked by red circles). In both cases, the many-body Chern numbers of the ground states are  $C_{\text{mb}} = 1/3$ . (d) Many-body gap  $\Delta E$  between the lowest energy excited state and highest energy ground state (among the three quasidegenerate ground state) as the ratio  $\beta/\alpha$  is varied along red line in (a).

### Appendix C: Particle entanglement spectra of the $\nu = 2/3$ FCI states in Figs. 3(g), 6(b), and 6(c)

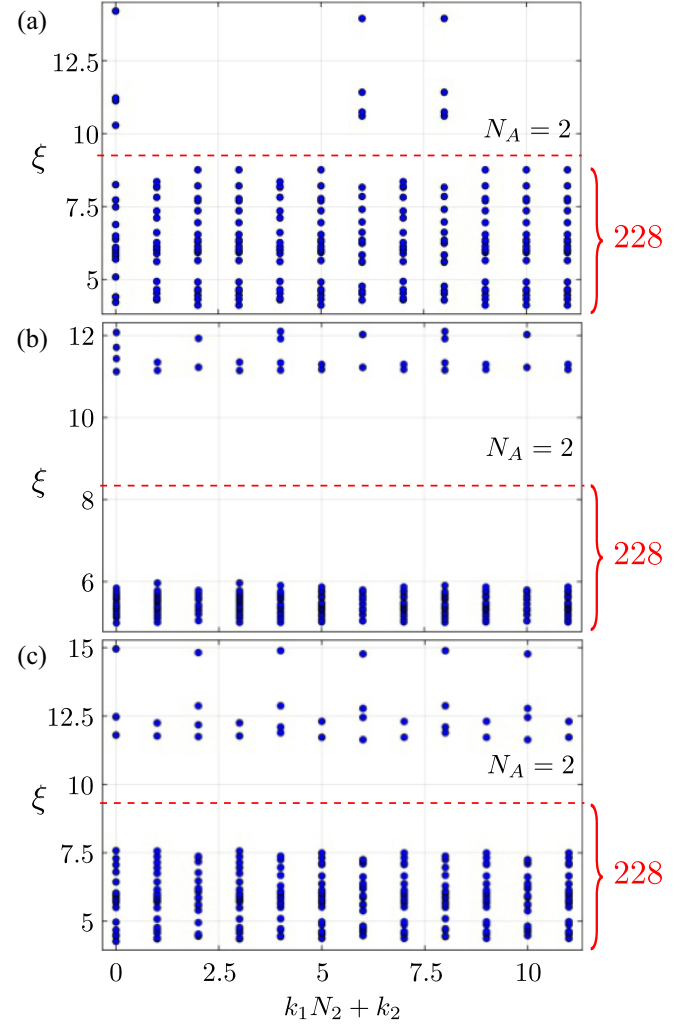


FIG. 7. Particle entanglement spectra of the  $\nu = 2/3$  FCI groundstates in Figs. 3(g), 6(b), and 6(c) for a particle cut of  $N_A = 2$  and  $N_B = 6$  (total number of electrons  $N_e = N_A + N_B = 8$ ) are shown in (a), (b), and (c), respectively (see Supplemental Material [41] for details). In all three cases, there are gaps marked by red dashed lines, below which there are 228 states. This number matches with the PES counting of  $\nu = 1/3$  FCI state for  $N_e = 8$  electrons in a  $C = 1$  band with  $N = 24$  single particle states partitioned into two groups with  $N_A = 2$  and  $N_B = 6$  electrons. In (a) and (c), there are small gaps above the lowest 66 states too. We showed that in Fig. 6 that the ground states corresponding to (b) and (c) are adiabatically connected. The fact that low lying spectra of (a) and (c) have the same counting indicates that the FCI state in (a) is the same as that in (c).

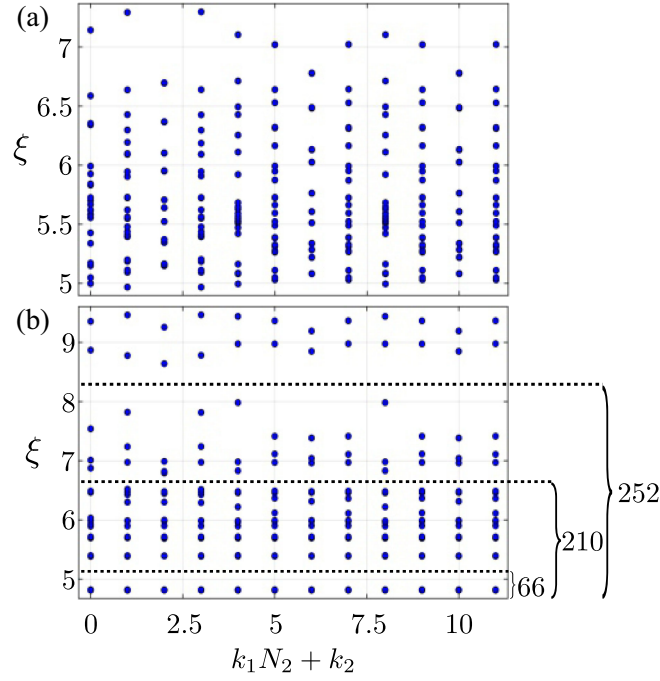
Appendix D: Particle entanglement spectra of the  $\nu = 4/3$  FCI states in Figs. 4(f) and 4(g)


FIG. 8. Particle entanglement spectra of the  $\nu = 4/3$  FCI ground-states in Figs. 4(f) and 4(g) for a particle cut of  $N_A = 2$  and  $N_B = 6$  (total number of electrons  $N_e = N_A + N_B = 8$ ) are shown in (a) and (b), respectively. The PES in (a) does not show any gap. The counting of the low-lying states of PES in (b) matches with a ground state with one fully filled Wannier band and a  $1/3$  filled Chern band (see Supplemental Material [41] for details).