

Accurate Boundary Bootstrap for the Three-Dimensional $O(N)$ Normal Universality Class

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The three-dimensional classical $O(N)$ model with a boundary has received renewed interest due to the discovery of the extraordinary-log boundary universality class for $2 \leq N < N_c$. The critical value N_c and the exponent of the boundary correlation function are related to certain amplitudes in the normal universality class. To determine their precise values, we revisit the 3D $O(N)$ boundary conformal field theory for $N = 1, 2, 3, 4, 5$. After substantially improving the accuracy of the boundary bootstrap, our determinations are in excellent agreement with the Monte Carlo results, resolving the previous discrepancies due to low truncation orders. We also use the recent bulk bootstrap results to deduce highly accurate Ising data. Many bulk and boundary predictions are obtained for the first time. Our results demonstrate the great potential of the η minimization method for many unexplored bootstrap problems in which positivity constraints are absent.

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Introduction—As codimension one defects, boundaries are ubiquitous and play an important role in condensed matter physics and high energy physics, ranging from edge states of topological materials to D-branes in string theory. In this Letter, we are interested in boundary critical phenomena [1,2]. The $O(N)$ model with a boundary provides a basic example of boundary criticality, which corresponds to the Ising, XY, and Heisenberg universality classes for $N = 1, 2, 3$. However, a complete understanding of the three-dimensional $O(N)$ boundary phase diagram remains elusive.

In [3], Metlitski pointed out that there exists a novel universality class of the extraordinary-log type for $2 \leq N < N_c$. See Refs. [4–13] for further developments. This phase is characterized by *logarithmic* decay of the boundary spin correlator,

$$\langle \vec{S}_{\vec{x}} \cdot \vec{S}_{\vec{y}} \rangle \sim \frac{1}{(\log |\vec{x} - \vec{y}|)^q}, \quad (1)$$

and its stability hinges on a delicate balance of universal parameters. The logarithmic form emerges from a marginally irrelevant coupling of the nonlinear sigma model for the Goldstone modes. The exponent $q = (N - 1)/(2\pi\alpha)$ is determined by the universal renormalization

group (RG) parameter

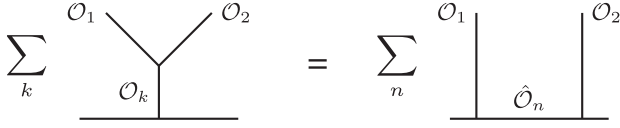
$$\alpha = \frac{1}{32\pi} \frac{a_\phi^2}{b_{\phi t}^2} - \frac{N - 2}{2\pi}, \quad (2)$$

where $(a_\phi, b_{\phi t})$ are amplitudes in the normal universality class [1,2,14–18]. The extraordinary-log phase is stable for $\alpha > 0$. The critical value N_c is defined by $\alpha|_{N=N_c} = 0$. Above N_c , the extraordinary fixed point may annihilate with the special fixed point or evolve into a power-law type [3]. For $N = 2, 3$, these amplitudes were extracted from Monte Carlo simulations [7]. Their precise values at larger N are crucial for the determination of N_c , but Monte Carlo results are not yet available. The previous conformal bootstrap study [6] provided important estimates for a larger range of N . In conformal field theory (CFT), the universal amplitudes $(a_\phi, b_{\phi t})$ are known as boundary operator expansion (BOE) coefficients. Due to the presence of an explicit symmetry breaking field, a normal boundary partly breaks not only the external conformal symmetry, but also the internal $O(N)$ symmetry. Thus, the normal boundary CFT (BCFT) is of interest on its own, and we also study the Ising case of $N = 1$.

Since the seminal work in [19], the conformal bootstrap program for $d > 2$ CFT has been revived by incorporating positivity constraints and efficient algorithms [20,21]. See Refs. [22–33] for the impressive progress on the 3D $O(N)$ bulk data based on the positive bootstrap. In [34], Liendo *et al.* extended the bootstrap program to BCFT. However, it is not clear if the powerful positive bootstrap is applicable, as the bulk-channel expansion of a two-point correlator is

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 FIG. 1. Crossing symmetry of $\langle \mathcal{O}_1 \mathcal{O}_2 \rangle$ in boundary CFT.

not quadratic. (See the left-hand side of Fig. 1.) This necessitates alternative, non-positivity-based methods.

In [35], Gliozzi proposed to solve the bootstrap equation by truncating to a finite number of operators, which does not rely on positivity. In [35–37], the truncated bootstrap constraints on operator dimensions are encoded in determinants or singular values. While early applications to the $O(N)$ BCFT in [6,38,39] were promising, these studies were limited to low truncation orders. For the normal universality class, the previous bootstrap studies [6,38] yielded results that were in noticeable tension with Monte Carlo estimates [7,40], casting doubt on the reliability of the truncated bootstrap approach. As the truncated bootstrap is not a rigorous method, the tension with [6] is not a mathematical contradiction. It was not clear if these discrepancies arose from uncontrolled systematic errors or low truncation orders. To address this question, we use the η minimization formulation of the truncated bootstrap [41], which also has a number of variants due to its flexibility, incorporating artificial intelligence [42–46], analytic input [47–50], and random weights [48–51].

By reformulating the truncated boundary bootstrap as a search for the zeros of a cost function, we reach significantly higher truncation orders Λ (see Table I) and substantially improve the accuracy of the bootstrap results. We introduce a new procedure to construct starting points for local minimizations and substantially enhance the efficiency of extracting sparse solutions. (Sparseness in the truncated bootstrap is the counterpart of positivity in the conventional bootstrap.) Then, the bootstrap estimates are in excellent agreement with the Monte Carlo results, resolving the previous discrepancies. Many results are more accurate or completely new. Furthermore, we assign reliable errors to the estimates associated with low-lying operators, which are mainly from uncertainties in bulk input parameters. To obtain highly accurate Ising results, all we need are precise bulk input and high enough truncation orders.

Since the truncated bootstrap was originally devised for nonunitary CFTs in [35], we anticipate that our

substantially improved version will overcome long-standing technical bottlenecks and find wide application in statistical mechanics, from random geometry to disordered systems. These systems involve more intricate theoretical descriptions such as logarithmic CFTs and supersymmetric nonlinear sigma models.

Boundary bootstrap with the η minimization—According to boundary conformal symmetry, the correlation function of two bulk scalar primaries reads

$$\langle \mathcal{O}_1(x) \mathcal{O}_2(y) \rangle = \frac{G(\xi)}{(2x_\perp)^{\Delta_1} (2y_\perp)^{\Delta_2}} \xi^{-\frac{\Delta_1 + \Delta_2}{2}}, \quad (3)$$

where $G(\xi)$ is an unknown function of the conformally invariant cross ratio $\xi = [(x-y)^2/(4x_\perp y_\perp)]$. We can decompose the two-point function, Eq. (3), into conformal blocks. In the bulk channel, we use the bulk operator product expansion (OPE)

$$\mathcal{O}_1(x) \mathcal{O}_2(y) = \sum_k \lambda_{12k} C_{12k}(x-y, \partial_y) \mathcal{O}_k(y), \quad (4)$$

so Eq. (3) is given by a summation of bulk one-point functions,

$$\langle \mathcal{O}_k(x) \rangle = \frac{a_k}{(2x_\perp)^{\Delta_k}}, \quad (5)$$

and their derivatives, where a_k vanishes for spinning primaries. In the boundary channel, we consider the boundary operator expansion or the bulk-to-boundary OPE

$$\mathcal{O}_k(x) = \sum_n b_{kn} \hat{C}_{kn}(x_\perp, \partial_{x_\parallel}) \hat{\mathcal{O}}_n(x_\parallel); \quad (6)$$

then, Eq. (3) becomes a summation of boundary two-point functions. The above one-point coefficients are the BOE coefficients associated with the boundary identity, i.e., $a_k = b_{kI}$. The agreement between the two decompositions implies the bootstrap or crossing equation (see also Fig. 1),

$$\sum_k \lambda_{12k} a_k f_{\Delta_k}^{\Delta_{12}}(\xi) - \xi^{\frac{\Delta_1 + \Delta_2}{2}} \sum_n \mu_{12n} \hat{f}_{\hat{\Delta}_n}(\xi) = 0, \quad (7)$$

where $\Delta_{12} = \Delta_1 - \Delta_2$ and $\mu_{12n} = b_{1n} b_{2n}$. The descendant contributions in $C(x-y, \partial_y)$ and $\hat{C}(x_\perp, \partial_{x_\parallel})$ are repackaged as the bulk and boundary channel conformal blocks [52]:

$$f_{\Delta}^{\Delta_{12}}(\xi) = \xi^{\Delta/2} {}_2F_1 \left[\frac{\Delta + \Delta_{12}}{2}, \frac{\Delta - \Delta_{12}}{2}; \Delta - \frac{d-2}{2}; -\xi \right],$$

$$\hat{f}_{\hat{\Delta}}(\xi) = \xi^{-\hat{\Delta}} {}_2F_1 \left[\hat{\Delta}, \hat{\Delta} - \frac{d}{2} + 1; 2\hat{\Delta} - d + 2; -1/\xi \right]. \quad (8)$$

As there is only one cross ratio, it is simpler to study Eq. (7) than the bulk four-point bootstrap equation without a

 TABLE I. The maximum truncation order $\Lambda_{\max}$ in this Letter and the previous truncated bootstrap studies [6,38]. The two numbers for $N = 1$ are associated with $\langle \sigma \epsilon \rangle$ and $\langle \sigma \sigma \rangle$.

	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$
This Letter	88, 68	76	68	60	88
[6,38]	9, 8	10	10	9	9

boundary. In the bulk channel, we make use of the previous accurate determinations of the bulk operator dimensions. In the boundary channel, the leading operators in the normal universality class have protected dimensions. Together with sparseness of the low-lying spectra, the normal boundary universality class is a natural target for the conformal bootstrap [6].

To discretize the bootstrap equation, Eq. (7), we take the m th derivative with respect to ξ and then set $\xi = 1$. We restrict the order of ξ derivatives to M , so we have a finite system. We truncate the bulk OPE and BOE in Eq. (7), i.e., $k = 1, 2, \dots, n_{\text{bulk}}$ and $n = 1, 2, \dots, n_{\text{bdy}}$. (See Supplementary Material [53] for more details.) We use the η function to encode these truncated bootstrap constraints,

$$\eta = \sum_j \sum_{m=0}^{M_j} \left| \partial_{\xi}^m (\text{bootstrap equation}_j) \right|_{\xi=1}^2, \quad (9)$$

where j labels the bootstrap equations under consideration. We impose that the number of bootstrap constraints is the same as that of free parameters, which is also referred to as the truncation order Λ . By construction, the η function can vanish only when all the truncated bootstrap equations are satisfied. Below, we systematically solve the truncated bootstrap equations by searching for the zeros of the η function, Eq. (9),

$$\eta(\{\Delta_i, \lambda_{ijk} a_k\}, \{\hat{\Delta}_n, b_{in}\}) = 0, \quad (10)$$

which are equivalent to the intersection points of certain vanishing loci of minors in Gliozzi's determinant formulation.

The $O(N)$ BCFT for $N = 2, 3, 4, 5$ —In the normal transition, the $O(N)$ symmetry is explicitly broken to $O(N-1)$. While the bulk operators are classified by $O(N)$ irreducible representations, the boundary operators are associated with $O(N-1)$. The two-point function of the lightest bulk $O(N)$ vector ϕ_a involves two $O(N-1)$ singlets, so we have two crossing equations. They correspond to

$$\langle \phi_N(x) \phi_N(y) \rangle, \quad \sum_i \langle \phi_i(x) \phi_i(y) \rangle, \quad (11)$$

where $a = (i, N)$ and $i = 1, 2, \dots, N-1$. The bulk fusion rule for the product of two $O(N)$ vectors reads

$$\phi_a \times \phi_b \sim \sum_S \delta_{ab} \mathcal{O} + \sum_T \mathcal{O}_{(ab)} + \sum_A \mathcal{O}_{[ab]}, \quad (12)$$

which involves the $O(N)$ singlets S , traceless symmetric tensors T , and antisymmetric tensors A . Only the first two types of representations can be scalar primaries and contribute to the boundary bootstrap equations. The boundary fusion rules associated with the $O(N-1)$ singlets $\hat{S}$

and vectors $\hat{V}$ are

$$\phi_N \sim 1 + D + \sum_{\hat{\Delta} > 3} \hat{\mathcal{O}}^{(\hat{S})}, \quad \phi_i \sim t_i + \sum_{\hat{\Delta} > 2} \hat{\mathcal{O}}_i^{(\hat{V})}, \quad (13)$$

where D is the displacement operator with $\hat{\Delta}_D = 3$, and t_i is the tilt operator with $\hat{\Delta}_t = 2$. See Eqs. (2.20) and (2.21) in [6] for the explicit crossing equations. The input parameters are the bulk dimensions $\{\Delta_\phi, \Delta_S, \Delta_{S'}, \Delta_T\}$ from Monte Carlo simulations [54–57] and the bulk bootstrap [23,30,31]. The operators in the fusion rules are ordered by scaling dimensions. The subleading ones are indicated by primes. For instance, the subleading $O(N)$ singlet is denoted by S' .

Using the η minimization method, we systematically increase the truncation order Λ . Surprisingly, the bootstrap results exhibit nice convergence patterns, so we further make some power-law fits in $1/\Lambda$ and extract the $\Lambda \rightarrow \infty$ extrapolations. We assume that the truncation errors vanish in the infinite Λ limit, so the errors are from the uncertainties in the bulk input and the $\Lambda \rightarrow \infty$ extrapolations. In Fig. 2, we compare our estimates for $(a_\phi, b_{\phi t})$ with the literature results. As substantial improvements of the previous results in [6], we significantly increase the bootstrap accuracy, and our results are in excellent agreement with the Monte Carlo estimates [7].

In [6], the T contributions in the bulk channel are projected out by considering a linear combination of the two crossing equations. Here, we take into account the T contributions and solve two crossing equations simultaneously. This difference is essential to the convergence of the bootstrap results and explains why the triangle points in Fig. 3 lie far from the extrapolation curves. In Fig. 3,

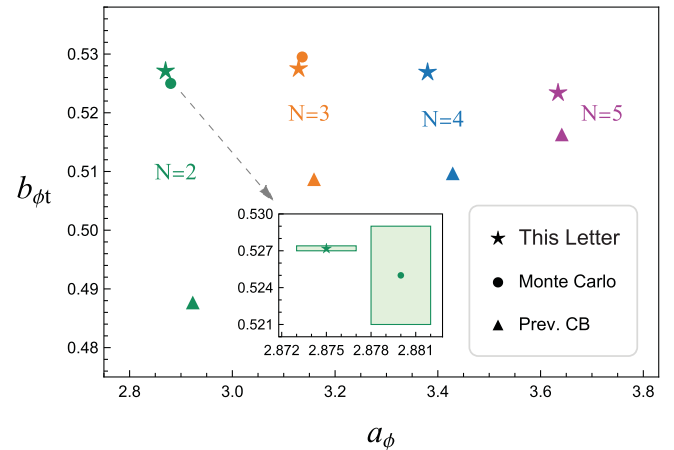


FIG. 2. The $N = 2, 3, 4, 5$ estimates for the universal amplitudes $(a_\phi, b_{\phi t})$. Our results (stars) show excellent agreement with the Monte Carlo estimates (dots) for $N = 2, 3$ [7] and resolve the discrepancies seen in the previous conformal bootstrap (CB) study [6] (triangles). The inset shows the $N = 2$ error bars from this Letter and [7].

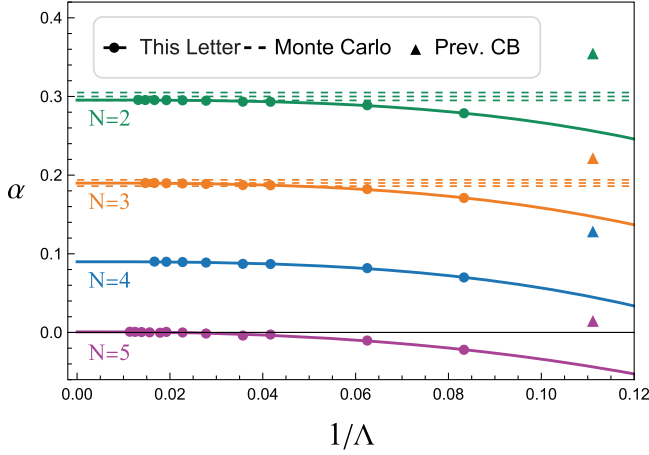


FIG. 3. The universal RG parameter α in Eq. (2) at various truncation orders Λ (dots). For comparison, we also plot the Monte Carlo results with errors [7] (dashed lines) and the previous conformal bootstrap results [6] (triangles, $\Lambda = 9$).

we present our results for α at various Λ , which allow for simple power-law fits. The infinite Λ extrapolations are well consistent with the Monte Carlo results [7]. On the other hand, the singlet projection leads to unstable, non-convergent results. (See Fig. 1 of Supplementary Material [53].) They do not follow the trend of the Fig. 3 curves because the presence of two boundary sectors leads to a less sparse spectrum, which is less suitable for the truncated bootstrap.

For $N = 5$, the sign of α is important for determining the critical value N_c . Previously, the boundary bootstrap study [6] obtained a result for $\alpha|_{N=5}$ around 0.017. Without assigning error, it is not clear if $\alpha|_{N=5}$ is really positive, i.e., if N_c is above 5. Our estimate for $\alpha|_{N=5}$ is 1 order of magnitude smaller, but still marginally positive. Based on the estimates $\alpha|_{N=2,3,4,5}$, we use power-law fits to extract the critical value

$$N_c^{\text{this Letter}} = 5.016(28), \quad (14)$$

which is surprisingly close to the integer value 5. For comparison, a fit of the previous results in [6] gives $N_c \approx 5.2$.

The use of two crossing equations also allows us to determine the bulk operator dimensions of the subleading $O(N)$ traceless-symmetric tensors. Using linear fits, we find

$$\Delta_{T'} = 3.6484(24), 3.559(4), 3.49(3), 3.354(17), \quad (15)$$

for $N = 2, 3, 4, 5$. Our $N = 2$ estimate is in nice agreement with the bulk bootstrap result 3.650(2) in [30]. The results for $N = 3, 4, 5$ are new. We also obtain rough estimates for the subleading boundary dimensions, $(\hat{\Delta}_{\hat{S}}, \hat{\Delta}_{\hat{T}}) \sim 5$. Using the bulk OPE coefficients $(\lambda_{\phi\phi_S}, \lambda_{\phi\phi_T})$ in [30,31], we further deduce the following one-point coefficients for the

first time:

$$N = 2: \quad a_S = 5.571(12), \quad a_T = 3.897(6), \quad (16)$$

$$N = 3: \quad a_S = 5.369(12), \quad a_T = 8.406(12). \quad (17)$$

The Ising BCFT—Using the η minimization, we can systematically increase the truncation orders, so the accuracy is mainly limited by the bulk input. To demonstrate this more clearly, we leverage the unprecedentedly precise determinations of the bulk Ising data in [33]:

$$\Delta_{\sigma}^{\text{input}} = 0.518148806(24), \quad \Delta_{\epsilon}^{\text{input}} = 1.41262528(29), \quad (18)$$

$$\lambda_{\sigma\sigma\epsilon}^{\text{input}} = 1.05185373(11). \quad (19)$$

In the normal transition, the $\mathbb{Z}_2$ symmetry of the Ising universality class is broken. The boundary fusion rules read

$$\sigma, \epsilon \sim 1 + D + \hat{N} + \hat{N}' + \dots \quad (20)$$

We consider two crossing equations in the Ising BCFT. The first one concerns the mixed spin-energy correlator $\langle \sigma(x)\epsilon(y) \rangle$, which is nonvanishing due to symmetry breaking. The bulk fusion rule is

$$\sigma \times \epsilon \sim \sigma + \sigma' + \sigma'' + \dots, \quad (21)$$

so there are only $\mathbb{Z}_2$ -odd operators. The dimension of the leading irrelevant operator is about $\Delta_{\sigma'} \approx 5.3$. As the bulk spectrum has a large gap [58], we expect to obtain highly accurate results. The second crossing equation is about the spin-spin correlator $\langle \sigma(x)\sigma(y) \rangle$, which is related to the bulk fusion rule

$$\sigma \times \sigma \sim 1 + \epsilon + \epsilon' + \epsilon'' + \dots \quad (22)$$

The scaling dimension of ϵ' is also an input parameter. We mainly use the rigorous result $\Delta_{\epsilon'}^{\text{input}} = 3.82951(61)$ in [32]. The uncertainty in $\Delta_{\epsilon'}$ is the main source of error in the Ising boundary bootstrap. For this reason, we solve the two crossing equations separately and choose a larger maximum truncation order Λ_{max} for the $\langle \sigma\epsilon \rangle$ crossing equation. (See Table I.)

Again, we observe nice convergence patterns as Λ grows. We also use the power-law fits to deduce the $\Lambda \rightarrow \infty$ extrapolations. In Fig. 4, we compare our estimates for the one-point coefficients of the bulk relevant operators with the literature results. Our accurate results,

$$a_{\sigma} = 2.6148(3), \quad a_{\epsilon} = 6.677424(16), \quad (23)$$

are in excellent agreement with the Monte Carlo results [40] and resolve the previous discrepancies due to low

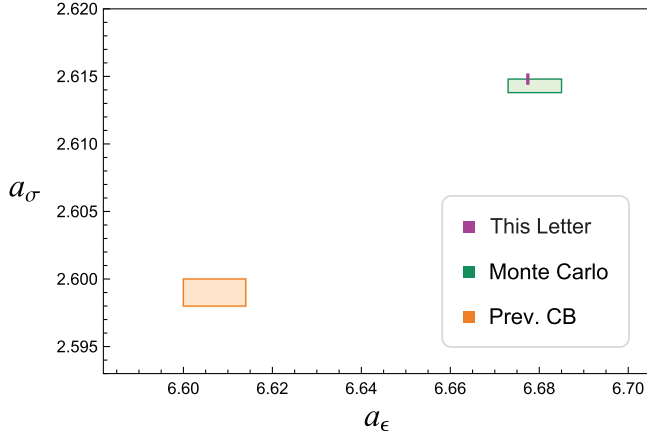


FIG. 4. The 3D Ising one-point coefficients (a_σ, a_ϵ) from this Letter (purple), Monte Carlo simulations [40] (green), and the previous conformal bootstrap [38] (orange). The uncertainty in our estimate for a_ϵ is smaller than the symbol size.

truncation orders in [38]. Using the bulk OPE coefficients from [29,32], we obtain the new one-point coefficients

$$a_{\epsilon'} = 42.46(14), \quad a_{\sigma'} = 110(4), \quad a_{\epsilon''} = 267.8(15). \quad (24)$$

We estimate some boundary dimensions and BOE coefficients:

$$\hat{\Delta}_{\hat{N}} = 5.8792(12), \quad \hat{\Delta}_{\hat{N}'} = 8.086(24), \quad (25)$$

$$b_{\epsilon\hat{N}} = 0.2147(23), \quad b_{\epsilon\hat{N}'} = 0.0464(42), \quad (26)$$

$$b_{\sigma\hat{N}} = 0.00946(11), \quad b_{\sigma\hat{N}'} = 0.00130(12). \quad (27)$$

The operator dimensions are derived from $\langle\sigma\epsilon\rangle$ due to smaller input uncertainties. Our estimate for $\hat{\Delta}_{\hat{N}}$ is consistent with the first two digits of the fuzzy sphere result, 5.858, in [60].

As a test of our error analysis, we extract the bulk OPE coefficient $\lambda_{\sigma\sigma\epsilon}$ from our boundary bootstrap results:

$$\lambda_{\sigma\sigma\epsilon}^{\text{this Letter}} = 1.05184(13). \quad (28)$$

The impressive agreement with the bulk bootstrap result, Eq. (19), suggests that our errors are reliable for low-lying operators. Another nontrivial check comes from the Ward identity associated with the displacement operator D . In a given boundary universality class, the Zamolodchikov norm of D ,

$$C_D = \left(\frac{\Delta_{\mathcal{O}a\mathcal{O}}}{4\pi b_{\mathcal{O}D}} \right)^2 = \begin{cases} 0.18966(10) & \mathcal{O} = \sigma \\ 0.18970(10) & \mathcal{O} = \epsilon, \end{cases} \quad (29)$$

should not depend on the choice of the bulk operator $\mathcal{O}$ [7,61]. The difference is consistent with error estimates and

much less than that in [38]. Our results are roughly compatible with the Monte Carlo results, 0.193(5) in [7] and 0.198(3) in [40].

Discussion—In summary, we resolved the previous discrepancies between the conformal bootstrap and Monte Carlo studies of the 3D $O(N)$ normal universality class. By implementing the η minimization that allows for a systematic increase in the truncation order Λ , we demonstrated that the earlier disagreements were artifacts of low truncation orders, not fundamental limitations of the truncated bootstrap approach. The truncated bootstrap results achieved excellent agreement with Monte Carlo benchmarks and are superior in several aspects. Some estimates are 2 orders of magnitude more accurate. Our bulk results also agree well with those from the conventional bootstrap method. Many bulk and boundary data are completely new. We obtain an accurate estimate for the critical value N_c , which is surprisingly close to 5.

It would be interesting to consider larger bootstrap systems, i.e., correlators of higher points, boundary operators, and other bulk operators. A promising future direction is to apply the η minimization method to other nonperturbative defect bootstrap problems [34,62,63]. It is also important to bootstrap nonunitary CFTs, which describe the critical behavior of the $O(N)$ loop model with noninteger or nonpositive N , the Yang-Lee edge singularity, percolation, and disordered systems [35,36,64–70]. When positivity violations are significant, the positive bootstrap methods may not be applicable even for the bulk crossing equation. But we can still use the truncation methods, such as the η minimization. It should also be helpful to revisit the bulk bootstrap of the standard Ising model, so our improved Gliozzi method can be tested against a known case. The convergent patterns were observed in other truncated bootstrap studies [47,71], which seem to be a general feature and should be useful to many unexplored nonpositive bootstrap targets.

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Data availability—The data that support the findings of this article are openly available [72].

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End Matter

More details about the $N = 2, 3, 4, 5$ results—In Table II, we list some of our boundary bootstrap results for $N = 2, 3, 4, 5$ and some literature results for comparison. While the accuracy of a_ϕ is comparable to that of the Monte Carlo results for $N = 2, 3$, our bootstrap

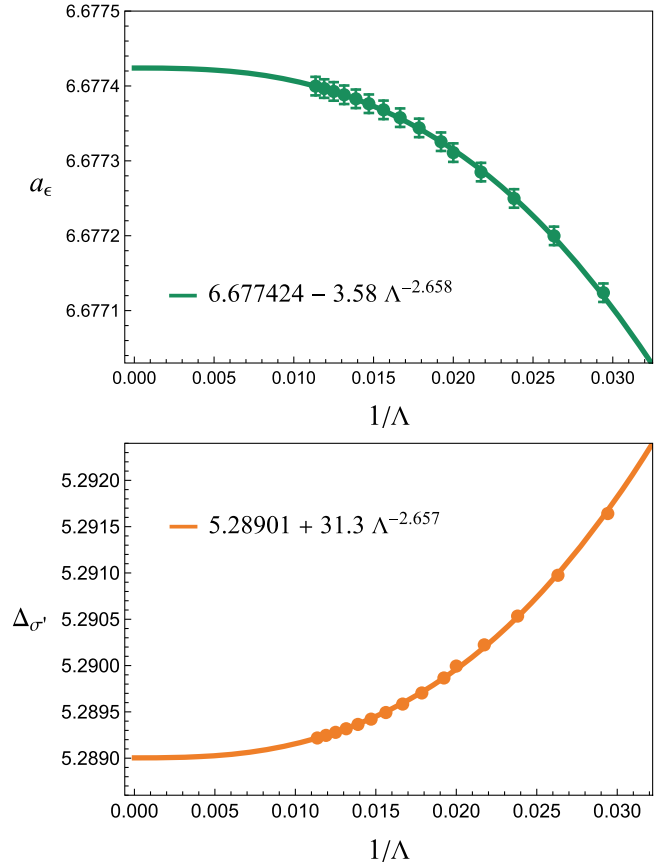
estimates for b_{ϕ_t} and α appear to be more accurate. Our new bootstrap result for $N = 4$ is also more compatible with the unpublished Monte Carlo result [76], $\alpha|_{N=4} = 0.097(3)$, mentioned in [13].

TABLE II. Some $N > 1$ BOE coefficients and the universal RG parameter α from this Letter, Monte Carlo (MC) simulations [7], and the previous conformal bootstrap study [6].

$N = 2$				
Method	a_ϕ	$b_{\phi t}$	$b_{\phi D}$	α
This Letter	2.875(2)	0.5272(2)	0.2440(4)	0.2957(6)
MC [7]	2.880(2)	0.525(4)		0.300(5)
CB [6]	2.923	0.4882	0.2701	0.3567
$N = 3$				
Method	a_ϕ	$b_{\phi t}$	$b_{\phi D}$	α
This Letter	3.129(3)	0.5278(2)	0.2406(6)	0.1903(7)
MC [7]	3.136(2)	0.529(3)		0.190(4)
CB [6]	3.159	0.5092	0.2690	0.2236
$N = 4$				
Method	a_ϕ	$b_{\phi t}$	$b_{\phi D}$	α
This Letter	3.380(7)	0.5272(13)	0.2369(24)	0.0906(35)
CB [6]	3.429	0.5105	0.2758	0.1304
$N = 5$				
Method	a_ϕ	$b_{\phi t}$	$b_{\phi D}$	α
This Letter	3.634(5)	0.5235(5)	0.2390(9)	0.002(2)
CB [6]	3.641	0.5166	0.2653	0.0166

TABLE III. Some 3D Ising data from this Letter, Monte Carlo simulations [7,40,73,74], fuzzy sphere (FS) regularization [75], and the previous conformal bootstrap studies [29,38].

Method	a_ϵ	a_σ	$b_{\epsilon D}$	$b_{\sigma D}$
This Letter	6.677424(16)	2.6148(3)	1.7234(5)	0.24757(4)
MC [7]		2.60(5)		0.244(8)
MC [40]	6.679(6)	2.6143(5)	1.69(1)	0.242(2)
FS [75]	6.4(9)	2.58(16)	1.74(22)	0.254(17)
CB [38]	6.607(7)	2.599(1)	1.742(6)	0.25064(6)
Method	$\Delta_{\sigma'}$	$\Delta_{\epsilon''}$	$\Delta_{\sigma''}$	
This Letter	5.28901(3)	6.873(7)	8.42915(36)	
CB [29]	5.2906(11)	6.8956(43)		
CB [38]	5.49(1)	7.27(5)	10.6(3)	


 FIG. 5. Power-law fits of some 3D Ising results from the $\langle \sigma \epsilon \rangle$ crossing equation. The input-induced errors are indicated by error bars, which are smaller than the plot symbols for $\Delta_{\sigma'}$.

More details about the Ising case—In Table III, we list some of our highly accurate boundary bootstrap results for the Ising BCFT and some literature results for comparison. In Fig. 5, we further present the power-law fits of our most accurate results, a_ϵ and $\Delta_{\sigma'}$, from the $\langle \sigma \epsilon \rangle$ crossing equation. Our prediction for a_ϵ appears to be 2 orders of magnitude more accurate than the latest Monte Carlo result in [40]. Our estimate for $\Delta_{\sigma'}$ is also well consistent with the previous bulk bootstrap result in [29] and the more rigorous result, 5.262(89), in [32].