

Nonlinear Odd Viscoelastic Effect

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We uncover a class of nonlinear odd viscoelastic effects in three spatial dimensions. We show that these dissipationless effects arise upon combining geometric distortions in two orthogonal directions, yielding momentum flow in the third direction. We demonstrate that the effect arises from nontrivial geometric tensors in quantum states, and can be scaled up with integer topological invariants. We further show that the effect fingerprints the multiband Hilbert-space geometry of the underlying quantum states, as encoded in the nonmetricity and three-state quantum geometric tensors. Our findings unravel the role of multistate geometry in viscoelastic phenomena, paving a path for experimental observation of uncharted nonlinear odd viscoelastic responses in quantum systems.

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Introduction—Viscosity and elasticity are bulk properties of matter that manifest upon application of physical deformations. They naturally arise in a wide range of physical systems at different scales spanning from cosmology and astrophysics [1,2], to particle physics [3], through mesoscopic fluids [4], glasses [5], and active condensed matter systems [6], among others. The corresponding transport coefficients that determine the responses to geometric perturbations imposed by normal and shear stresses are encoded in viscoelastic tensors, which are in turn captured by stress correlators [7–9]. Beyond their distinct emergence in classical fluids and solids, viscoelastic tensors are also realized by quantum fluids of electrons in crystals. In particular, viscoelastic tensors also include dissipationless flows of momentum orthogonal to perturbation directions, commonly known as odd viscosity and odd elasticity [10–14]. In condensed matter systems, odd viscosity arises in viscous Hall currents, that is, as Hall viscosity [15–20]. Odd viscosity has been experimentally measured in crystals such as graphene [21], and was further recognized to contribute corrections to spatially dispersive conductivities [17,22].

Hall phenomena have, among many other impactful directions, been related to two themes of study. First, there is a primary role for topology, well exemplified by various detailed responses of quantum Hall fluids [23–28]. Second, it has been demonstrated that intrinsic electronic Hall effects occur not only at linear order in perturbing electric fields [29], but are also manifested at higher orders [30–38]. In addition, electronic nonlinear Hall responses are an increasingly studied topic of recent interest in quantum Hall fluids and topological phases, from both experimental and theoretical angles [39–48]. Universally, these responses depend heavily on the Hilbert-space geometry of quantum states, which is intricately linked to, and can be controlled by, the topology of wave functions [49–54].

Given the well-established understanding of these fields and the rich quantum geometric structure of nonlinear electromagnetic responses beyond the extensively studied Berry curvature and quantum metric tensors, a natural question emerges as to whether new insights can emerge upon exploiting the relation between multiband geometry, topology, higher-order responses, and odd viscoelasticity. Notably, a nonlinear viscoelastic effect that might accompany, or more importantly, arise even in the absence of, electronic Hall currents has to our knowledge not been explored to date, particularly within these quantum geometric frameworks.

In this Letter, we predict the emergence of such an effect and, in doing so, also address the problem of the existence of nonlinear *odd* viscoelastic effects (NOVEs) in quantum fluids. We demonstrate that the nonlinear correlators of elastic stress and modulus operators are uniquely constituted by two-state and three-state quantum geometric tensors (QGTs), revealing the crucial role of multistate

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geometry in viscoelastic responses. We also show that the quantum nonmetricity tensor and three-state QGT can be isolated via NOVEs in the flat-band limit, and that two- and three-state geometric responses arise naturally in magnetic topological phases, establishing these systems as strong candidates for measuring viscoelastic nonlinearities.

Moreover, we demonstrate the interplay of nonlinear elasticity and ground-state topologies via qualitative changes of nonlinear stress correlators, culminating in marked variations of NOVEs across topological phase transitions. Finally, we showcase the nonlinear viscoelastic phenomenology via deformations of real-space Wannier functions, subject to nonlinear couplings to normal and shear stresses. Our findings establish a set of unique viscoelastic effects and experimental protocols to measure them in a broad range of phases of matter. Beyond allowing to access and fingerprint exotic magnetic topological invariants, our results establish viscoelastic responses as probes of nontrivial multistate geometries for electronic states in solids.

Nonlinear odd viscoelastic response—To retrieve the NOVE, presented schematically in Fig. 1, we consider the nonlinear response of the system to the elastic distortion fields $w_{ij} = \partial \mathbf{u}_i / \partial x_j$, with $\mathbf{u}$ denoting local displacements in the system, $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{u}$.

In a crystalline solid, the geometric deformations of the system can be captured by minimally coupling the distortion fields to momenta in the lattice Hamiltonian as $H(k_j) \rightarrow H\{k_j - w_{ij}[\sin(k_i a_0)/a_0]\}$, with a_0 the lattice constant [55]. At arbitrary frequencies ω_1, ω_2 of the applied stress perturbations, the nonlinear elastic current density takes the general form

$$J_{ij}(\omega) = \eta_{ij;kl,mn}(\omega; \omega_1, \omega_2) w_{kl}(\omega_1) w_{mn}(\omega_2), \quad (1)$$

with $\{i, j, k, l, m, n\} = \{x, y, z\}$. To evaluate the nonlinear elastic response tensor $\eta_{ij;kl,mn}$ in the dc limit $\omega_1, \omega_2 \rightarrow 0$, we employ quadratic Kubo response theory generalized to higher-order tensor distortion field perturbations. In the systematic perturbative expansion, in addition to the stress tensor operator $T_{ij} = \delta H / \delta w_{ij}$, we identify higher-order operators, i.e., the elastic modulus tensor operator $U_{ij,kl} = \delta^2 H / \delta w_{ij} \delta w_{kl}$ and the nonlinear elastic modulus tensor operator $V_{ij,kl,mn} = \delta^3 H / \delta w_{ij} \delta w_{kl} \delta w_{mn}$, to be essential for a comprehensive derivation of nonlinear viscoelastic responses. The tensor U encodes a viscoelastic analog of diamagnetic response. The tensor V captures the contact terms in quadratic response theory, although these contributions vanish for odd responses. We express T , U , and V in terms of band-geometric quantities in the End Matter, following derivations in Supplemental Material (SM) [56].

The evaluation of the second-order correlators for odd viscoelastic responses reveals a rich quantum geometric structure. Specifically, we find that the dc quadratic response tensor naturally decomposes into gauge-invariant two-state and three-state geometric terms as

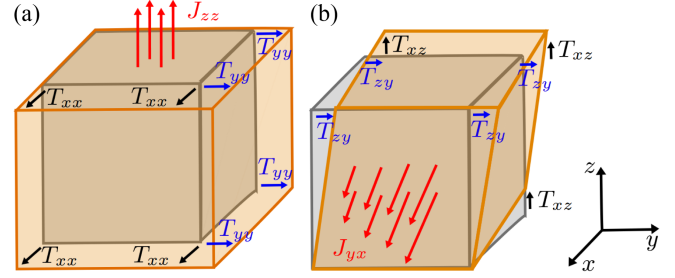


FIG. 1. Nonlinear odd viscoelastic effects. Application of (a) orthogonal normal distortions w_{xx}, w_{yy} coupling to stress-momentum currents T_{xx}, T_{yy} , and (b) shear distortions w_{xz}, w_{yz} coupling to currents T_{xz}, T_{yz} , result in dissipationless nonlinear viscoelastic current responses $J_{zz} = \langle T_{zz} \rangle$ and $J_{yx} = \langle T_{yx} \rangle$ along the normal to the stress plane, respectively.

$$\eta_{ij;kl,mn} = \int_{\text{BZ}} d^3 \mathbf{k} [\eta_{ij;kl,mn}^{2\text{-band}}(\mathbf{k}) + \eta_{ij;kl,mn}^{3\text{-band}}(\mathbf{k})]. \quad (2)$$

Here, we integrate over lattice momenta $\mathbf{k}$ in the three-dimensional Brillouin zone (BZ). The first term, $\eta_{ij;kl,mn}^{2\text{-band}}(\mathbf{k})$, captures the two-band contributions to viscoelastic effects at zero temperature in systems with a finite energy gap, such as trivial and topological insulators. This measures the response arising from the quantum state geometry of virtual transitions of occupied states to the unoccupied state manifold induced by the stress tensor and elastic moduli operators. The second term, $\eta_{ij;kl,mn}^{3\text{-band}}(\mathbf{k})$, arises from the three-band contributions, which emerge from (i) triplets of virtual transitions between an occupied state and two virtual unoccupied energy levels and (ii) additional contributions with two occupied states and one empty virtual energy level [56].

We further characterize the quantum geometries underlying the individual NOVE terms. The two-band contributions, resolved in momentum space, compactly read

$$\eta_{ij;kl,mn}^{2\text{-band}}(\mathbf{k}) = \sum_{a,b} f_{ba} \left[\frac{4i\tau}{3} g_{[(kl,mn)]}^{ba} \partial_{ij} \Delta^{ba} + \frac{\Omega_{ij,(kl)}^{ba} \partial_{mn} \Delta^{ba}}{\Delta^{ba}} + \frac{32i}{3} N_{[(kl,mn),ij]}^{ba} \right], \quad (3)$$

where $\partial_{kl} \equiv \partial / \partial w_{kl}$ are distortion derivatives, $\Delta^{ba} \equiv E_{\mathbf{k}}^b - E_{\mathbf{k}}^a$ are eigenstate energy differences, $f_{ba} = f_b - f_a$ is the difference of band occupation factors, and τ is the scattering time that enters as the inverse of the Lorentzian broadening in the electron propagators in the dc limit. The adiabatic curvature, $\Omega_{ij,kl}^{ba} = -2 \text{Im}(A_{ij}^{ba} A_{kl}^{ab})$, is a two-form in the parameter space of deformations, $g_{ij,kl}^{ba} = \text{Re}(A_{ij}^{ba} A_{kl}^{ab})$ is the corresponding metric, $N_{ij,kl,mn}^{ba} = -\tilde{\nabla}_{mn} g_{ij,kl}^{ba}$ is the quantum nonmetricity tensor (see End Matter).

(...) and [...] denote normalized symmetrizations and antisymmetrizations of spatial index pairs, respectively. Here, $A_{ij}^{ab} = i\langle u_{\mathbf{k}}^a | \partial_{ij} u_{\mathbf{k}}^b \rangle = [-\sin(k_i a_0)/a_0] A_j^{ab}$ is the non-Abelian Berry connection in the parameter space of deformations, with A_j^{ab} the momentum-space Berry connection.

Physically, the quantum nonmetricity tensor N captures the failure of the Hermitian connection to preserve the two-state quantum metric under parallel transport. Equivalently, it quantifies the covariant variation of the matrix-valued inner product between quantum-state bundle vectors along momentum-space geodesics.

Additionally, the three-band contributions to the dc nonlinear elastic response tensor are concisely expressed as

$$\eta_{ij;kl,mn}^{3\text{-band}}(\mathbf{k}) = \frac{2}{3} \sum_{a,b,c} \frac{\text{Re } Q_{ij,(kl,mn)}^{abc}}{\Delta^{ba}} \left(2 + \frac{(\Delta^{ba})^2}{\Delta^{bc} \Delta^{ca}} \right) \times (f_a \Delta^{bc} + f_b \Delta^{ca} + f_c \Delta^{ab}), \quad (4)$$

with $Q_{ij,kl,mn}^{abc} = A_{ij}^{ab} A_{kl}^{bc} A_{mn}^{ca}$ a three-state QGT [37,38] in the deformation parameter space. The three-state QGT captures the amplitudes of interband transitions between three states [33,37,52], which reflects the nonlinear character of the odd viscoelastic effect. For full derivations of $\eta_{ij;kl,mn}^{2\text{-band}}(\mathbf{k})$, $\eta_{ij;kl,mn}^{3\text{-band}}(\mathbf{k})$ and a detailed analysis of the symmetry transformations and of the independent components of the response tensors, see SM [56].

Example I: Nonlinear viscoelasticity of Hopf bands—We now turn to concrete model settings to exemplify the controllable effect in two topological bands. The NOVES under consideration require inherently three-dimensional quantum geometry, which mixes all three spatial dimensions. Thus, we appeal to basic three-dimensional models of magnetic topological insulators with nontrivial topologies, which naturally provide nontrivial quantum geometries [69]. To this end, consider a two-band Moore-Ran-Wen (MRW) Hopf insulator model [70]; see End Matter. The integer topological Hopf invariant, $\chi \in \mathbb{Z}$, reads $\chi = -(1/4\pi^2) \int_{\text{BZ}} d^3\mathbf{k} \epsilon^{ijk} A_i^{aa} \Omega_{jk}^{aa}$, with $\Omega_{jk}^{aa} = \partial_{k_j} A_k^{aa} - \partial_{k_k} A_j^{aa}$ the band-diagonal Berry curvature tensor. The invariant requires nontrivial Berry curvature Ω_{jl} , and by extension, $\Omega_{ij,kl}$ tensors, and thus serves as an effective source of nontrivial quantum geometry required for $\eta_{ij;kl,mn}^{2\text{-band}}$, as we demonstrate in Fig. 2(a). Here, we can access an arbitrarily high Hopf invariant $\chi \rightarrow p\chi$ upon scaling $k_z \rightarrow pk_z$, with $p \in \mathbb{Z}$ [71]. In addition, we may control topological phases across topological phase transitions by varying a topological mass parameter m . In Fig. 2(a), we show the changes and the integer (p) scaling of the $\eta_{xx;yy,zz}$ nonlinear viscoelastic tensor component. Because of symmetry-enforced selection rules, under the combined momentum-space mirror and time reversal symmetry $\mathcal{M}_{xz}\mathcal{T}$, this model selectively realizes the first

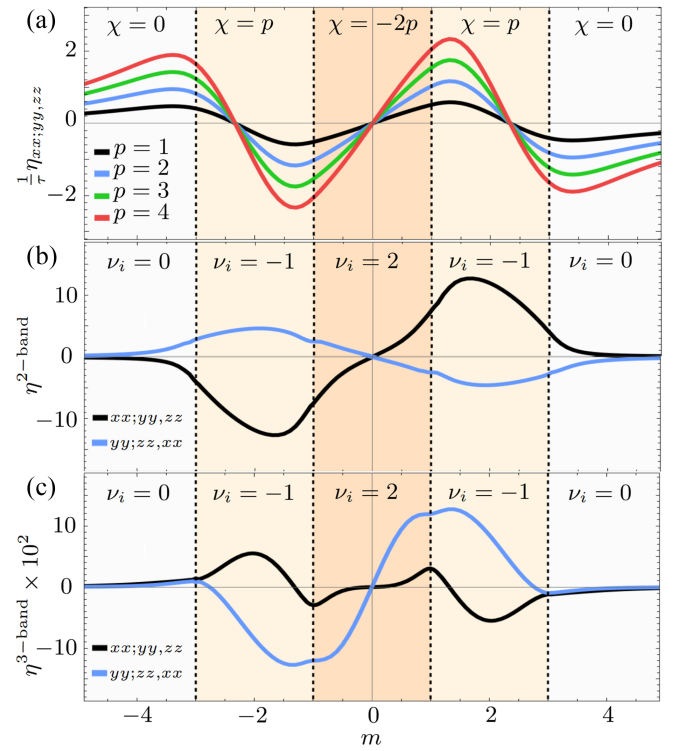


FIG. 2. Two- and three-band contributions to the nonlinear odd elasticity. (a) Two-band $\eta_{xx;yy,zz}$ for the MRW model with mass m , showcasing a sustained response in the phases with nonzero Hopf invariant magnitudes $|\chi|$ and a decaying response in the trivial phase ($\chi = 0$). Here, $p \in \mathbb{Z}$ denotes the momentum scaling factor $k_z \rightarrow pk_z$, which leads to integer scaling of the invariant and the response. (b) Two-band and (c) three-band $\eta_{xx;yy,zz}$ and $\eta_{yy;zz,xx}$ realized in a flattened three-dimensional chiral model with chiral invariant $|\nu_i|$ that trivializes upon adding a symmetry-breaking perturbation. The $\eta_{zz;xx,yy}$ response component follows from the sum rule $\eta_{xx;yy,zz} + \eta_{yy;zz,xx} + \eta_{zz;xx,yy} = 0$.

term in Eq. (3) [56]. Notably, the trivial phase $\chi = 0$ ($|m| > 3$) exhibits decaying odd viscoelastic effects, unlike its topological counterparts. We find that the response scales exactly with p in the model, i.e., $\int_{\text{BZ}} d^3\mathbf{k} \eta(k_x, k_y, pk_z) = p \int_{\text{BZ}} d^3\mathbf{k} \eta(k_x, k_y, k_z)$ [56].

Example II: Multiband nonlinear viscoelasticity—We now focus on the demonstration of two- and three-band contributions to the elasticity in multiband systems, and highlight deeper connections of NOVES to internal geometry. As a starting point, we adapt chiral three-dimensional models in the Altland-Zirnbauer class AIII [72]. We start with flattened perturbed three-band systems characterized by a chiral invariant, $\nu_i \in \mathbb{Z}$, [73,74]; see End Matter. The flat-band limit allows to isolate and probe $\eta_{ij;kl,mn}$ contributions from the quantum nonmetricity tensor $\sum_{a,b} f_{ba} N_{[(kl,mn),ij]}^{ba}$ in Eq. (3), and the three-band responses due to the three-state QGTs, $Q_{ij,(kl,mn)}^{abc}$, in Eq. (4). In this limit, the $\mathbf{k}$ dependence of both two- and three-band responses becomes entirely geometric and deconvolved

from the band dispersion contributions, showcasing that NOVEs directly reflect the internal geometry of non-metricities and three-state QGTs. Further retaining the dispersionless limit, we break the $\mathcal{M}_{xz}\mathcal{T}$ symmetry of the model, which otherwise causes the integrated N - and Q -dependent responses to vanish as a result of the aforementioned selection rules. This symmetry-breaking perturbation also trivializes the invariant, and demonstrates unambiguously that the retrieved effect is intrinsically geometric in nature, and goes beyond nontrivial topologies per se. We present numerical results for the two- and three-band contributions in this flattened perturbed chiral model in Figs. 2(b) and 2(c), respectively.

We observe that the responses are significantly weaker in the phase $|m| > 3$, owing to the suppression of the geometric tensors by the virtual transition matrix elements. In the considered model, this suppression further results in a 2 order of magnitude difference between the two- and three-band contributions. While this chiral setup effectively demonstrates the geometric nature of the effect, we note in passing that the three-band effect can also be retrieved in models with a preserved invariant, analogously to the two-band Hopf case. Indeed, when all bands are gapped and characterized by a classifying space $U(3)/U(1)^3$, this allows for an integer invariant ($\pi_3[U(3)/U(1)^3] = \mathbb{Z}$) that does not rely on additional symmetries beyond translational symmetry and particle number conservation [69,75].

Nonlinear viscoelasticity in Wannier basis—Having retrieved the two- and three-band viscoelastic momentum-space contributions in topological and geometrically nontrivial phases, we now characterize NOVEs in the real-space formalism. Correspondingly, we demonstrate how the nonlinear viscoelastic effects are manifested in the real-space Wannier function representation. To this end, we transform the single-particle states to the localized Wannier basis $|a\mathbf{R}\rangle = \int_{\text{BZ}} [d^3\mathbf{k}/(2\pi)^3] e^{-i\mathbf{k}\cdot\mathbf{R}} |\psi_{\mathbf{k}}^a\rangle$ [76,77], and deduce their centers of mass $x_c \equiv \langle a\mathbf{0} | \hat{x} | a\mathbf{0} \rangle$ for the lowest band $a = 1$, subject to perturbations w_{kl}, w_{mn} .

In Fig. 3, we show the normal and shear stress induced Wannier velocities dx_c/dw_{kl} in different topological phases, which are consistent with the elastic response currents $J_{ij} = \langle T_{ij} \rangle$ shown in Fig. 1. Hence, we show that the localized wave packet basis reflects the nonlinear odd viscoelastic transport induced by the deformations, consistently with the values of nonlinear response functions $\eta_{ij;kl,mn}$ demonstrated in Fig. 2. As such, we show that NOVEs naturally arise in momentum-space and real-space representations of fermionic wave functions realizing three-dimensional quantum-state geometries.

Discussion and conclusion—We first highlight and discuss characteristic properties of NOVEs, which lie in the symmetry transformations of the nonlinear viscoelastic response tensors, the unique roles of quantum geometry underpinned by the Hilbert-space nonmetricity and three-state QGTs, and their dissipationless nature.

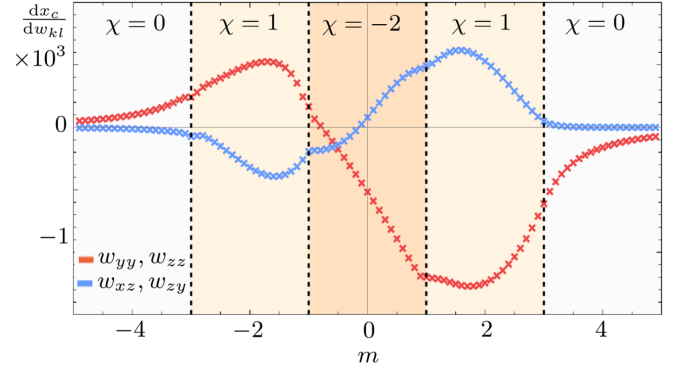


FIG. 3. Real-space representation of the NOVE in the MRW model for $p = 1$. As a function of the topological mass m , we show the dependence of the ratio x_c/w_{kl} of the Wannier center x velocity $v_c = dx_c/dt$ to the distortion rate dw_{kl}/dt that induces it. We consider here equal applied normal stresses w_{yy}, w_{zz} (red) and equal shear stresses w_{xz}, w_{zy} (blue) that lead to momentum currents J_{xx} and J_{yx} in the x direction. The viscoelastic response of the Wannier functions changes across topological phase transitions, displaying correspondence to the momentum-space results shown in Fig. 2.

The symmetry transformations of NOVEs account for the tensorial, as opposed to vectorial, nature of distortions and culminate in (i) distinct sum rules for odd responses, (ii) correspondences between normal and shear stress responses, and (iii) symmetry selection rules for response coefficients and their contrast with electromagnetic analogs; see SM [56]. The nonmetricity and three-state QGTs capture the intrinsic geometry of two-band and three-band viscoelastic tensor contributions, as exemplified by the flat-band limit. As such, these constitute geometric higher-order analogs of the Berry curvature in the linear Hall viscosity of incompressible quantum Hall fluids [15]. Further, analogously to the linear viscous Hall responses, NOVEs dissipate no energy upon actions of strains and shears, as we demonstrate in SM [56].

We stress, however, that unlike the linear viscoelasticity, the nonlinear viscoelastic effects realize a unique interplay of two-state and three-state contributions. In particular, the vanishing of the three-band term under different symmetries reflects the chirality of the electronic ground states supporting multiband NOVEs. These physical implications arise from the fact that the three-state tensors manifest in the skewness of the Wannier functions [78]. Specifically, the three-band contribution $\eta_{ij;kl,mn}^{3\text{-band}}$ is intrinsically three-dimensional, due to the symmetries of the three-state QGT, $Q_{ij;kl,mn}^{abc}$. Moreover, in this interpretation, we find that increasing the values of these QGTs, and thus the skewness [78] and chirality of the Wannier states, enables the distortions to couple more strongly, within a direct proportionality to the higher geometric tensors.

We briefly comment on the implementation of distortions in theoretical contexts of nonlinear viscoelastic

effects. For second-order responses, the adapted Peierls gauge formalism for stress tensors $T_{ij} = \delta H / \delta w_{ij} = -[\sin(k_i a_0) / a_0] \partial H / \partial k_j$ [55], and elastic moduli tensors, is justified for small deformations ($|u|/a_0 \ll 1$), consistently with the approach of Ref. [55]. We note that from a theoretical angle, the formulation of the effect suggests that generalizations to higher dimensions, with n -band terms, provides a fruitful avenue for further studies.

Finally, we address how to isolate the nonlinear viscoelastic effects experimentally. To realize NOVEs, a protocol to generate shear forces should be adapted similarly to linear responses [17,21]. However, in the presence of nontrivial $\eta_{ij;kl,mn}$ correlators, the scaling of the current response would be expected to deviate from linear. The degree of nonlinearity, combined with the transverse character of the response of systems subject to orthogonal distortions, e.g., w_{xx}, w_{yy} , should precisely reflect the qualitative phenomenological features of the odd effects identified in this Letter. Concerning material candidates, we anticipate that magnetic topological insulators, e.g., axion insulators, could naturally realize NOVEs, due to the three-dimensional multiband quantum geometry accessible in these phases. Consequently, effective θ vacua of axion insulators under realistic material setups [79–81] provide for a promising route to not only generate, but also amplify NOVE responses.

In summary, we uncovered a class of exotic nonlinear odd viscoelastic effects. We showed that these effects naturally arise in magnetic three-dimensional topological phases, and also in topologically trivial phases with nontrivial three-dimensional and three-state quantum geometries. Hence, if measured in a bulk material, the response provides an experimental fingerprint for the intrinsic quantum state geometries of higher geometric tensors and unconventional topological invariants. Finally, our work on dc nonlinear viscoelastic effects paves the way for further studies of analogous time-dependent responses and resultant exotic phenomenologies in future theoretical and experimental pursuits.

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Data availability—The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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End Matter

Quantum nonmetricity tensor—Here, we detail the formal definition of the quantum nonmetricity tensor [61]. The nonmetricity is expressed as $N_{ij,kl,mn}^{ba} = -\tilde{\nabla}_{mn} g_{ij,kl}^{ba}$, with $\tilde{\nabla}_{mn} = -\frac{\sin(k_m a_0)}{a_0} \tilde{\nabla}_n$, under the minimal coupling prescription introduced in the main text. Here, $\tilde{\nabla}_k \mathcal{O}_{ij} = \partial_k \mathcal{O}_{ij} - \tilde{\Gamma}_{lik} \mathcal{O}_j^l - \tilde{\Gamma}_{ljk} \mathcal{O}_i^l$ is the covariant derivative with respect to the Hermitian connection $\tilde{\Gamma}_{ijk}^{ba} = A_i^{ab} (\mathcal{D}_j A_k)^{ba}$, $\mathcal{D}$ is the Berry covariant derivative defined as $\mathcal{D}_i O^{ab} = \partial_i O^{ab} - i[A_i + \mathcal{A}_i, O]^{ab}$ for an arbitrary operator O [82,83], and $\mathcal{A}$ the band-diagonal Berry connection.

Elastic tensor forms—In the following, we provide the explicit expressions for the matrix elements of the stress tensor T_{ij} and its higher order analogs $U_{ij,kl}$ and $V_{ij,kl,mn}$, which read

$$T_{ij}^{ab} = \delta^{ab} \partial_{ij} E_{\mathbf{k}}^b + i \Delta^{ab} A_{ij}^{ab}, \quad (\text{B1})$$

$$\begin{aligned} U_{ij,kl}^{ab} &= \mathcal{D}_{ij} T_{kl}^{ab}, \\ &= \partial_{ij} T_{kl}^{ab} + i T_{kl}^{ab} (\mathcal{A}_{ij}^b - \mathcal{A}_{ij}^a) \\ &\quad + i \sum_{c \neq a,b} (T_{kl}^{ac} A_{ij}^{cb} - A_{ij}^{ac} T_{kl}^{cb}), \end{aligned} \quad (\text{B2})$$

$$\begin{aligned} V_{ij,kl,mn}^{ab} &= \mathcal{D}_{ij} U_{kl,mn}^{ab}, \\ &= \partial_{ij} U_{kl,mn}^{ab} + i U_{kl,mn}^{ab} (\mathcal{A}_{ij}^b - \mathcal{A}_{ij}^a) \\ &\quad + i \sum_{c \neq a,b} (U_{kl,mn}^{ac} A_{ij}^{cb} - A_{ij}^{ac} U_{kl,mn}^{cb}), \end{aligned} \quad (\text{B3})$$

where $\mathcal{D}_{ij} O^{ab} = \partial_{ij} O^{ab} - i[A_{ij} + \mathcal{A}_{ij}, O]^{ab}$ represents the distortion-space covariant derivative.

Details on models with NOVE—In the following, we further detail the constructions and exact parametrizations of model Hamiltonians realizing NOVEs. We consider a two-band Hopf insulator model that admits an adaptable tight-binding representation in real space [70]. This can

be expressed as $H_0 = \mathbf{d} \cdot \boldsymbol{\sigma}$, with $d_i = \mathbf{z}^\dagger \boldsymbol{\sigma}_i \mathbf{z}$ and a vector $\mathbf{z}$,

$$\mathbf{z} = \begin{pmatrix} \sin(k_x a_0) + i \sin(k_y a_0) \\ \sin(k_z a_0) + i(\sum_i \cos(k_i a_0) - m) \end{pmatrix}. \quad (\text{C1})$$

Here, $\mathbf{z}$ realizes nontrivial Hopf invariant $\chi = 1$ for $1 < |m| < 3$ and $\chi = -2$ for $|m| < 1$.

For the three-band case, we begin with the model that possesses an invariant protected by chiral symmetry [74]. The Hamiltonian reads $H = \mathbf{d} \cdot \boldsymbol{\lambda}$, with $\mathbf{d} = [0, 0, 0, \sin(k_x a_0), \sin(k_y a_0), \sin(k_z a_0), m - \sum_i \cos(k_i a_0), 0]^T$, where $i = x, y, z$, and $\boldsymbol{\lambda}$ is the eight-vector of Gell-Mann matrices that represents a basis of 3×3 Hermitian traceless

matrices. In order to isolate the nonmetricity and three-state QGT contributions, we set the energies of this model explicitly to $(-1, 0, 1)$. This flat-band limit causes the energy derivatives, and thus the metric- and curvature-dependent two-band terms, to vanish. This model, however, still possesses a $\mathcal{M}_{xz} \mathcal{T}$ symmetry, which nullifies the response. To this end, keeping the bands flat, we add a three-band perturbation,

$$\Delta H = \delta \sin[(k_x + k_y) a_0] \lambda_1, \quad (\text{C2})$$

with a perturbation strength δ . The perturbation breaks the chiral symmetry and trivializes the chiral invariant ν_i , but retains the nontrivial geometry of the unperturbed model.