

Self-Similar Phase Diagram of the Fibonacci-Driven Quantum Ising Model

Harald Schmid¹, Yang Peng², Gil Refael³, and Felix von Oppen¹

¹*Dahlem Center for Complex Quantum Systems and Fachbereich Physik, Freie Universität Berlin, 14195 Berlin, Germany*

²*Department of Physics and Astronomy, California State University, Northridge, California 91330, USA*

³*Institute of Quantum Information and Matter and Department of Physics, California Institute of Technology, Pasadena, California 91125, USA*



(Received 2 November 2024; revised 24 March 2025; accepted 9 June 2025; published 20 June 2025)

We study a stroboscopic quantum Ising model with Fibonacci dynamics. We use its boundary spin correlation functions in long but finite chains to identify regions in the phase diagram which exhibit Majorana zero modes (MZM) as well as Majorana golden-ratio modes (MGM). We find that these regions evolve in a self-similar manner with increasing simulation time and identify the self-similarity transform which governs this evolution of the phase diagram. Integrability-breaking perturbations lead to a temporal decay of the boundary spin correlations, ultimately limiting the self-similarity of the phase diagram. Our predictions are testable with current quantum information processors.

DOI: [10.1103/hn66-j8pt](https://doi.org/10.1103/hn66-j8pt)

Introduction—Quasiperiodicity has been a remarkably rich source of intriguing physics. In quantum systems, it is known to lead to self-similar spectra as exemplified by the Hofstadter butterfly for Bloch electrons in strong magnetic fields [1,2]. Moreover, quasiperiodic potentials are intermediate between periodic and disordered, which makes them model systems in relation to Anderson localization [3–5]. In quantum dynamics, quasiperiodic driving introduces dynamical localization in the energy domain, which protects systems such as the paradigmatic kicked rotor [6–9] against heating. Quasiperiodic driving with multiple incommensurate frequencies realizes dynamical localization transitions between phases with and without protection against heating in the universality class of the 3D Anderson transition [10,11].

More recent research on quantum dynamics with quasiperiodic driving focuses on quantum many body systems. This is motivated by fundamental interest as well as implementations on quantum information processors. In particular, driven spin chains are readily implemented and raise numerous fascinating and challenging questions. These include the fate of many body localization and glassy dynamics in quasiperiodically driven systems [12], the possibility of discrete-time quasicrystals [13], as well as the heating behavior of critical and noncritical systems [14–18]. Recent work has also focused on the quantum

dynamics of topological phases in quasiperiodically driven systems [19–26].

Here, we introduce and study the concept of a scale-dependent dynamical phase diagram of quasiperiodically driven many body systems. While their longtime dynamics has been extensively studied, little attention has been paid to phase diagrams. We address this question for a quantum Ising model stroboscopically driven with a quasiperiodic Fibonacci drive. The quantum Ising model and its phase diagram are already well understood both in the Hamiltonian [27,28] and the Floquet incarnations [29–32], which made them central to Floquet time crystals [33–40] and frequently implemented models on current quantum information processors [40,41].

In our context, boundary modes in finite quantum Ising chains (related to Majorana modes in the fermionized version aka Kitaev chain) provide a convenient diagnostic of the phase. We show that these can be used to uncover and understand the scale-dependent and self-similar phase diagram of the Fibonacci-driven quantum Ising model. Moreover, the quantum Ising model readily admits introducing symmetry- and integrability breaking perturbations. We find that breaking of integrability leads to a decay of boundary correlations, providing an intrinsic cutoff of the self-similarity of the phase diagram.

Fibonacci-driven quantum Ising model—Our driven quantum Ising model obeys the Fibonacci rule [12,24]

$$U(f_{n-1})U(f_n) = U(f_{n+1}), \quad (1)$$

with initial condition $U(f_0) \rightarrow U_0$ and $U(f_1) \rightarrow U_1$. Here, $U(f_n)$ denotes the time evolution operator at time f_n , with f_n being the n th Fibonacci number ($f_0 = f_1 = 1$). The basic unitaries

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

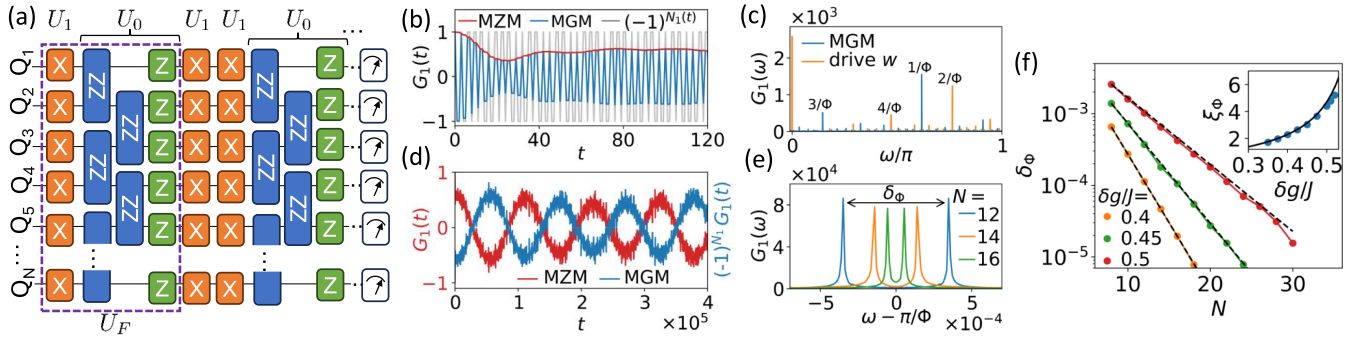


FIG. 1. Quasiperiodically driven quantum Ising model. (a) Quantum circuit for Fibonacci drive. Purple box: Floquet operator U_F of Floquet quantum Ising model. (b) Boundary correlation function $G_1(t) = \langle Z_1(t)Z_1(0) \rangle$ without longitudinal fields in the MZM (red) and MGM (blue) regime (gray: ideal MGM case $g = 1$). (c) Fourier transform $G_1(\omega)$ (MGM) peaks at $2\pi(n + 1/2)/\Phi$ (blue), while driving sequence w peaks at $2\pi n/\Phi$ (orange). (d) $G_1(t)$ for longer simulation times with oscillations due to Majorana hybridization (superimposed on fast quasiperiodic pattern for MGM). (e) Hybridization splitting of dominant MGM peak. (f) Splitting δ_Φ decreases exponentially with system size N . Inset: Majorana localization length ξ_Φ extracted from splittings diverges at $J = \delta g \Phi$. Parameters: $J = 0.1$ (all), $g = 0.04$ (MZM), $g = 0.96$ (MGM), (b,c) $N = 100$, (d) $N = 16$.

$$U_0 = e^{\frac{i\pi J}{2} \sum_{j=1}^{N-1} Z_j Z_{j+1}}, \quad U_1 = e^{\frac{i\pi g}{2} \sum_{j=1}^N X_j}, \quad (2)$$

respectively implement the two-qubit exchange and the single-qubit transverse-field gates on the qubits of the chain (Pauli operators X_j and Z_j at site j). As $U(f_n)$ has f_n factors, we define $U(t) = \dots U_0 U_1 U_1 U_0 U_1$ for general (integer) times $t < f_n$ by retaining the t rightmost factors in $U(f_n)$ [see quantum circuit in Fig. 1(a)].

We extract the phase diagram of the model from numerical results for the infinite-temperature correlation function $G_j(t) = \langle Z_j(t)Z_j(0) \rangle$, which averages over the entire many-body spectrum, $\langle \dots \rangle = 2^{-N} \text{tr}[\dots]$. This is motivated by the Floquet quantum Ising model [time evolution operator $U(t) = [U_F]^t$ with Floquet operator $U_F = U_0 U_1$ as illustrated in Fig. 1(a)]. Its phases [Fig. 2(a)] are characterized by the appearance of Majorana zero modes (MZMs) and Majorana π modes (MPMs) at the

ends of a finite chain. This is reflected in characteristic behavior of the boundary correlation functions $G_1(t)$ and $G_N(t)$ [31,32,41]. After an initial transient, the boundary correlation function plateaus in the MZM case and exhibits period doubling $G_1(t) \propto (-1)^t$ in the MPM case. This is reflected in $\omega = 0$ and $\omega = \pi$ peaks in the Fourier transform $G_1(\omega)$, respectively. Bulk spin correlations $G_j(t)$ ($j \neq 1, N$) decay rapidly.

Figure 2(b) shows a phase diagram of the Fibonacci-driven quantum Ising model obtained in an analogous way. Unlike for classical integrable spin models on quasicrystals [42,43], quasiperiodic driving alters the phase diagram qualitatively from its Floquet counterpart in Fig. 2(a). For transverse fields near $g = 0$, we find that the boundary correlation functions exhibit plateau-like behavior and hence an $\omega = 0$ peak similar to the Floquet quantum Ising model. Figure 2(b) (lower half) maps the height of

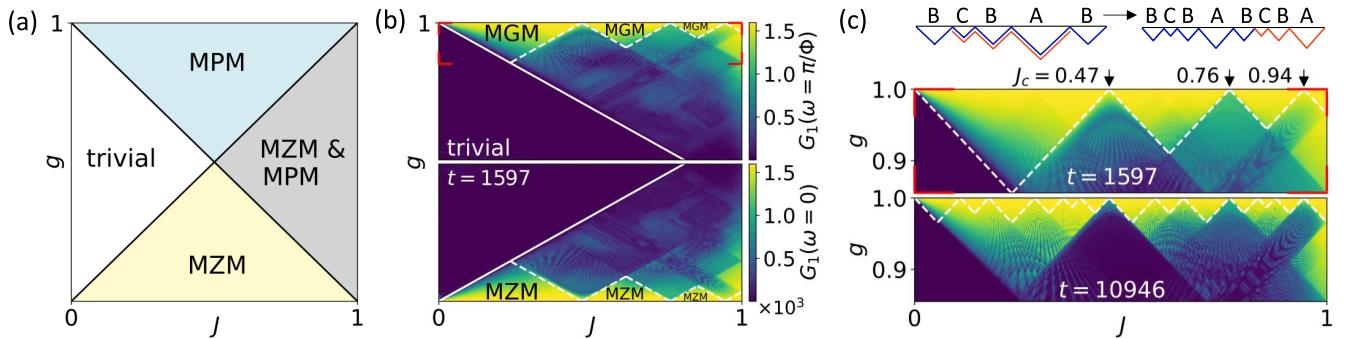


FIG. 2. Phase diagrams of driven quantum Ising models. (a) Floquet model with MZM and MPM. (b) Quasiperiodic Fibonacci drive. Map of peak heights of $G_1(\omega)$ at $\omega = 0$ (lower half) and $\omega = \pi/\Phi$ (upper half), indicating stable/unstable regions of MZM and MGM. White solid lines: dominant phase boundaries $J = g\Phi$ and $J = (1 - g)\Phi$. Dashed cones originating from unstable points J_c show additional phase boundaries for $t = 1597$ time steps. (c) Zoom of stable/unstable regions in the MGM phase for $t = 1597$ and $t = 10946$. Longer simulations reveal a self-similar structure with more instability points. Top: self-similarity transformation describing the evolution of stable/unstable regions with simulation time t . Parameter: $N = 40$.

this peak in the g - J plane, revealing triangular regions exhibiting MZMs. Cones with slopes of about ± 0.62 originate from unstable points at $g = 0$, tracing critical lines. $G_1(\omega = 0)$ drops sharply across these lines.

Near the line $g = 1$, the dominant Fourier peak of the boundary correlation function has frequency $\omega \simeq 0.62\pi$ [Figs. 1(b) and 1(c)]. This contrasts with the Floquet case, where the dominant peak appears at $\omega = \pi$. For reasons that will become clear below, we refer to the underlying boundary mode with frequency $\omega \simeq 0.62\pi$ as the Majorana golden-ratio mode (MGM). We map the MGM peak in Fig. 2(b) (upper half). As emphasized by the zoom in Fig. 2(c), the fine structure of the MGM regions is due to unstable points at $J_c \simeq 0.76, 0.47, \dots$ on the line $g = 1$. The fine structure around $g = 1$ [Figs. 2(b) and 2(c)] mirrors the one near $g = 0$ for MZMs.

Remarkably, we find that the fine structure depends on the simulation time, i.e., the time interval over which the Fourier transform $G_1(\omega)$ is computed. This is illustrated in Fig. 2(c), which shows phase diagrams obtained at two Fibonacci times $f_{16} = 1597$ and $f_{20} = 10946$. At larger times, the phase diagram exhibits more unstable points with their associated phase boundaries. We find by inspecting a sequence of phase diagrams for consecutive Fibonacci times that up to boundary effects, the structure of the stable triangular regions is generated by iterating the initial self-similarity transformation shown in Fig. 2(c) (top). For each Fibonacci time, there are three triangle types (labeled A, B, C) with distinct base lengths (ratios $\simeq 0.62$), but identical slopes of the isosceles. In each step, there are f_n triangles (starting with $f_4 = 5$), which one concatenates with a copy truncated by f_{n-3} (f_{n-4}) triangles on the left (right).

Quasiperiodic driving and boundary correlation functions—We turn to understanding this self-similar phase diagram. An explicit expression for the time-evolution operator of the Fibonacci-driven quantum Ising model is $U(t) = U_{w(t)} \dots U_{w(1)}$ with driving pattern $w(t) = \lfloor (t+1)/\Phi \rfloor - \lfloor t/\Phi \rfloor$, where $\Phi = (1 + \sqrt{5})/2$ is the golden ratio, see Ref. [44]. The number of U_1 factors in $U(t)$ equals $N_1(t) = \sum_{t'=1}^t w(t') = \lfloor (t+1)/\Phi \rfloor$. The Fourier transform of the driving pattern $w(t)$ has peaks at $2\pi n/\Phi$ ($n \in \mathbb{Z}$) [Fig. 1(c)]. We note that due to the stroboscopic time evolution, these frequencies can be folded back into the basic frequency interval $[-\pi, \pi]$. The irrational nature of Φ implies that the model involves quasiperiodic driving, with a dense spectrum of measure zero in $[-\pi, \pi]$.

A Jordan-Wigner transformation to Majorana operators, $a_{2j-1} = (\prod_{l<j} X_l) Z_j$ and $a_{2j} = (\prod_{l<j} X_l) Y_j$, maps the basic unitaries into the free-fermion unitaries

$$U_0 = e^{\frac{\pi i}{2} \sum_{j=1}^{N-1} a_{2j} a_{2j+1}}, \quad U_1 = e^{\frac{\pi i}{2} \sum_{j=1}^N a_{2j-1} a_{2j}}. \quad (3)$$

The transverse field couples Majoranas from the same site, the exchange coupling from different sites. The

free-fermion nature implies that the time evolution of the Majorana operators, $a_j(t) = U^\dagger(t) a_j U(t) = \sum_l W_{jl}(t) a_l$, can be described in terms of the Bogoliubov-de Gennes time evolution operator $W(t)$. This allows for efficient computation of boundary spin correlations as $G_1(t) = \langle a_1(t) a_1(0) \rangle = W_{11}(t)$ [44].

We gain intuition for the boundary correlation function by considering special cases. First consider $J = 0$, where $U_0 = 1$. Then, eigenstates of $U(t) = U_1^{N_1(t)}$ are eigenstates of U_1 , which dimerizes Majoranas of the same site. One readily finds $G_j(t) = \cos(\pi g N_1(t))$ for all j . Its Fourier transform $G_j(\omega)$ has a dense spectrum of peaks at frequencies $(\pm \pi g + 2\pi n)/\Phi$. The peak replicas reflect the incommensurate frequencies $2\pi n/\Phi$ of the quasiperiodic driving pattern $w(t)$. Consistent with the phase diagram, there are no stable boundary modes along the line $J = 0$.

Next, consider $g = 0$, where $U_1 = 1$. Thus, eigenstates of $U(t) = U_0^{t-N_1(t)}$ are eigenstates of U_0 , which dimerizes Majoranas of neighboring sites. One finds $G_j(t) = \cos(\pi J(t - N_1(t)))$ with a dense Fourier spectrum for all bulk sites ($j \neq 1, N$). In contrast, the boundary correlation function becomes $G_1(t) = 1$, since the two end Majoranas decouple and turn into Majorana zero modes (MZMs) [44]. Unlike the finite-frequency peaks of $G_j(\omega)$ for bulk sites, the $\omega = 0$ MZM peak of $G_1(\omega)$ does not have replicas due to the quasiperiodicity.

Finally, for $g = 1$, the transverse-field unitary flips all spins, $U_1 = P = \prod_j X_j$, so that both U_0 and U_1 are nontrivial. Due to the decoupling of the end Majoranas and the identification $a_1 = Z_1$, we find the boundary correlation function $G_1(t) = (-1)^{N_1(t)}$, while the bulk dynamics depends on the value of J [44]. Unlike for $g = 0$, $G_1(\omega)$ exhibits a dense spectrum after backfolding, with peaks at the half-integer harmonics $2\pi(n + 1/2)/\Phi$ of the golden ratio. This motivates the designation as MGM. Importantly, the MGM spectrum differs from the driving pattern $w(t)$ with peaks at $2\pi n/\Phi$ [Fig. 1(c)].

The dominant phase boundaries [full lines in Fig. 2(b)] can be understood by writing $U(t) \simeq e^{-iH_{\text{eff}}^{\text{MZM}} t}$ (g small) and $U(t) \simeq P^{N_1(t)} e^{-iH_{\text{eff}}^{\text{MGM}} t}$ ($g \approx 1$). For g and J small, higher commutators can be neglected, and the effective Hamiltonian becomes the quantum Ising model,

$$H_{\text{eff}}^{\text{MZM}} \simeq -\frac{\pi g}{2\Phi} \sum_j X_j - \frac{\pi J}{2\Phi^2} \sum_j Z_j Z_{j+1}. \quad (4)$$

$H_{\text{eff}}^{\text{MGM}}$ is obtained by replacing $g \rightarrow 1 - g$ in $H_{\text{eff}}^{\text{MZM}}$. The Φ dependence of the prefactors reflects the multiplicities of U_0 and U_1 in $U(t)$. The dominant lines correspond to the self-duality lines $J = g\Phi$ (MZM) and $J = (1 - g)\Phi$ (MGM) of the effective quantum Ising Hamiltonian.

The fine structure of the phase diagram can be understood from the dynamics of the single-particle bulk modes.

At the unstable points, the bulk gap collapses and the edge modes are suppressed. This can be seen from the single-particle time-evolution operator $W(t)$. The bulk dynamics is captured by a chain of N sites with periodic boundary conditions, so that modes can be labeled by momentum k . In the Majorana basis, the elementary single-particle unitaries read [44],

$$W_0(k) = e^{i\frac{k}{2}\sigma^z} e^{i\pi J\sigma^y} e^{-i\frac{k}{2}\sigma^z}, \quad W_1(k) = e^{-i\pi g\sigma^y}. \quad (5)$$

At $g = 0$, we find $W(k, t) = [W_0(k)]^{\lfloor t/\Phi^2 \rfloor} \approx e^{i\pi J(t/\Phi^2)\sigma^y}$. Here, the last step is accurate when t is a Fibonacci time [44]. The unstable points can now be identified as $J_c = 2m\Phi^2$ with $m \in \mathbb{Z}$. At these points, we have $W(k, t) \approx 1$, corresponding to a gapless bulk single-particle spectrum for all k . At small but nonzero g , the gap closes only at $k = 0$, where $W(k = 0, t) \approx e^{i\pi[J(t/\Phi^2) - g(t/\Phi)]\sigma^y}$. In view of the symmetries of the phase diagram under $J \rightarrow J \pm 2$ and $J \rightarrow -J$, this reproduces phase boundaries at

$$J \mp g\Phi = 2m\Phi^2 \pmod{2}, \quad (6)$$

i.e., the dashed lines in Fig. 2(b). This uses that for $1 < J < 2$, the symmetries imply a phase boundary at $2 - J$. For MGM modes near $g = 1$, one replaces $g \rightarrow 1 - g$. This follows by analogous arguments, using that the gap now closes at $k = \pi$ [44]. Equation (6) implies that all phase boundaries have slopes of the same magnitude $1/\Phi$.

All unstable points have irrational J_c , so that they eventually cover the entire lines $g = 0, 1$. To estimate the number of unstable points at a given simulation time t , we note that at Fibonacci times, t/Φ^2 approximates $\lfloor t/\Phi^2 \rfloor$ with a precision which increases $\sim 1/t$. This implies that $W(k = 0, t) \simeq 1$ provided that $\pi J_c/(\sqrt{5}t) \ll 1$, i.e.,

for $t \gg 2\pi|m|\Phi^2/\sqrt{5}$ [44]. Thus, the number of unstable points which are resolved increases linearly with simulation time t , consistent with our numerical results. The $t \rightarrow \infty$ limit at fixed g is discussed further in [44].

Majorana modes—The interpretation in terms of Majorana modes is supported by the long-time dynamics. For long simulation times, the boundary correlation functions display sinusoidal oscillations [Fig. 1(d)]. For MGMs, these oscillations are superimposed on the quasiperiodic pattern $(-1)^{N_1(t)}$. The oscillations translate to splittings $\delta_{0,\Phi}$ of the Fourier peaks in $G_1(\omega)$ [Fig. 1(e)]. The splitting can be interpreted as hybridization of the Majorana modes at the two ends. Indeed, we find an exponential dependence of the splittings $\delta_{0,\Phi} \sim e^{-N/\xi_{0,\Phi}}$ on N . The localization lengths ξ_0 (ξ_Φ) of the MZM (MGM) mode extracted from fits of the splittings diverge at the phase transition lines $J = g\Phi$ (MZM) and $J = (1 - g)\Phi$ (MGM) [Fig. 1(f)] and are consistent with the Majorana localization lengths $\xi_0 \simeq -1/\ln(g\Phi/J)$ ($\xi_\Phi \simeq -1/\ln((1 - g)\Phi/J)$) predicted by the effective Hamiltonian in Eq. (4) [inset Fig. 1(f)]. In keeping with the Majorana interpretation, we also identify a single-particle gap that changes sign across the phase transitions [44].

The presence of MGM can be viewed as a generalization of strong modes [31,36,45–50] to quasiperiodic systems. The instantaneous many-body spectrum of the circuit, $U(t)|n(t)\rangle = e^{-iE_n(t)}|n(t)\rangle$ [Fig. 3(e)], has paired eigenphases $E_n(t) \in [-\pi, \pi]$. The pairs are degenerate for even and π -paired for odd $N_1(t)$, with the pairing being strongest at Fibonacci times. This implies that the corresponding mode operator Ψ_Φ satisfies $\Psi_\Phi U(t) \simeq e^{i\pi N_1(t)} U(t) \Psi_\Phi$.

Random longitudinal field—We investigate the robustness of the self-similar phase diagram against breaking of integrability and spin-flip symmetry P by considering quenched random longitudinal fields. This modifies the

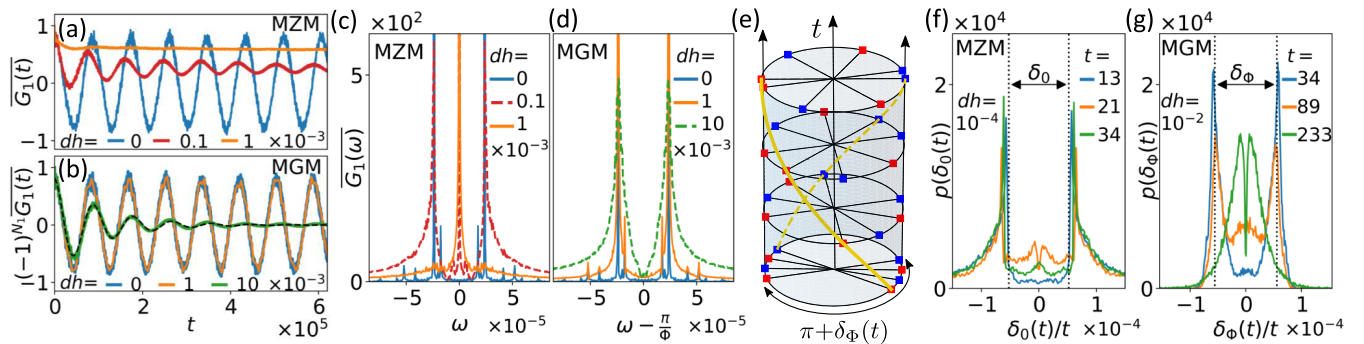


FIG. 3. Boundary correlations with random longitudinal fields [full circuit in Fig. 1(a)]. (a) MZM. Longitudinal fields rapidly suppress hybridization-induced oscillations in disorder-averaged $G_1(t)$ and polarize edge spins. (b) MGM. Oscillations remain robust up to much larger dh . Dashed line: fit to exponential decay. (c), (d) Fourier transforms $G_1(\omega)$. (e) For MGMs, many-body eigenphases of $U(t)$ are π -paired when $e^{i\pi N_1(t)} = -1$, up to a state-dependent splitting $\delta_\Phi(t)$ which grows linearly with t . (f) Splitting distribution of zero pairs, normalized to splitting for $dh = 0$ and rescaled by t . Longitudinal fields ($dh = 10^{-4}$) rapidly broaden the distribution. (g) Distribution of normalized splittings of π pairs. Splittings mostly decrease for nonzero $dh = 10^{-2}$. Parameters: $N = 10$, $\mathcal{N} = 100$ realizations, $g = 0.03$ (MZM), $g = 0.97$ (MGM), $J = 0.1$ (both).

exchange gates $U_1 \rightarrow e^{\frac{i\pi}{2} \sum_j Z_j Z_{j+1}} e^{\frac{i\pi}{2} \sum_j h_j Z_j}$ with spatially disordered $h_j \in [-dh, dh]$. In computing $G_1(t)$, we replace the trace by sampling over random initial product states in the $\{Z_j\}$ -basis.

Breaking of integrability introduces an intrinsic cutoff of the self-similar structure of the phase diagram. Due to the dense spectra, it generically opens scattering channels, which induce a decay of the boundary spin correlations. This effectively provides an upper limit for the simulation time and hence the structural details of the phase diagram. Figure 3 shows simulation results for the disorder-averaged boundary correlation function. In the MZM case, the amplitude of the hybridization-induced oscillations decreases with increasing disorder strength dh and crosses over into a slowly decaying nonoscillatory dependence; see Fig. 3(a). In the MGM case, the envelope oscillations of the boundary spin correlation persist to larger disorder strength, but decay exponentially at long times [Fig. 3(b)]. Corresponding Fourier transforms $G_1(\omega)$ are shown in Figs. 3(c) and 3(d).

Breaking of the spin-flip symmetry P by the random longitudinal field allows for asymmetry between the MZM and MGM regions. A pronounced asymmetry is seen in the dependence of the distribution of zero and π splittings on the strength of the longitudinal field; see Figs. 3(f) and 3(g). This reflects the distinct response of degenerate and π -paired many body states to the longitudinal field [32]. The longitudinal field couples the two (zero or π) paired many-body eigenstates. This induces a splitting for degenerate levels, while the coupling between π -paired levels pushes them closer to π .

Conclusions—We find that quasiperiodically driven quantum many body systems can exhibit dynamical phase diagrams with self-similar scale dependence. While generically requiring protection by integrability, some features such as the Majorana-like golden-ratio modes which we find exhibit remarkable resilience against integrability-breaking disorder. Interestingly, self-similar phase diagrams should be experimentally accessible on current quantum processors [44]. While our results for the Fibonacci-driven quantum Ising model establish the existence of self-similar phase diagrams, it will be interesting to delineate the class of models which exhibit such phase diagrams and to explore whether extended models have more elaborate fractal phase diagrams.

Acknowledgments—We thank Kang Yang for useful discussions. Research at Freie Universität Berlin was supported by Deutsche Forschungsgemeinschaft through CRC 183 as well as by the Einstein Research Unit on Quantum Devices. Work at CSUN and Caltech was supported by NSF PREP Grant No. PHY-2216774. G. R. is grateful for support through AFOSR MURI Grant No. FA9550-22-1-0339, as well as the Simons Foundation and the Institute of Quantum Information

and Matter, an NSF Frontier Center. We thank the HPC service of ZEDAT, Freie Universität Berlin, for computing time [51].

Data availability—The supporting data and codes for this article are available from Zenodo [52].

-
- [1] P. G. Harper, *Proc. Phys. Soc. London Sect. A* **68**, 874 (1955).
 - [2] D. R. Hofstadter, *Phys. Rev. B* **14**, 2239 (1976).
 - [3] S. Aubry and G. André, *Ann. Isr. Phys. Soc.* **3**, 133 (1980).
 - [4] M. Kohmoto, L. P. Kadanoff, and C. Tang, *Phys. Rev. Lett.* **50**, 1870 (1983).
 - [5] S. Ostlund, R. Pandit, D. Rand, H. J. Schellnhuber, and E. D. Siggia, *Phys. Rev. Lett.* **50**, 1873 (1983).
 - [6] D. L. Shepelyansky, *Physica (Amsterdam)* **8D**, 208 (1983).
 - [7] D. R. Grempel, R. E. Prange, and S. Fishman, *Phys. Rev. A* **29**, 1639 (1984).
 - [8] F. L. Moore, J. C. Robinson, C. Bharucha, P. E. Williams, and M. G. Raizen, *Phys. Rev. Lett.* **73**, 2974 (1994).
 - [9] P. M. Blekher, H. R. Jauslin, and J. L. Lebowitz, *J. Stat. Phys.* **68**, 271 (1992).
 - [10] G. Casati, I. Guarneri, and D. L. Shepelyansky, *Phys. Rev. Lett.* **62**, 345 (1989).
 - [11] J. Chabé, G. Lemarié, B. Grémaud, D. Delande, P. Szriftgiser, and J. C. Garreau, *Phys. Rev. Lett.* **101**, 255702 (2008).
 - [12] P. T. Dumitrescu, R. Vasseur, and A. C. Potter, *Phys. Rev. Lett.* **120**, 070602 (2018).
 - [13] H. Zhao, F. Mintert, and J. Knolle, *Phys. Rev. B* **100**, 134302 (2019).
 - [14] B. Lapiere, K. Choo, A. Tiwari, C. Tauber, T. Neupert, and R. Chitra, *Phys. Rev. Res.* **2**, 033461 (2020).
 - [15] X. Wen, R. Fan, A. Vishwanath, and Y. Gu, *Phys. Rev. Res.* **3**, 023044 (2021).
 - [16] S. Nandy, A. Sen, and D. Sen, *Phys. Rev. B* **98**, 245144 (2018).
 - [17] S. Maity, U. Bhattacharya, A. Dutta, and D. Sen, *Phys. Rev. B* **99**, 020306(R) (2019).
 - [18] V. Tiwari, D. S. Bhakuni, and A. Sharma, *Phys. Rev. B* **109**, L161104 (2024).
 - [19] I. Martin, G. Refael, and B. Halperin, *Phys. Rev. X* **7**, 041008 (2017).
 - [20] Y. Peng and G. Refael, *Phys. Rev. B* **97**, 134303 (2018).
 - [21] Y. Peng and G. Refael, *Phys. Rev. B* **98**, 220509(R) (2018).
 - [22] D. V. Else, W. W. Ho, and P. T. Dumitrescu, *Phys. Rev. X* **10**, 021032 (2020).
 - [23] D. M. Long, P. J. D. Crowley, and A. Chandran, *Phys. Rev. Lett.* **126**, 106805 (2021).
 - [24] A. J. Friedman, B. Ware, R. Vasseur, and A. C. Potter, *Phys. Rev. B* **105**, 115117 (2022).
 - [25] D. M. Long and D. V. Else, *arXiv:2408.00825*.
 - [26] Z. Qi, I. Na, G. Refael, and Y. Peng, *Phys. Rev. B* **110**, 014309 (2024).
 - [27] E. Lieb, T. Schultz, and D. Mattis, *Ann. Phys. (N.Y.)* **16**, 407 (1961).
 - [28] P. Pfeuty, *Ann. Phys. (N.Y.)* **57**, 79 (1970).

- [29] L. Jiang, T. Kitagawa, J. Alicea, A. R. Akhmerov, D. Pekker, G. Refael, J. I. Cirac, E. Demler, M. D. Lukin, and P. Zoller, *Phys. Rev. Lett.* **106**, 220402 (2011).
- [30] B. Bauer, T. Pereg-Barnea, T. Karzig, M.-T. Rieder, G. Refael, E. Berg, and Y. Oreg, *Phys. Rev. B* **100**, 041102(R) (2019).
- [31] D. J. Yates, F. H. L. Essler, and A. Mitra, *Phys. Rev. B* **99**, 205419 (2019).
- [32] H. Schmid, A.-G. Penner, K. Yang, L. Glazman, and F. von Oppen, *Phys. Rev. Lett.* **132**, 210401 (2024).
- [33] F. Wilczek, *Phys. Rev. Lett.* **109**, 160401 (2012).
- [34] D. V. Else, B. Bauer, and C. Nayak, *Phys. Rev. Lett.* **117**, 090402 (2016).
- [35] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi, *Phys. Rev. Lett.* **116**, 250401 (2016).
- [36] D. V. Else, P. Fendley, J. Kemp, and C. Nayak, *Phys. Rev. X* **7**, 041062 (2017).
- [37] K. Sacha and J. Zakrzewski, *Rep. Prog. Phys.* **81**, 016401 (2017).
- [38] V. Khemani, R. Moessner, and S. L. Sondhi, *arXiv*: 1910.10745.
- [39] D. V. Else, C. Monroe, C. Nayak, and N. Y. Yao, *Annu. Rev. Condens. Matter Phys.* **11**, 467 (2020).
- [40] X. Mi *et al.*, *Nature (London)* **601**, 531 (2021).
- [41] X. Mi *et al.*, *Science* **378**, 785 (2022).
- [42] V. E. Korepin, *Commun. Math. Phys.* **110**, 157 (1987).
- [43] N. V. Antonov and V. E. Korepin, *Theor. Math. Phys.* **77**, 1282 (1988).
- [44] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/hn66-j8pt> for details on Fibonacci words, single-particle evolution, self-similar phase diagram, and effective Hamiltonians.
- [45] P. Fendley, *J. Stat. Mech.* (2012) P11020.
- [46] A. S. Jermyn, R. S. K. Mong, J. Alicea, and P. Fendley, *Phys. Rev. B* **90**, 165106 (2014).
- [47] P. Fendley, *J. Phys. A* **49**, 30LT01 (2016).
- [48] G. J. Sreejith, A. Lazarides, and R. Moessner, *Phys. Rev. B* **94**, 045127 (2016).
- [49] J. Kemp, N. Y. Yao, C. R. Laumann, and P. Fendley, *J. Stat. Mech.* (2017) 063105.
- [50] H.-C. Yeh, A. Rosch, and A. Mitra, *Phys. Rev. B* **108**, 075112 (2023).
- [51] L. Bennett, B. Melchers, and B. Proppe, *Curta: A General-purpose High-Performance Computer at ZEDAT* (Freie Universität, Berlin, 2020).
- [52] H. Schmid, Y. Peng, G. Refeael, and F. von Oppen, Self-similar phase diagram of the Fibonacci- driven quantum Ising model, <https://doi.org/10.5281/zenodo.15065090>.