

Systematic Search for Conformal Field Theories in Very Small Spaces

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Ground states of $1 + 1d$ conformal field theories (CFTs) satisfy a local entropic condition called the vector fixed point equation. This condition is surprisingly well satisfied by ground states of quantum critical lattice models even at small system sizes. We perform a search in the space of states of very small systems (4 qubits and 4 qutrits) and examine the states that satisfy this condition. By reconstructing a local Hamiltonian from each state, we are able to identify many of these solutions with known CFTs; others are gapped fixed points or involve large relevant perturbations, and others are CFTs we have not yet identified. These ideas are also useful for identifying continuous quantum phase transitions in a given family of Hamiltonians and for identifying the nature of the critical theory in small systems.

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Introduction—Conformal field theories (CFTs) are the end points of renormalization group flows of relativistic quantum field theories, govern the universal information at continuous phase transitions of lattice models, and in the case of $1 + 1$ dimensions, are a crucial ingredient in perturbative string vacua. The space of CFTs remains an object of mystery. Even in $1 + 1$ dimensions, where conformal symmetry implies that a theory represents an infinite-dimensional Virasoro algebra [1], a classification of unitary CFTs only exists when the Virasoro central charge c is less than 1 [2]. Constraints from modular bootstrap [3–16] notwithstanding, our knowledge of CFTs with $c \geq 1$ relies largely on *ad hoc* constructions and solvable models with extra symmetries (see, e.g., Ref. [17]). As a result, all well-understood explicit constructions produce CFTs that are rational with respect to some chiral algebra (Virasoro or one of its extensions). Despite a general belief that the generic CFT is irrational, and evidence from holographic duality for the existence of many irrational CFTs, explicit examples of irrational CFTs that are not connected to rational CFTs by marginal deformation are scarce [18] (see, e.g., Refs. [19–24] and references therein for candidate constructions).

The Entanglement Bootstrap is a program to understand the universal properties of quantum many-body states from their entanglement structure, and it provides a new perspective on quantum field theory (QFT). Let us define a universal property of a state to be a property of a

renormalization group fixed point, shared by its whole basin of attraction. The idea is that the universal information in a ground state is that of its low-energy QFT. In the case of quantum critical lattice models, this low-energy QFT is a CFT. The thus-far-most-well-developed aspect of Entanglement Bootstrap is its application to liquid gapped bosonic phases [25–32], where the low-energy field theory is a topological quantum field theory (TQFT). A key ingredient of Entanglement Bootstrap for gapped states is a set of local conditions on the state that guarantee that the state is a good representative of a gapped liquid topological order. From these local conditions, much of the machinery of TQFT has been constructed.

In [33], we proposed an analogous local condition for $1 + 1$ -dimensional CFT ground states $|\psi\rangle$,

$$K_{\Delta}|\psi\rangle \propto |\psi\rangle, \quad K_{\Delta} \equiv \eta \hat{\Delta}(A, B, C) + (1 - \eta) \hat{I}(A : C | B), \quad (1)$$

$$\hat{\Delta}(A, B, C) \equiv K_{AB} + K_{BC} - K_A - K_C, \quad (2)$$

$$\hat{I}(A : C | B) \equiv K_{AB} + K_{BC} - K_B - K_{ABC}, \quad (3)$$

where $K_A = -\log \rho_A$ is the entanglement Hamiltonian from reduced density matrix $\rho_A = \text{Tr}_{\bar{A}} |\psi\rangle\langle\psi|$, A , B , C are three adjacent intervals, and η is the cross ratio specified by their four end points. The fact that a CFT ground state satisfies this condition, which we call the vector fixed point equation (VFPE), follows from known properties of the CFT ground state entanglement Hamiltonian of an interval [34,35]. Conversely, there is evidence that a state satisfying the VFPE actually must be a fixed point of the renormalization group. (We note that not all such fixed points are CFTs; some of them have zero correlation length, rather than infinite correlation length. We will explain below how to distinguish these two classes of states.) Among the

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evidence for this conjecture [33], we mention its strong consequences for the form of the entanglement Hamiltonian [36] that a reformulation of this condition [37] can logically result in the emergence of conformal geometry, as well as the ability to extract expressions for all six global conformal generators (including the Hamiltonian) and Virasoro generators [38]. This VFPE, although it is derived from continuum CFT, works surprisingly well for critical lattice models, even in very small system sizes with small local Hilbert spaces. This is not true for all results derived from field theory, due to strong finite size effects. The VFPE is designed in such a way that finite size effects and UV physics are suppressed to some extent. This is why we can impose this condition on small systems to study universal properties of phases of matter.

The VFPE turns out [33] to be the condition that the state is a critical point of a certain linear combination of entropies of three contiguous regions, $S_\Delta \equiv \langle \psi | K_\Delta | \psi \rangle$. This suggests the possibility that S_Δ can behave like a Morse function on the space of $1 + 1d$ states, similarly to the famous Zamolodchikov c function [39], which is instead defined (for relativistic QFT) in terms of correlation functions of local operators, and in terms of which the renormalization group is gradient flow. The relationship between our entropy function and the renormalization group is still open, but this picture has an exciting practical consequence: it suggests a systematic algorithm to enumerate CFTs. A nice feature of this algorithm is that it requires no input about symmetry or a choice of Hamiltonian. In particular, there is no reason that CFTs found this way should be rational for some chiral algebra.

In this Letter, we take steps toward implementing this algorithm. A crucial ingredient is the ability [33] to reconstruct a local Hamiltonian from the ground state. The reconstructed Hamiltonian, on a circle with L sites, is

$$H_{\text{rec}}^\psi = \frac{\cot \frac{\pi}{L}}{L} \sum_{i=1}^L (K_{i,i+1} - K_i). \quad (4)$$

This Hamiltonian has several virtues. First, for $L = 4$, it satisfies $H_{\text{rec}}^\psi |\psi\rangle \propto K_\Delta |\psi\rangle$, with $\eta = 1/2$, by a purification move (e.g., $K_{ABC} |\psi\rangle = K_D |\psi\rangle$). Second, it is a local Hamiltonian; in fact, it involves only nearest-neighbor interactions. This means that, from a small system, we can reconstruct the Hamiltonian for a large system. Third, H_{rec} is a non-negative operator. This can be seen by writing it as

$$H_{\text{rec}}^\psi = \frac{\cot \frac{\pi}{L}}{2L} \sum_i \hat{\Delta}(i-1, i, i+1), \quad (5)$$

where $\hat{\Delta} \geq 0$ by operator weak monotonicity [40]. Fourth, if the state is the ground state of a CFT, and each site is identified with an interval in the continuum, it is exactly the Hamiltonian of the CFT on the circle, with the

normalization chosen so that the eigenvalues are $2\pi\Delta_i/L$, where $\{\Delta_i\}$ are the CFT scaling dimensions,

$$(H_{\text{rec}}^\psi - E_0)|\Delta_i\rangle = \frac{2\pi}{L} \Delta_i |\Delta_i\rangle, \quad (6)$$

where E_0 is the ground state energy. Elsewhere [36], we will show that a state $|\psi\rangle$ satisfying a strong version of the fixed point equation is the ground state of its reconstructed Hamiltonian H_{rec}^ψ . (The assumptions under which we currently know how to show the ground state result are stronger than those used to enumerate CFT in this Letter. Specifically, proving the ground state result requires taking the limit of many sites, $L \rightarrow \infty$, and demands that the fixed point equation hold for all possible three intervals, whereas in this Letter, we rely only on the case $L = 4$.) The spectrum of H_{rec} computed from the ground states of a critical quantum lattice model matches surprisingly well with that of the original Hamiltonian even for small system sizes. The fixed point equation for $|\psi\rangle$ also implies scale invariance of H_{rec}^ψ with dynamical exponent $z = 1$ [33].

Strategy for enumeration of CFTs—With confidence from the numerical tests in [33,41], we are ready to describe an algorithm that potentially enumerates all unitary CFTs. The algorithm contains two parts. The first part generates approximate CFT ground states $|\psi\rangle_{ABCD}$ on a circle of four evenly partitioned intervals, A, B, C, D . We replace each interval with a single “site,” a Hilbert space of finite dimension, d , which plays the role of a control parameter in our search. We proceed by finding states that are the critical points of S_Δ with $\eta = 1/2$. We search within the set of translation-invariant states on four sites. For now, we

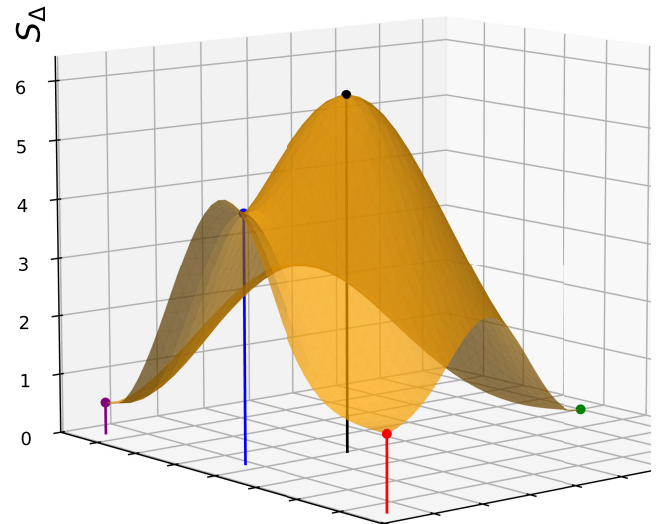


FIG. 1. A two-dimensional cross section of the function S_Δ on the space of translation-invariant real wave functions on 4 qubits (see End Matter for details). Remarkably, five of the critical points listed in Table I can be seen, including trivial (green), Ising (purple), Heisenberg (red), Fibonacci (blue), and maximum (black).

also restrict our search to time-reversal-invariant, i.e., real, states (see Fig. 1).

The second part of the procedure generates approximate CFT Hamiltonians from the ground states using (4). Once we have the Hamiltonian, we can obtain the CFT data, including scaling dimensions and operator product expansions, through known algorithms [42]. Note that the purpose of this algorithm is not to give accurate values for the CFT data. The purpose is to generate as many CFTs as possible, including previously unknown ones. If one wants accurate values, improvements to the Hamiltonians can be made, and conformal bootstrap [43] can be used afterward to turn these approximate values into tighter bounds.

Before continuing, let us introduce the operator $\hat{c}_\Delta$ proportional to H_{rec} :

$$\hat{c}_\Delta(\psi) \equiv \frac{3}{2 \log 2} (K_{AB} - K_B + K_{BC} - K_C + K_{CD} - K_D + K_{DA} - K_A), \quad (7)$$

with expectation value $c_\Delta(\psi) \equiv \langle \psi | \hat{c}_\Delta(\psi) | \psi \rangle$. Since $\hat{c}_\Delta \propto H_{\text{rec}}$, we have $\hat{c}_\Delta | \psi \rangle \propto H_{\text{rec}} | \psi \rangle \propto K_\Delta (\eta = 1/2) | \psi \rangle \propto | \psi \rangle$ for CFT ground states. The advantage of this normalization is that, when $| \psi \rangle$ is the ground state of the field theory, c_Δ is exactly the central charge of the corresponding CFT. Thus, equivalently, the goal in the first part is to find all the critical points of c_Δ .

Search: The process of finding critical points consists of two steps. First, we find an initial point close to a critical point. To do so, we cannot just do gradient descent on c_Δ because that will give us only the local minima, whereas here we want to find all the critical points. [If instead the goal is to find CFTs with spectral gap > 2 (also known as dead-end CFTs since they cannot be perturbed by relevant operators), then it may be sufficient to study the local minima.] Instead, we perform gradient descent over the error function, which is the standard deviation of $\hat{c}_\Delta$ (note

that our choice of normalization differs from that used in [33]):

$$\text{err} \equiv \sigma(\hat{c}_\Delta) \equiv \sqrt{\langle \psi | (\hat{c}_\Delta(\psi))^2 | \psi \rangle - \langle \psi | \hat{c}_\Delta(\psi) | \psi \rangle^2}. \quad (8)$$

Second, we use a second-order Newton's method to converge to the critical point. See Supplemental Material [44] for further details of the search procedure.

Note that c_Δ is invariant under local unitary transformations. Thus, each critical point is actually an orbit of this gauge redundancy.

Analysis: Once the critical points are identified, we perform a series of diagnostics to determine whether their properties are consistent with those of a conformal field theory. As a preliminary step, we discard states whose H_{rec} has degenerate ground states since a CFT has a unique ground state. Next, we extend the reconstructed Hamiltonian from four sites to L sites with the same $K_{i,i+1} - K_i$ and use periodic matrix product states to obtain the low-energy spectrum and corresponding eigenstates. We wish to check whether the low-energy spectrum of H_{rec} exhibits universal behavior as the system size L increases. Let $E_{L,i}$ denote the i th eigenvalue of H_{rec} on L sites. If H_{rec} is described by a CFT without relevant perturbations, one expects that, for fixed i , the limit $\lim_{L \rightarrow \infty} L \cdot E_{L,i}$ exists. If a small relevant perturbation is present, $L \cdot E_{L,i}$ converges when L is less than the correlation length ξ , as illustrated in [44]. Thus, the diagnostic is to test whether the low-lying spectrum decreases at the rate $1/L$ for $L < \xi$.

We also examine whether the ground state exhibits the entanglement entropy scaling expected of a CFT ground state, namely $S(\ell) = (c/3) \log \sin(\pi \ell / L) + \text{const}$. The diagnostic is to test whether a plot of $S(\ell)$ vs $\log \sin(\pi \ell / L)$ yields a straight line.

Results—Table I and Fig. 2 list the critical states we found over 4 qubits and 4 qutrits, respectively. In the case of qutrits, we perform sampling—it might not be a complete

TABLE I. All critical 4-qubit states. Actual CFTs are in blue. We defined $|\alpha\rangle \equiv (|0011\rangle + |0110\rangle + |1100\rangle + |1001\rangle)/2$ and $|\beta\rangle \equiv (|1010\rangle + |0101\rangle)/\sqrt{2}$ as elements of our basis of translation-invariant states. The coefficients for the Ising solutions are $[a_1, a_2, a_5, a_6] = [0.11036, 0.80152, 0, 0, -0.51103, 0.29023]$. In fact, there are two inequivalent maximum point states with the same H_{rec} and S_Δ that are not related by on-site unitaries.

c_Δ	Description	Explicit form up to on-site unitary
0	Cat states	$a 0000\rangle + b 1111\rangle$
0.526	Ising CFT	$a_1 0000\rangle + a_2 1111\rangle + a_5 \alpha\rangle + a_6 \beta\rangle$
1.132	W state	$\frac{1}{2}(0001\rangle + 0010\rangle + 0100\rangle + 1000\rangle)$
1.211	XX model	$-\frac{1}{\sqrt{2}} \beta\rangle + \frac{1}{\sqrt{2}} \alpha\rangle$
1.245	Heisenberg	$-\sqrt{\frac{2}{3}} \beta\rangle + \frac{1}{\sqrt{3}} \alpha\rangle$
3.510	Ferromagnet	$\frac{1}{\sqrt{3}} 0000\rangle + \frac{1}{\sqrt{6}}(0111\rangle + 1011\rangle + 1101\rangle + 1110\rangle)$
4.165	Fibonacci chain	$\frac{1}{\sqrt{45}}(0000\rangle + \sqrt{\sqrt{5}-1} \beta\rangle)$
6.000	Maximum point	$\frac{1}{2}(0000\rangle \pm 1111\rangle + 0101\rangle + 1010\rangle)$

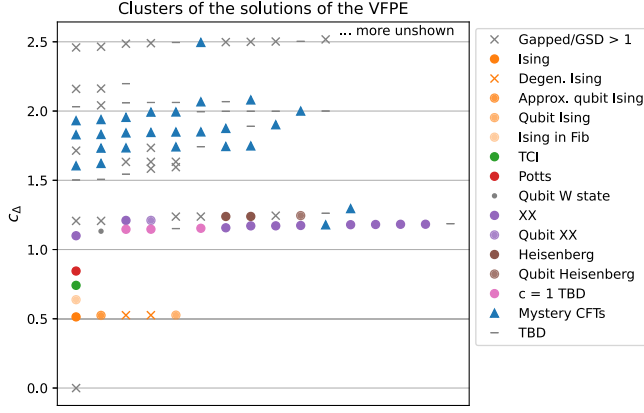


FIG. 2. Critical 4-qutrit states. We do not display the states with larger values of c_Δ . “Degen. Ising” refers to a state with the spectrum of the critical Ising model, but with twofold degeneracy. “Ising in Fib.” refers to a state whose spectrum on L sites is the first ℓ_L states of the Ising model below a gap, where ℓ_L is the L th Lucas number.

list, but a list of solutions we have found. In both cases, we restrict our search to translation-invariant and time-reversal-invariant states. For qubits and qutrits, respectively, this space has 5 and 23 dimensions.

Qubits: The entries in the table are all RG fixed points, but some of them are zero-correlation-length fixed points. Actual CFTs are in blue. (There exist translation-invariant states that are equivalent under an on-site unitary, but not under a translation-invariant on-site unitary. For example, two realizations of the Ising CFT, $a_1|0000\rangle + a_2|1111\rangle + a_5|\alpha\rangle \pm a_6|\beta\rangle$ are related by the unitary $I \otimes Z \otimes I \otimes Z$.) These two classes of states are distinguished by reconstructing the local Hamiltonian from the 4-qubit state and studying it on larger system sizes as described above. For the XX and Heisenberg solutions, we discover that the reconstructed Hamiltonians are exactly the well-known Hamiltonians $H = XX + YY$ and $H = XX + YY + ZZ$ up to a prefactor and an additive constant. (These correspond to $c = 1$ free boson CFTs with radius $r = 1, 1/\sqrt{2}$, respectively.) In fact, this prefactor, which gives the exactly correct scaling behavior of the Hamiltonian, is a bonus from using the reconstructed Hamiltonian—one does not need to manually scale the spectrum. The reconstructed Hamiltonians from the gapped fixed point states on qubits all have a ground state degeneracy.

Here are some comments about the zero-correlation-length fixed points that appear in the table: The ground state degeneracy and c_Δ have a peculiar relation, i.e., $\log \text{degeneracy} = 0.46c_\Delta$. There are only two eigenstates with finite energy for the reconstructed Hamiltonian of the W state: one ground state is the W state, and the other is the product state $|0\rangle^{\otimes n}$. This is consistent with the theorem of [46], which forbids a local Hamiltonian whose *unique* ground state is the W state. The reconstructed Hamiltonians

for the states labeled “ferromagnet” and “Fibonacci” are (up to an additive constant)

$$H_{\text{Ferromagnet}} \propto -\sum_i (X_i X_{i+1} + Y_i Y_{i+1} + Z_i Z_{i+1}), \quad (9)$$

$$H_{\text{Fibonacci}} \propto \sum_i (|11\rangle\langle 11|)_{i,i+1}, \quad (10)$$

which explains the nomenclature. The latter has been well known to produce the golden chain [24,47–49], which at system size $L = 2, 3, 4, 5, 6, 7, \dots$ has a ground state degeneracy $\ell_L = 3, 4, 7, 11, 18, 29, \dots$, where ℓ_L are the Lucas numbers. The maximum point of S_Δ on 4 qubits is something like a perfect tensor, where $S_2 = 2 \log 2$ attains its maximal value for any two contiguous sites.

It is not a coincidence that the three CFTs that fit into the 4-qubit Hilbert space in Table I are ones for which we know a parent Hamiltonian acting on nearest-neighbor qubits. Our reconstructed Hamiltonian always acts on two sites at a time. The question of which CFTs will fit in a given local Hilbert space, with our nearest-neighbor reconstructed Hamiltonian, is closely related in spirit to the c - d conjecture of [50], which says that a lattice model of local Hilbert space dimension d and nearest-neighbor interactions can only realize a CFT with $c \leq d - 1$. The results so far are consistent with that conjecture. In fact, our search strategy can in principle provide a rather stringent test of that conjecture: in our search with local dimension d , we should find no CFTs with $c > d - 1$.

Qutrits: Compared to Table I, when searching in the space of translation-invariant real states on 4 qutrits, we find more CFTs (Fig. 2). In the list for qutrits, there are multiple representatives of the Ising model. One is the same state as realized on qubits, making no use of the third register (labeled “Ising qubits”), and the other is a new state whose value of c_Δ is closer to $\frac{1}{2}$. Interestingly, there is a solution with $c_\Delta \approx 0.638$ that encodes the Ising spectrum in the Fibonacci chain ground state subspace.

We find representatives of the next two unitary minimal models after Ising, with $c_\Delta \approx 0.744$ (tricritical Ising has $c = 7/10$) and $c_\Delta \approx 0.844$ (three-state Potts has $c = 4/5$), as well as many more points along the $c = 1$ line. At larger c , we find many more (of order 25) states whose reconstructed spectrum looks like that of a CFT, possibly with small relevant perturbation. In [44], we give details of the information extracted from the solutions of the VFPE on qutrits.

Further discussion: What determines the order in which CFTs appear in our search as we increase the local Hilbert space dimension? There is a preference for small values of c , which, since the reconstructed Hamiltonian always acts on precisely two sites, is consistent with the spirit of the c - d conjecture of [50]. We observe that CFTs with primaries of small dimensions do not fit in small Hilbert spaces; this phenomenon is illustrated in Fig. 4 by varying the radius of the $c = 1$ CFTs. This observation is

consistent with the slow appearance of the unitary minimal models in our search as we increase d .

Discussion—

c_Δ as a c function?: Starting at a saddle point $|\psi\rangle$ of c_Δ identified with a CFT, we can parametrize nearby states as $|\psi\rangle + |d\psi\rangle$. In turn we can expand $|d\psi\rangle$ in a basis of eigenstates of H_{rec}^ψ , which, by the state-operator correspondence of CFT, correspond to operators of definite scaling dimension, $|\Delta_i\rangle$. Thus, we can ask about the change in c_Δ upon deforming the state by a given $|\Delta_i\rangle$. It is natural to ask about a possible relation between the spectrum of H_{rec}^ψ and the spectrum of the Hessian of c_Δ . The example of CFTs with exactly marginal operators shows that the match is not precise. However, we observe numerically in examples that the number of relevant operators is equal to the number of downhill directions of c_Δ , that is, negative eigenvalues of its Hessian.

Toward the set of CFT: Does a solution of the VFPE on four sites encode all the data of the associated CFT? This seems plausible, though turning this into a universal algorithm remains challenging. An overly optimistic possibility is that the Hamiltonian reconstructed from such a state, when extended to large system size, would automatically realize the CFT in the IR limit. While this is true for some of the states (such as Heisenberg and XX on qubits), most of the states we find produce H_{rec} with a finite correlation length (of order 15–30 lattice spacings), even when the spectrum of H_{rec} for small sizes agrees with the known list of scaling dimensions. However, using the four-site state as a starting point, we can systematically refine the information in many ways. A simple method is as follows: from the four-site solution with local dimension d , one can extend the reconstructed Hamiltonian to eight sites and block the sites into four supersites of dimension d^2 . Starting with its ground state, one can use Newton’s method to find a lower-error state on four sites with local dimension d^2 . One can repeat this process to progressively improve the reconstructed Hamiltonian.

In addition to well-understood states with $c < 1$ and $c = 1$, in the qutrit search, we identify several candidate CFT states in the range $1 < c < 2$ (see Ref. [44]). These states, the spectra of whose appropriately normalized reconstructed Hamiltonians appear to converge with system size, clearly deserve further attention.

Criticality in small spaces: We should emphasize an important practical point. H_{rec} is a useful tool for learning from small systems: the normalization of $H_{\text{rec}} = \sum_i \lambda_i K_i$ is completely fixed by $\sum_i \lambda_i \beta(x) \stackrel{!}{=} 1$, as in (4). This obviates the need to identify the stress tensor in order to fix the normalization. Moreover, H_{rec} for the optimized states, we find, gives the best realization of the continuum CFT in the given Hilbert space, often better than our favorite lattice model. We have, thus, provided a method to improve upon a given quantum critical Hamiltonian with a given local Hilbert space dimension.

Entanglement Bootstrap for CFT: One important goal of Entanglement Bootstrap is to use local entanglement conditions to prove structural properties of the universal data. We believe that the VFPE, for every set of three contiguous intervals, is such an axiomatic criterion for the state to be a CFT ground state, from which, for example, the structure of the Virasoro algebra should follow. An important future task is to study the behavior of its violation under various improvements discussed above, starting from the special states we have found, analogous to the robust version of Entanglement Bootstrap for gapped states [51].

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*Data availability—*The data that support the findings of this article are openly available [54].

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End Matter

Axes of Fig. 1—Figure 1 shows a two-dimensional slice of the translation-invariant and time-reversal-invariant subspace of 4 qubits on a ring. This slice is carefully chosen to show five of the critical points of the function S_Δ . We choose three critical states (the product state $|0000\rangle$, the Heisenberg state, and the Ising state) and associate them with the corners of a simplex. Each superposition of these three states is then mapped to the plane using barycentric coordinates.

Approximate solutions of the VFPE—In addition to the exact or numerically exact solutions found above, it is also interesting to study approximate solutions of the VFPE. One reason we are led to study such approximate solutions is the behavior of CFTs with exactly marginal operators in our search: on 4 qubits, we find two special points in the $c = 1$ moduli space, as isolated critical points of S_Δ , despite the fact that the XXZ model

$$H_{\text{XXZ}} = \sum_i (X_i X_{i+1} + Y_i Y_{i+1} + \Delta_{\text{XXZ}} Z_i Z_{i+1}) \quad (\text{A1})$$

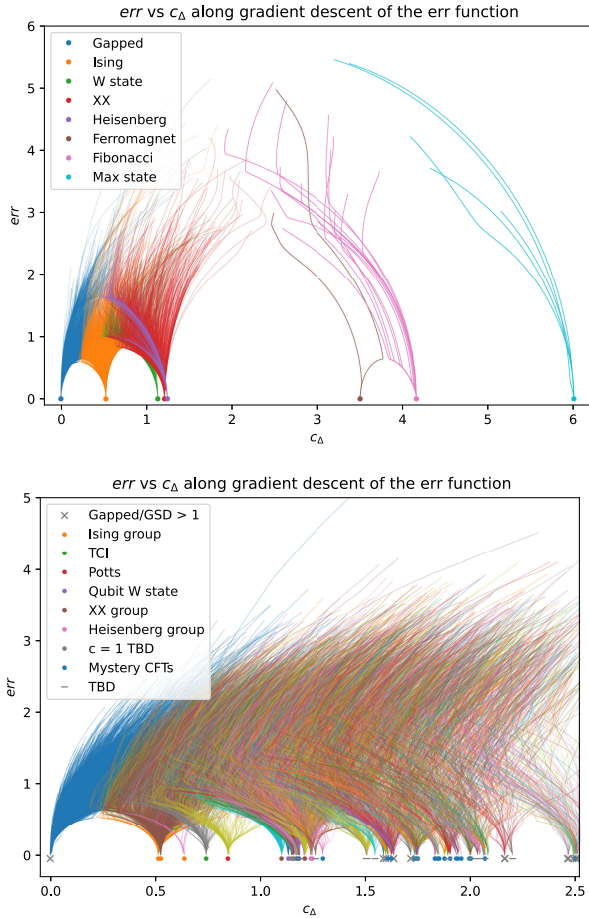


FIG. 3. Error of the VFPE vs c_Δ along the trajectory of the gradient descent of the error function on 4 qubits (top) and 4 qutrits (bottom). We first do a uniform sampling in the space of translation-invariant and time reversal symmetric states. Then we use the ground states of their reconstructed Hamiltonians H_{rec} [Eq. (4)] as the initial states in the gradient descent. There are forbidden regions for c_Δ when the errors of VFPEs are small. The names in the figure legends will be explained below.

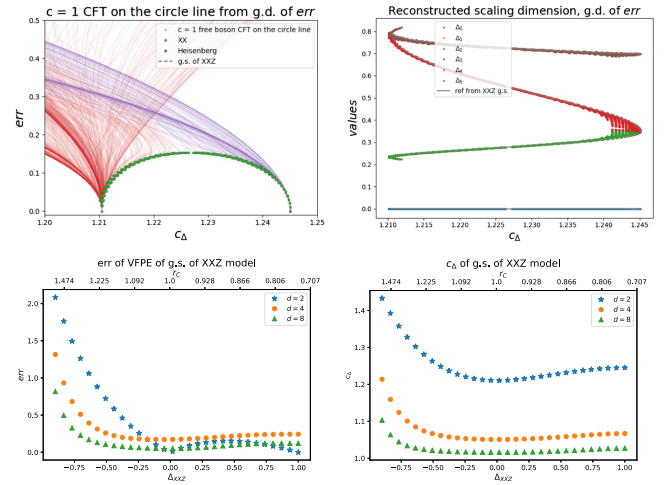


FIG. 4. Top: gradient descent on the error near the $c = 1$ fixed points on qubits, XX ($c_\Delta = 1.211$) and Heisenberg ($c_\Delta = 1.245$). The void between these two points is a finite size effect. Also shown (in gray) are (c_Δ, err) in ground states of the XXZ model as Δ_{XXZ} varies from 0 to 1, which interpolate between XX and Heisenberg; for intermediate Δ_{XXZ} on four sites, there is a finite error. The green dots are obtained by selecting the states whose (c_Δ, err) is close (with deviation less than 10^{-3}) to that of an XXZ ground state and whose reconstructed Hamiltonians have approximate degeneracy of the first excited states. In this way, we match a value of c_Δ with a value of the parameter Δ_{XXZ} in H_{XXZ} . Top right: the spectra of H_{rec} from states at the boundary of the void of the top left plot. These spectra agree with those of the $c = 1$ theory on the circle line and that of the XXZ model Eq. (A1) at the values of Δ_{XXZ} determined by the green dots. Bottom left: the error of the VFPE for ground states of the XXZ model Eq. (A1) as a function of Δ_{XXZ} , which determines the value of the radius of the $c = 1$ boson (indicated at the top of the plot). We see that, when the radius grows, so that there are low-dimension primaries, the error of the VFPE (with fixed local Hilbert space) grows. Bottom right: the corresponding values of c_Δ .

is a nearest-neighbor Hamiltonian on qubits that realizes a whole family of $c = 1$ CFTs with different radii. We conclude that the error of the VFPE in these states is a finite size effect. By relaxing our search criteria to states with *small* error of the VFPE, we can indeed see the whole $c = 1$ moduli space (though we have not yet found the Ginsparg archipelago [55,56], likely because they have several primaries with small scaling dimensions); see Ref. [44]. The pattern of approximate solutions also contains information about the space of CFTs. In particular, one can see voids in the set of states in the plot of error vs c_Δ around known forbidden regions; see Fig. 3 and [44]. There could be some quantum-information-theoretic reason that forbids states in these areas: we conjecture that, when $\sigma(\hat{c}_\Delta) \leq \epsilon$, the values of c_Δ are constrained to take a set of specific values with a small window of deviations. It is also suggestive that we can observe similar voids in regions for $c > 1$, where there is so far no known condition forbidding unitary CFTs. It might also be interesting to study the state that minimizes the error with a given c_Δ , i.e., the states at the boundary of the voids.

In Fig. 4, we show the error of the VFPE in the XXZ model as a function of Δ_{XXZ} , which determines the radius of the $c = 1$ boson. We observe that the states in the whole range $\Delta_{\text{XXZ}} \in (0, 1)$ have small error. When the radius takes extreme values (near 0 or ∞), there are light primaries, and the error grows. The figure shows, furthermore, that the low-

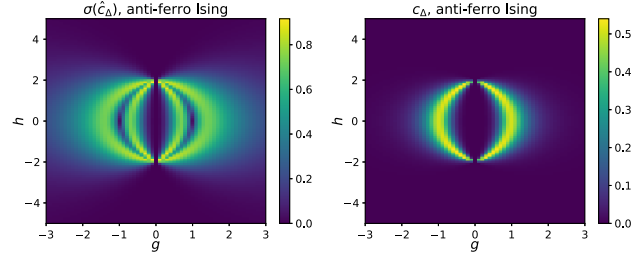


FIG. 5. The error of the VFPE and the value of c for the *anti* ferromagnetic Ising chain with transverse and longitudinal fields: $H = + \sum_{j=1}^L Z_j Z_{j+1} - g \sum_j X_j - h \sum_j Z_j$, with periodic boundary conditions, $L = 8$, and $A = \{1, 2\}$, $B = \{3, 4\}$, $C = \{5, 6\}$. We can see two gapped fixed points with $c = 0$ separated by a wall of Ising transitions with $c \approx 0.5$. After a sublattice rotation to map to a ferromagnetic Ising model in a staggered field, it has a hidden Ising symmetry acting by $\prod_j X_j \otimes$ translation by one site.

lying spectrum of the reconstructed Hamiltonian still matches that of the corresponding CFT for small but nonzero error.

Error of VFPE as a mapping tool—The error of the VFPE provides a simple practical tool for drawing phase diagrams of a given family of Hamiltonians. Scanning a phase diagram for minimal error of the VFPE detects fixed points of the RG. In Fig. 5, we give an example of this method. More examples of this application of the VFPE can be found in [57].