

Far-from-Equilibrium Thermodynamics of Non-Abelian Thermal States

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Noncommutativity of observables is a defining feature of quantum physics. It plays a fundamental role in the formulation of the uncertainty principle for complementary variables and strongly affects the thermodynamic behavior of systems with noncommuting, that is, non-Abelian, conserved quantities. We here derive extensions of the first and second laws of thermodynamics, including nonequilibrium fluctuation relations, for driven quantum systems. We identify an additional non-Abelian contribution to the energy and entropy balances, without which these relations would be violated. This contribution can be harnessed to enhance both work extraction and nonequilibrium currents compared to what is achieved in commuting thermodynamics. These results demonstrate that noncommutativity is a potentially valuable thermodynamic resource.

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Statistical mechanics offers a generic probabilistic framework to describe the equilibrium thermodynamics of systems with conserved quantities. In this theory, average macroscopic properties are determined from the probability distribution (often called Gibbs distribution) of the corresponding microscopic states [1]. Such distributions may be obtained by maximizing the information entropy of the system with the constraints of normalization and average conserved quantities [2–5]. A prominent example is provided by the grand canonical distribution, which is associated with conserved average energy and average particle number [1]. More generally, the powerful maximum entropy approach has been extended to an arbitrary number of conserved quantities [2–5], leading to the concept of generalized Gibbs ensembles, which play a central role in the description of relaxation dynamics in isolated integrable quantum systems [6].

Whereas conserved quantities always commute in classical statistical physics, this is not necessarily the case in the quantum domain [7–15]. Owing to their noncommutation, conserved quantum observables do not in general have well-defined simultaneous values. The resulting thermal states are frequently referred to as non-Abelian thermal states to emphasize their noncommutative feature [7–16]. They have been shown to exhibit unconventional

nonclassical features that differ from those of common thermal states [7–15], such as the absence of microcanonical subspaces [7]. The experimental observation of the nonstandard thermalization toward a non-Abelian thermal state has recently been reported for a long-range Heisenberg chain in an ion-trap setup [17].

Nonequilibrium properties of small quantum systems are dominated by thermal and quantum fluctuations [18,19]. The second law needs therefore to be generalized to account for fluctuating thermodynamic variables. One important such stochastic extension is provided by fluctuation relations of the form, $P(\Sigma)/P(-\Sigma) = \exp(\Sigma)$, for the random entropy production Σ with probability distribution $P(\Sigma)$ [18–21]. Fluctuation theorems imply the second law on average, $\langle \Sigma \rangle \geq 0$, and quantify the occurrence of negative entropy production events. They are moreover valid arbitrarily far from equilibrium. They have, for this reason, found widespread applications in the study of nonequilibrium microscopic systems [18,19]. One usually distinguishes work fluctuation relations, with entropy production $\Sigma = \beta(W - \Delta F)$, for systems that are mechanically driven from a thermal state at inverse temperature β (ΔF here denotes the associated free energy difference) [22] and heat exchange fluctuation relations, with entropy production $\Sigma = \Delta\beta Q$, for two thermally coupled systems with inverse temperature difference $\Delta\beta$ [23]. Fluctuation theorems have been obtained for generalized Gibbs ensembles with commuting conserved quantities [24–26]. Moreover, fluctuation relations for non-Abelian transport have lately been derived using quasiprobabilities [15] and a collisional approach [27]. However, general quantum fluctuation relations for

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arbitrarily driven systems with noncommuting conserved quantities are currently missing.

We here investigate the far-from-equilibrium thermodynamics of generalized Gibbs ensembles with noncommuting conserved quantities by deriving quantum fluctuation theorems for non-Abelian thermal states. Since there is no joint eigenbasis for noncommuting observables, off-diagonal density matrix elements, associated with quantum coherence [28], appear in the eigenbasis of the conserved observables. Such quantum coherence strongly impacts the nonequilibrium properties of non-Abelian thermal states. However, standard methods, such as the two-point-measurement scheme, which has been extensively used to study quantum fluctuations relations [18,19], are not applicable in this case. Owing to their projective nature, they indeed cannot account for nondiagonal matrix elements of initial or final states of a quantum process [18,19]. In the following, we employ the formalism of dynamic Bayesian networks that allows one to analyze conditional dependencies in a general set of time-dependent random variables via Bayes' rule [29,30]. This approach is able to capture off-diagonal density matrix elements at all times [31–36]. We identify an additional non-Abelian contribution to the first and second laws, and show that the latter term can be exploited to enhance both work extraction and exchange currents, as well as their thermodynamic efficiency, defined by their respective ratio to their maximal value [37].

Generalized Gibbs ensemble—We consider a small quantum system that exchanges an arbitrary number of possibly noncommuting extensive quantities A_k , often called charges in analogy to electric charges [7–16], with a weakly coupled large reservoir R , such that the global charges, $A_k^{\text{tot}} = A_k + A_k^R$, are conserved with respect to the total Hamiltonian [38]. The local equilibrium state of the system takes the form [7–16]

$$\rho = \exp(F - \lambda \cdot \mathbf{A}), \quad (1)$$

with affinities (or generalized chemical potentials) λ_k . To simplify the notation, we have introduced the two vectors λ and $\mathbf{A}$, with respective components λ_k and A_k , such that $\lambda \cdot \mathbf{A} = \sum_k \lambda_k A_k$, as well as the generalized free entropy $F = -\ln \text{Tr}[\exp(-\lambda \cdot \mathbf{A})]$, also known as the Massieu potential [1]. The generalized Gibbs state (1) generalizes the usual canonical distributions of statistical mechanics [1]. For the standard grand canonical distribution, the charges are the energy and the particle number operators, whereas the corresponding affinities are given by the inverse temperature and its product with the chemical potential [1]. Since the observables A_k do not necessarily commute, $[A_k, A_l] \neq 0$, eigenbases associated with different charges are not always mutually orthogonal, contrary to the case of commuting observables. As a result, state (1) will exhibit quantum coherences in the basis of each individual A_k , which will be important in the following. In the grand canonical ensemble,

energy and particle number operators always commute, and therefore have a common eigenbasis [1].

First law for noncommuting charges—Let us begin by deriving a generalized first law of thermodynamics for non-Abelian charge transfer. To that end, we drive the system, initially prepared in state (1) at time $t = 0$, via a nonequilibrium protocol of duration τ , parametrized by time-dependent charges $\mathbf{A}^t$. During the entire protocol, the system is weakly coupled to a reservoir prepared in a non-Abelian thermal state, $\rho^R = \exp(F^R - \lambda^R \cdot \mathbf{A}^R)$ [13], through an interaction Hamiltonian V that satisfies strict total charge conservation at all times, $[V, \mathbf{A}^t + \mathbf{A}^R] = 0$. The latter condition extends the conventional strict energy conservation equality, $[V, H^t + H^R] = 0$, from the theory of open quantum systems [39,40]. Such reservoirs are often referred to as non-Abelian reservoirs [16]. We additionally denote the projectors on the respective instantaneous eigenstates of system and reservoir as Π_i^t and Π_μ^R , and express the corresponding generalized Gibbs states as $\rho^t = \sum_i p_i^t \Pi_i^t$ and $\rho^R = \sum_\mu p_\mu^R \Pi_\mu^R$, with associated probabilities p_i^t and p_μ^R ; we will use Latin (Greek) indices for system (reservoir) operators throughout.

A stochastic first law for a single realization of the nonequilibrium quantum process can be obtained in the presence of noncommuting charges by introducing the total (inclusive) work performed on the composite system, defined as the difference between the total charges at the beginning and at the end of the nonequilibrium driving protocol, $\mathbf{w} = \langle j_\tau, \nu | \mathbf{A}^{\text{tot}}(\tau) | j_\tau, \nu \rangle - \langle i_0, \mu | \mathbf{A}^{\text{tot}}(0) | i_0, \mu \rangle$. This definition can be regarded as a generalization of the usual (inclusive) mechanical work which is equal to the change of the total energy of the composite system [18,19]. We can relate it to the local charge variations, $\Delta \mathbf{a}(i^0, j^\tau) = \langle j | \mathbf{A}^\tau | j \rangle - \langle i | \mathbf{A}^0 | i \rangle$, and the corresponding heat, $\mathbf{q}(\mu, \nu) = -(\langle \nu | \mathbf{A}^R | \nu \rangle - \langle \mu | \mathbf{A}^R | \mu \rangle)$, exchanged with the reservoir, in complete analogy with the standard derivation of the first law for thermal Gibbs states [18,19]. We find (see Supplemental Material [41])

$$\begin{aligned} \mathbf{w}(i^0, \mu, j^\tau, \nu) &= \mathbf{w}_C(i^0, \mu, j^\tau, \nu) + \boldsymbol{\varepsilon}(i^0, \mu, j^\tau, \nu) \\ &= \Delta \mathbf{a}(i^0, j^\tau) - \mathbf{q}(\mu, \nu), \end{aligned} \quad (2)$$

where we have identified the commuting part $\mathbf{w}_C(i_0, \mu, j_\tau, \nu)$ of the stochastic work, solely due to the time-dependent driving, and a new non-Abelian contribution $\boldsymbol{\varepsilon}(i^0, \mu, j^\tau, \nu)$ given by

$$\begin{aligned} \boldsymbol{\varepsilon} &= - \frac{\langle j_\tau, \nu | \mathbf{A}(\tau) \mathcal{O}^\tau V - V \mathcal{O}^0 \mathbf{A}(0) | i_0, \mu \rangle}{2 \langle j_\tau, \nu | V | i_0, \mu \rangle} \\ &\quad - \frac{\langle j_\tau, \nu | \mathbf{A}^R \mathcal{O}^\tau V - V \mathcal{O}^0 \mathbf{A}^R | i_0, \mu \rangle}{2 \langle j_\tau, \nu | V | i_0, \mu \rangle}, \end{aligned} \quad (3)$$

where $\mathcal{O}^0 = \mathbb{1} - \Pi_{i,\mu}^0$ and $\mathcal{O}^\tau = \mathbb{1} - \Pi_{j,\nu}^\tau$ are projectors outside of the initial and final states of the trajectory,

respectively. Expression (3) vanishes for commuting charges. It is hence of purely quantum origin and absent in the standard formulation of the energy balance. Conversely, the Abelian part of the work w_C vanishes, when there is no external driving, in contrast to the non-Abelian contribution ϵ which survives even on average.

A physical intuition for the noncommuting work ϵ may be developed by considering the special case of a harmonic oscillator interacting with a squeezed bath [46–49], which acts as a non-Abelian reservoir on average [50]. Work can be performed on the quantum oscillator either through an external time-dependent forcing [this corresponds to the commuting part of the work w_C in Eq. (2)], or by squeezing it via the coupling to the bath (which can be interpreted as a frequency change of the oscillator) [50]. The non-Abelian work ϵ may then be viewed as the generalization of this squeezing work to arbitrary non-Abelian reservoirs. An alternative view is to associate ϵ with the mechanical work performed on the system by a coherent reservoir [51]. As mentioned above, the noncommutativity of the charges induces coherences in the basis of each operator A_k , mimicking the coupling to a coherent reservoir. Such coherences have been shown in a collisional framework to coherently drive the system, even when its Hamiltonian is time-independent, leading to a worklike contribution in the energy balance [51]. Equation (2) extends and unifies both approaches: it indicates that noncommutativity, and the induced quantum coherence, is a potential thermodynamic resource.

Non-Abelian work fluctuation relation—We next derive a generalization of the second law in the form of a Jarzynski-Crooks-type fluctuation relation [18,19]. Since for GGEs the initial system state ρ^0 is in general coherent with respect to the eigenbasis Π_m^0 set by the environment, the usual two-point measurement scheme [18,19] is inappropriate to study the fluctuations of the protocol A^t , since it destroys non-diagonal density matrix elements, owing to its projective nature [52]. We address this issue by using the tools of dynamic Bayesian networks, which allow one to account for off-diagonal matrix elements at all times, by specifying the dynamics of a system in an eigenbasis conditioned on the evolution in an incompatible eigenbasis [31–36]. Dynamic Bayesian networks are commonly used to describe the relationship between dynamical variables through conditional probabilities evaluated via Bayes’ rule [29,30]; they may be considered as an extension of hidden Markov models.

The conditional probability of measuring the system in state $|m^t\rangle$ at time t given that it is in state $|i^t\rangle$ is given by Born’s law, $p^t(m|i) = |\langle m^t|i^t\rangle|^2$; these constitute the conditional prior probabilities, which are to be determined beforehand. If the environment is thermal, this amounts to energy measurements on the system. By performing conditional measurements at the beginning and at the end of the nonequilibrium driving protocol, we may introduce a conditional trajectory $\Gamma = (i, \mu, m, j, \nu, n)$ for the composite system, with path probability [31–36]

$$P(\Gamma) = p_i^0 p_\mu^R p^0(m|i) p^{0,\tau}(j, \nu|i, \mu) p^\tau(n|j), \quad (4)$$

where $p^{0,\tau}(j|i) = |\langle j^\tau, \nu|U|i^0, \mu\rangle|^2$ is the (transition) probability of measuring state $|j^\tau\rangle$ after evolving state $|i^0\rangle$ through the joint system-environment time evolution operator $U = U(\tau, 0)$. We emphasize that the measurement at the end of the protocol is performed in the eigenbasis of the instantaneous generalized Gibbs state $\pi^t = \exp(F - \lambda \cdot A^t)$ (and not in the instantaneous eigenbasis of ρ^t) in accordance to the original spirit of the Jarzynski-Crooks relation [53].

We may similarly define the probability of the reversed trajectory $\Gamma^\dagger = (j, \nu, n, i, \mu, m)$ as

$$P^\dagger(\Gamma^\dagger) = p_j^\tau p^\tau(\nu) p^\tau(n|j) p^{\tau,0}(i, \mu|j, \nu) p^0(m|i), \quad (5)$$

where the initial state of the reservoir is chosen as the final instantaneous generalized Gibbs state of the system, $\rho^{R,0} = \pi^\tau$, and where $p^{\tau,0}(i, \mu|j, \nu) = |\langle i^0, \mu|U^\dagger|j^\tau, \nu\rangle|^2$, with the time-reversed evolution operator $U^\dagger$. We note that path probabilities of a dynamic Bayesian network can be determined experimentally [34]. By now taking the ratio of Eqs. (4) and (5), and inserting the microscopic first law (2), we obtain the generalized non-Abelian detailed fluctuation relation

$$\frac{P(\Gamma)}{P^\dagger(\Gamma^\dagger)} = \exp\{\lambda \cdot (w_C + \epsilon) + \Delta\lambda \cdot q\} \\ \times \exp\{-\Delta F^R - \Delta c - \Delta d\}, \quad (6)$$

where $\Delta\lambda = \lambda - \lambda^R$ is the difference between system and reservoir affinities, and $\Delta F^R = F^{r,\tau} - F^{r,0}$ is the change of free entropy, $F^{r,t} = -\ln \text{Tr}[\exp(-\lambda^R \cdot A^t)]$, associated with the instantaneous generalized Gibbs state $\pi^{r,t} = \exp(F^{r,t} - \lambda^R \cdot A^t)$, which acts as a reference state, with spectral decomposition $\pi^{r,t} = \sum_m p_m^{r,t} \Pi_m^{r,t}$. This state generalizes the instantaneous equilibrium state in the standard derivation of the quantum fluctuation relation [20]. We have moreover introduced the variation of the stochastic relative entropies of coherence [28], $\Delta c(i, j, m, n) = \ln p_j^\tau / p_n^d - \ln p_i^0 / p_m^d$, and athermality [54], $\Delta d(m, n) = \ln p_n^d / p_r^n - \ln p_m^d / p_r^m$, of state π with respect to state π^r at the endpoints of the protocol; here p^d refers to the probability of measuring the state π in the eigenbasis of π^r . The relative entropies of coherence and athermality respectively quantify the difference between the two states along the off-diagonal and diagonal [28,54].

Equation (6) is an extension of the quantum Crooks relation for noncommuting conserved quantities. It represents one of the most general nonequilibrium formulations of the quantum second law of thermodynamics to date, incorporating not only the quantum properties of the states (such as discrete spectra) and of the dynamics (coherent time evolution), but also accounting for the presence of non-Abelian charges and induced quantum coherences. It is

important to realize that not including the non-Abelian term ϵ would lead to a violation of the fluctuation theorem, as well as of the first law (2). The latter contribution is therefore essential for a correct analysis of the far-from-equilibrium thermodynamics of non-Abelian systems. Integrating Eq. (6) over all forward trajectories further leads to the Jarzynski-like integral quantum fluctuation relation

$$\langle \exp\{-\lambda \cdot (\mathbf{w}_C + \epsilon) - \Delta\lambda \cdot \mathbf{q} + \Delta c + \Delta d\} \rangle = e^{-\Delta F^R}. \quad (7)$$

Applying Jensen's inequality to Eq. (7), we additionally have the generalized second-law like inequality

$$\lambda \cdot (\mathbf{W}_C + \epsilon) + \Delta\lambda \cdot \mathbf{Q} \geq \Delta F^R + \Delta C + \Delta D, \quad (8)$$

with the averaged thermodynamic variables $\mathbf{W}_C = \langle \mathbf{w}_C \rangle$, $\epsilon = \langle \epsilon \rangle$ and $\mathbf{Q} = \langle \mathbf{q} \rangle$, and the two information-theoretic quantities $\Delta C = \langle \Delta c \rangle$ and $\Delta D = \langle \Delta d \rangle$ that correspond to the respective difference in relative entropy of coherence and athermality between the two generalized Gibbs states. We also note that the right-hand side of Eq. (8) can be compactly written as the difference of the nonequilibrium free entropy defined as $\mathcal{F} = F^R + C + D$.

When system and environment have equal affinities, $\lambda = \lambda^R$, $\mathcal{F}$ reduces to F , and there is no heat exchange between the two since $\Delta\lambda = 0$. If the charges moreover commute, the non-Abelian work contribution vanishes, $\epsilon = 0$, and we recover the inequality $\lambda \cdot \mathbf{W} \geq \Delta F$ obtained in Ref. [8] for generalized Gibbs ensembles with commuting conserved quantities. Furthermore, following Ref. [21], we may write the average nonequilibrium entropy production, that characterizes the irreversible character of the quantum process, in the form

$$\langle \Sigma \rangle = \lambda \cdot (\mathbf{W}_C + \epsilon) + \Delta\lambda \cdot \mathbf{Q} - \Delta\mathcal{F} \geq 0. \quad (9)$$

Equations (6)–(9) are important since they enable the identification, and proper quantification, of available thermodynamic resources. In equilibrium thermodynamics,

for instance, the free energy difference upper bounds the amount of mechanical work that can be extracted from a system [1], while in standard nonequilibrium thermodynamics, nonzero affinities that drive heat, particle or charge currents constitute a supplemental resource for work production [1]. Moreover, the resource theory of quantum states out of equilibrium [55] has specified two additional thermodynamic resources, athermality connected with the departure from thermal equilibrium of the populations of a state [54], as well as quantum coherence associated with off-diagonal elements of the density matrix of the state [28]. Equations (6)–(9) further reveal that charge noncommutativity leads to another potential thermodynamic resource, in addition to the ones already found in the literature, through the appearance of the non-Abelian work contribution ϵ .

The special case of an undriven system, $\mathbf{W}_C = 0$, including the corresponding non-Abelian exchange fluctuation relations, are discussed in the End Matter.

Example of a Heisenberg chain—In order to gain deeper insight into the nonequilibrium thermodynamics of non-Abelian systems, we examine a model of two spins coupled via a Heisenberg interaction of the form $V = J\sigma \cdot \sigma^R$, where σ is a vector of Pauli matrices and J is the coupling strength. This interaction preserves the average of $\sigma_i + \sigma_i^R$ for each Pauli matrix σ_i , thus constituting our charges. The local Hamiltonians of system and environment are respectively taken as $H = \omega(\cos\theta\sigma_z + \sin\theta\sigma_x)/2$ and $H^R = \omega\sigma_z^R/2$, with frequency ω and angle θ between the two spins. The corresponding initial states of system and reservoir further read $\rho = \exp(F - \beta H)$ and $\rho^R = \exp(F^R - \beta^R H^R)$. The state ρ is a non-Abelian generalized Gibbs state when the two spins are not parallel, that is, for $\theta \neq n\pi$, while ρ^R is a standard thermal state. The associated generalized chemical potentials are respectively given by $\lambda_x = \omega \sin\theta\beta/2$ and $\lambda_z = \omega \cos\theta\beta/2$.

We mechanically drive the system via $H(t) = \omega[g(t)\cos\theta\sigma_z + \sin\theta\sigma_x]/2$, with a time-dependent protocol $g(t) = g_0(1-t)/\tau + g_\tau t/\tau$ of duration τ . Figure 1(a) displays the extracted work, $-\lambda \cdot (\mathbf{W}_C + \epsilon)$, and the

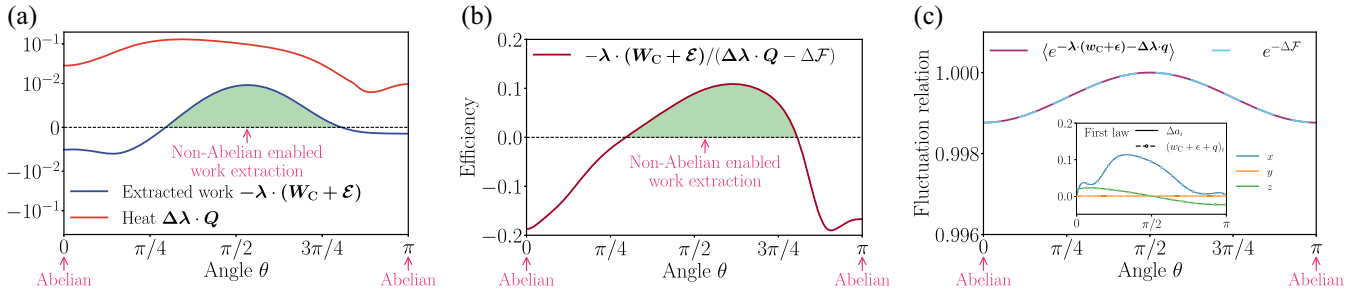


FIG. 1. Non-Abelian nonequilibrium thermodynamics. (a) Extracted work $-\lambda \cdot (\mathbf{W}_C + \epsilon)$ and heat $\Delta\lambda \cdot \mathbf{Q}$ for a linearly driven spin of a two-qubit Heisenberg chain, as a function of the angle θ between the spin states. Work is done on the system in the Abelian case $\theta = (0, \pi)$. In the intermediate non-Abelian regime, work can be extracted from the system (shaded area), indicating that noncommutativity is a thermodynamic resource. (b) The efficiency η of work extraction is maximum close to $\theta = \pi/2$, where the two spins are initially orthogonal. (c) The first law (2) and the work fluctuation relation (7) are obeyed when the non-Abelian term ϵ (3) is included. Parameters are $\omega = 1$, $J = 0.15$, $\beta = 0.1$, $\beta^R = 1$, $\tau = \pi/2J$, $g_0 = 1$, and $g_\tau = 0.1$.

exchanged heat, $\Delta\lambda \cdot \mathbf{Q}$, as a function of the angle θ . We observe that the extracted work is negative in the Abelian case ($\theta = 0$ or π), meaning that the external driving performs work on the system in this instance. By contrast, the extracted work becomes positive in the intermediate non-Abelian regime (shaded area), reaching a maximum close to $\theta = \pi/2$, when the two spins are initially orthogonal. This result confirms that non-commutativity is a thermodynamic resource that can boost work production. Equation (9) suggests the definition of an efficiency of work extraction, $\eta = -\lambda \cdot (\mathbf{W}_C + \mathcal{E})/(\Delta\lambda \cdot \mathbf{Q} - \Delta\mathcal{F})$, which is equal to one when $\langle \Sigma \rangle = 0$. This efficiency is maximum close to $\theta = \pi/2$ [Fig. 1(b)], when the coherence of the initial state in the eigenbasis of σ_z is most pronounced (the position of the maximum depends on the form of the driving). Finally, Fig. 1(c) shows that the first (inset) and second laws [Eqs. (2) and (7)] are verified, when the non-Abelian work contribution is included.

Conclusions—The laws of thermodynamics provide a powerful framework for identifying diverse energy forms and resources, and for analyzing their interconversion. We have derived extensions of these laws for generic non-Abelian thermal states with noncommuting conserved quantities subject to time-dependent driving. In particular, we have obtained generalized nonequilibrium fluctuation relations using dynamic Bayesian networks, explicitly accounting for the quantum coherence associated with noncommutativity. In doing so, we have determined an additional non-Abelian contribution to both energy and entropy which is essential to make correct thermodynamic predictions. By analyzing the example of a Heisenberg spin chain, we have shown that this term can be tuned to enhance both work extraction and charge transport. These findings suggest that non-Abelian states are a potential, currently untapped, physical resource for quantum thermodynamic applications.

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Data availability—The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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End Matter

Non-Abelian exchange fluctuation relation—We here examine the fluctuations of the exchange of charges between two non-Abelian bodies with different generalized Gibbs states in the absence of external

driving, and derive the corresponding transport fluctuation theorem. This situation can be viewed as a special case of the previous analysis in which the system plays the role of one body and the reservoir the role of

the second one. In this instance, the two systems share a common eigenbasis, meaning that the use of dynamical Bayesian networks is not necessary. Without mechanical driving, $\mathbf{W}_C = 0$, the first law (2) simplifies to

$$\Delta \mathbf{a}(i^0, j^\tau) = \mathbf{q}(\mu, \nu) + \boldsymbol{\varepsilon}(i^0, \mu, j^\tau, \nu). \quad (\text{A1})$$

Since there is no time dependence in the local charges of the system, the reference states are the same for the forward and backward protocols, implying that $\Delta \mathcal{F} = 0$. As a result, we have the non-Abelian detailed exchange fluctuation relation

$$\frac{P(\Gamma)}{P^\dagger(\Gamma^\dagger)} = \exp\{\boldsymbol{\lambda} \cdot \boldsymbol{\varepsilon} + \Delta \boldsymbol{\lambda} \cdot \mathbf{q}\}, \quad (\text{A2})$$

as well as the corresponding integral version

$$\langle \exp\{-\boldsymbol{\lambda} \cdot \boldsymbol{\varepsilon} - \Delta \boldsymbol{\lambda} \cdot \mathbf{q}\} \rangle = 1. \quad (\text{A3})$$

Equations (A2) and (A3) are extensions of the Jarzynski-Wojcik fluctuation theorem for heat transport [23] valid for arbitrary non-Abelian states and associated charges. Applying again Jensen's inequality, we find in this case the mean nonequilibrium entropy production

$$\langle \Sigma \rangle = \boldsymbol{\lambda} \cdot \boldsymbol{\varepsilon} + \Delta \boldsymbol{\lambda} \cdot \mathbf{Q} \geq 0. \quad (\text{A4})$$

The above expressions reveal that both charge transport and nonequilibrium entropy production are controlled by the non-Abelian work term $\boldsymbol{\varepsilon}$. They reduce to the results concurrently obtained in Ref. [27] using a collisional approach. While this Letter has focused on current reversals and on thermodynamic uncertainty relations, we next show that the non-Abelian property can enhance the nonequilibrium charge current.

To that end, we consider the two-qubit Heisenberg chain example in the absence of driving ($g(t) = 0$). Figure 2(a)

represents the extracted non-Abelian work contribution $-\lambda_z \mathcal{E}_z$ as well as the current $\Delta \lambda_z Q_z$ (for simplicity, we focus on the z components). Remarkably, noncommutativity leads here to both work extraction and enhanced nonequilibrium current compared to the Abelian case $\theta = (0, \pi)$ (shaded areas), confirming that noncommutativity is a useful thermodynamic resource. Figure 2(b) further shows that the non-Abelian work contribution $\mathcal{E}_z$ increases the ratio between nonequilibrium current and entropy production, $\Delta \lambda_z Q_z / \langle \Sigma \rangle_z$ in the non-Abelian regime close to $\theta = \pi/2$, revealing that more charges can be transported for a given dissipation value, compared to what is achieved in commuting thermodynamics. Finally, Fig. 2(c) validates the first and second laws [Eqs. (A1)–(A3)] when the non-Abelian work term $\boldsymbol{\varepsilon}$ is included.

It is also interesting to compare these results with those obtained in Ref. [15] using quasiprobabilities. The derived correction to the transport fluctuation theorem is there expressed in terms of commutators of the non-Abelian charges, which makes explicit the crucial role of non-commutativity. The non-Abelian work contribution $\boldsymbol{\varepsilon}$ is, however, not captured in this approach.

Next-nearest-neighbor Heisenberg chain—We finally consider a larger interacting example in order to demonstrate that the non-Abelian work fluctuation relation is not restricted to a single-spin working medium. We take an open chain of four spin-1/2 particles and divide it into a system S , formed by spins 1 and 2, and a finite environment R , formed by spins 3 and 4. The local spin charges are $\mathbf{A}^S = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2$ and $\mathbf{A}^R = \boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_4$, where $\boldsymbol{\sigma}_k = (\sigma_k^x, \sigma_k^y, \sigma_k^z)$. In particular, $[A_x^\alpha, A_z^\alpha] \neq 0$ for $\alpha = (S, R)$, so that states containing both components are genuinely non-Abelian.

In contrast to the two-spin example discussed in the main text, the local generalized Gibbs states now include the internal Hamiltonian of each pair, which are defined as

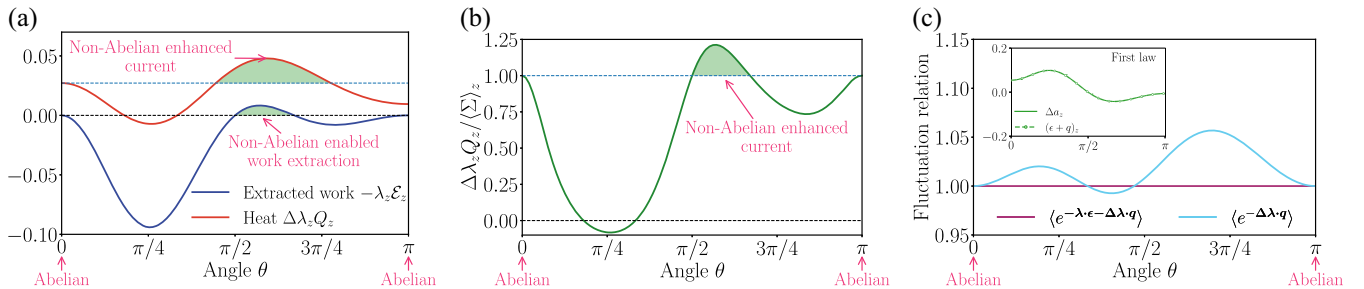


FIG. 2. Non-Abelian exchange fluctuation relation. (a) Extracted non-Abelian work $-\lambda_z \mathcal{E}_z$ and current $\Delta \lambda_z Q_z$, for a two-qubit Heisenberg chain, as a function of the angle θ between the spin states, without driving ($g(t) = 0$). In the non-Abelian regime, noncommutativity enhances both work extraction and nonequilibrium current compared to the Abelian case $\theta = (0, \pi)$. (b) The ratio of current and dissipation is boosted close to $\theta = \pi/2$ thanks to the non-Abelian work $\mathcal{E}_z$, indicating that noncommutativity is a thermodynamic resource. (c) The first law (A1) (inset) and the exchange fluctuation relation (A3) are satisfied, when the non-Abelian contribution $\boldsymbol{\varepsilon}$, Eq. (3), is included. Parameters are $\omega = 1$, $J = 1$, $\beta = 1$, $\beta^R = 0.5$, and $\tau = 3\pi/(4J)$.

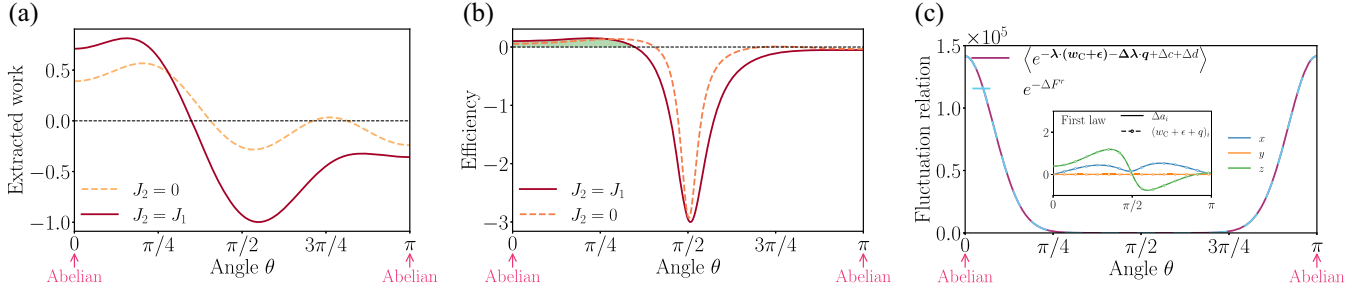


FIG. 3. A next-nearest-neighbor four-spin open Heisenberg chain is divided into a two-spin system and a two-spin environment. The nearest-neighbor coupling J_1 enters both the local generalized Hamiltonians and the system–environment interaction, whereas the next-nearest-neighbor coupling J_2 appears only in the interaction. (a) Extracted work as a function of the non-Abelian angle θ for $J_2 = 0$ and $J_2 = J_1$. For the parameters considered, the additional next-nearest-neighbor exchange enhances positive work extraction over part of the non-Abelian regime. (b) Corresponding work-extraction efficiency, showing a small enhancement due to the additional J_2 coupling. (c) Verification of the dynamic-Bayesian-network fluctuation relation [Eq. (7)]. Since the endpoint generalized Gibbs states do not depend on J_2 at fixed J_1 , the right-hand side is independent of J_2 , while the dynamical dependence cancels from the trajectory average. The inset verifies the component-resolved first law, $\Delta \mathbf{a} = \mathbf{w}_C + \mathbf{e} + \mathbf{q}$. Parameters are $\omega = 1$, $J_1 = 0.3$, $\beta_S = 3$, $\beta_E = 5$, $\lambda_i = 0.01$, $\lambda_f = 3$, and $\tau = 0.15\pi/J_1$. For (c), $J_2 = J_1$.

$$H_S^t = \frac{J_1}{2} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - \frac{\omega}{2} [\sin \theta A_x^S + g(t) \cos \theta A_z^S],$$

$$H_R = \frac{J_1}{2} \boldsymbol{\sigma}_3 \cdot \boldsymbol{\sigma}_4 - \frac{\omega}{2} A_z^R, \quad (\text{B1})$$

with the linear protocol $g(t) = g_0(1 - t/\tau) + g_\tau t/\tau$. The endpoint states are therefore $\rho_S^t = e^{-\beta H_S^t}/Z_S^t$ and $\rho_R = e^{-\beta H_R}/Z_R$. These states have the generalized Gibbs form $\exp[-\beta(H - \sum_k \mu_k A_k)]/Z$. The isotropic intra-pair exchange commutes with every component of the corresponding total pair spin, whereas A_x^a and A_z^a do not commute with each other. The two spin pairs are coupled through the open-chain interaction

$$V = \frac{J_1}{2} \boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_3 + \frac{J_2}{2} (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_4). \quad (\text{B2})$$

Together with the two local terms in Eq. (B1), this gives nearest-neighbor bonds (1,2), (2,3), and (3,4) with strength J_1 , and next-nearest-neighbor bonds (1,3) and (2,4) with strength J_2 . Each isotropic bond conserves the three global spin components, $[V, A_k^S + A_k^R] = 0$. The next-nearest-neighbor coupling guarantees that the total Hamiltonian is not integrable. As before, the process starts from the product state $\rho_S^0 \otimes \rho_R$ and evolves under $H_{\text{tot}}(t) = H_S^t + H_R + V$.

An important feature of this construction is the different role played by J_1 and J_2 . The nearest-neighbor coupling J_1 enters both local generalized Hamiltonians

in Eq. (B1); hence the free entropy difference ΔF^R depend on J_1 . By contrast, J_2 occurs only in the system–environment interaction (B2). At fixed J_1 , changing J_2 modifies the dynamics and the individual trajectory probabilities, but not the endpoint generalized Gibbs states. The right-hand side of Eq. (7) is therefore independent of J_2 , while the same dependence cancels from the trajectory average on the left-hand side. Work (and heat), however, needs not be independent of J_2 , since it depends on the actual nonequilibrium evolution, as can be seen in Fig. 3(a). Also, for the parameter set shown, the additional J_2 coupling can enhance the positive extracted work in part of the non-Abelian angular interval, showing even a slightly higher efficiency as shown by Fig. 3(b). This enhancement is not a general monotonic property of J_2 : the extra coupling changes the coherent exchange frequencies and may either improve or detune the driving protocol. It should therefore be understood as a controllable dynamical resource rather than as an unconditional advantage of longer-range interactions. Finally, Fig. 3(c) verifies the component-resolved first law, $\Delta \mathbf{a} = \mathbf{w}_C + \mathbf{e} + \mathbf{q}$, and the fluctuation relation (7). The equality is evaluated with the full dynamic Bayesian network, including the coherence and athermality endpoint corrections. This second example therefore confirms that the non-Abelian contribution is required beyond the minimal two-qubit setting and remains well defined in the presence of local interactions, several exchange pathways, and a larger trajectory space.