

Flavor Nonsinglet Splitting Functions at Four Loops in QCD: Fermionic Contributions

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We have determined the fourth-order n_f contributions to the two splitting functions governing the evolution of all flavor differences of quark distributions of hadrons in perturbative quantum chromodynamics with n_f light flavors. The analytic forms of these functions are presented in both Mellin N space and momentum-fraction x space for a general gauge group. In the small- x limit double logarithms occur, but the small- x rise of both splitting functions is confined to extremely small- x values, $x \lesssim 10^{-6}$. The large- x limit includes the n_f -part of the four-loop quark virtual anomalous dimension. Using this result we obtain also the n_f contributions to the corresponding gluonic quantity and the complete threshold-enhanced logarithms from soft-gluon emission for a large class of inclusive observables, including Higgs boson production in gluon-gluon fusion.

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Splitting functions are universal quantities in quantum chromodynamics (QCD), the gauge theory of the strong interaction. They describe collinear parton dynamics and govern the scale dependence of parton distribution functions (PDFs) of hadrons, playing a crucial role in predicting scattering processes with initial-state hadrons. They are computed in perturbation theory and are currently fully known at three-loop accuracy [1,2], corresponding to the next-to-next-to-leading order. The precision measurements achieved now at the Large Hadron Collider [3], as well as anticipated advancements at the forthcoming Electron Ion Collider [4], challenge the existing theoretical accuracy. This development requires the inclusion of the next quantum loop, specifically four loops (N^3 LO), in QCD evolution equations.

For the nonsinglet quark PDFs $q_{\text{ns}}^{\pm}(x, \mu^2)$ of hadrons, given by flavor differences of quark–antiquark sums (+) and differences (−), the evolution equations read as

$$\frac{d}{d \ln \mu^2} q_{\text{ns}}^{\pm} = P_{\text{ns}}^{\pm} \otimes q_{\text{ns}}^{\pm}, \quad (1)$$

with $P_{\text{ns}}^{\pm}(x, \alpha_s) = \sum_{n=0} a_s^{n+1} P_{\text{ns}}^{(n),\pm}(x)$ denoting the splitting functions in an expansion in the strong coupling

$a_s = \alpha_s(\mu^2)/(4\pi)$. The convolution $\otimes$ is taken in the momentum-fraction variable x , and the anomalous dimensions $\gamma_{\text{ns}}^{\pm}(N)$ are obtained by a Mellin transform,

$$\gamma_{\text{ns}}^{(n),\pm}(N) = - \int_0^1 dx x^{N-1} P_{\text{ns}}^{(n),\pm}(x), \quad (2)$$

where the relative sign is a standard convention. Note that the total valence distribution evolves with $P_{\text{ns}}^v(x) = P_{\text{ns}}^-(x) + P_{\text{ns}}^s(x)$; the latter function is not considered here. For the current state-of-the-art regarding the evolution of the singlet quark and gluon PDFs, see [5–8] and the references therein.

Exact results for $P_{\text{ns}}^{(3),\pm}$ are available in the large- n_c limit of the color $\text{SU}(n_c)$ gauge group [9] and for the (small) leading and subleading contributions in the limit of a large number of flavors n_f [10–12]. In addition, the $n_f C_F^3$ contribution is now known [13], as are all terms in $\gamma_{\text{ns}}^{(3),\pm}$ which are proportional to values of the Riemann ζ function [9,14–16]. Beyond these all- N results, the fixed- N values up to $N = 22$ have been computed for the complete anomalous dimensions $\gamma_{\text{ns}}^{(3),\pm}$ [9,16], together with partial results reaching higher values of N .

In this Letter we provide, for the first time, the complete fermionic contributions, i.e., all terms proportional to n_f . The analytic forms of both $\gamma_{\text{ns}}^{(3),\pm}(N)$ and, by virtue of Eq. (2), of $P_{\text{ns}}^{(3),\pm}(x)$ are presented for a general gauge group. The determination of all n_f terms in $\gamma_{\text{ns}}^{(3),\pm}(N)$ is accomplished by reconstructing the entire functional form from fixed- N results using techniques developed by Lenstra, Lenstra and Lovász (LLL) [17,18] for systems of Diophantine equations. This all- N analytic reconstruction

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can be restricted to those parts of the anomalous dimensions which are invariant under conformal symmetry. This significantly reduces the necessary number of fixed- N values.

Our new results have wide-ranging implications due to the universality of soft and collinear parton dynamics. The n_f terms in $P_{\text{ns}}^{(3),\pm}$ enable the analytic determination of the quark virtual anomalous dimension at four loops, i.e., the terms proportional to $\delta(1-x)$ in $P_{\text{ns}}^{(3),\pm}$, except for one coefficient known numerically to high accuracy. Exploiting the known correspondence between collinear gluon radiation off quarks and gluons, we also find the gluon virtual anomalous dimension at this order. In addition, due to the universal structure of subleading infrared poles [19,20], the complete soft corrections at four loops for a large class of inclusive observables are now known analytically up to the same one numerical coefficient. These include the coefficient functions for single-inclusive electron-positron annihilation (SIA) [21,22], inclusive deep-inelastic scattering (DIS) [23], the Drell-Yan (DY) process and Higgs production in gluon-gluon fusion (ggF) [24,25].

Fixed- N values of γ_{ns} in Eq. (2)—even- N for γ_{ns}^+ and odd- N for γ_{ns}^- —can be obtained from matrix elements of leading-twist spin- N local nonsinglet quark operators,

$$O_{\{\mu_1, \dots, \mu_N\}}^{\text{ns}} = \bar{\psi} \lambda^\alpha \gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}} \psi, \quad (3)$$

where $\psi(\bar{\psi})$ are (anti)quark fields, D_{μ_i} are covariant derivatives and λ^α are the flavor $SU(n_f)$ generators with $\alpha = 3, 8, \dots, (n_f^2 - 1)$. The symmetric and traceless parts of operators O^{ns} , denoted by $\{\dots\}$ in Eq. (3), are inserted into quark two-point functions, and the workflow for the computation of the operator matrix elements (OMEs), depicted in Fig. 1, follows [9,16]. The Feynman diagrams for the calculation of the OMEs are generated using Qgraf [26] and then processed by the computer algebra system Form [27–29], which classifies them according to their color factors [30] and topologies. The resulting propagator-type integrals for fixed N are computed with the Forcer package [31]. In this manner, the complete $\gamma_{\text{ns}}^{(3),\pm}(N)$ are obtained at $N \leq 22$ and their n_f parts to at least $N = 24$ [9,16]. The computations are limited by the run-time (up to several weeks of wall-time for individual diagrams) and the size of intermediate expressions [up to

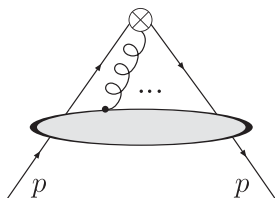


FIG. 1. Two-point functions with an insertion of the operator $O_{\{\mu_1, \mu_2, \dots, \mu_n\}}^{\text{ns}}$ (represented by $\otimes$), computed up to four loops.

$\mathcal{O}(20)$ TByte] in the four-loop integral reductions of the Forcer program.

The analytic expressions for $\gamma_{\text{ns}}^{(3),\pm}(N)$ can be written in terms of powers of N and $N+1$ and harmonic sums [32],

$$S_{\pm m_1, m_2, \dots, m_d}(N) = \sum_{j=1}^N (\pm 1)^j j^{-m_1} S_{m_2, \dots, m_d}(j), \quad (4)$$

of depth d and weight $w = \sum_{i=1}^d |m_i| \leq 7$, where $m_1 > 0$. Sums with index -1 do not occur, leaving $1, 3, 7, 17, 41, 99, \dots$ sums at $w = 1, 2, 3, 4, 5, 6, \dots$, i.e., their number is given by the Pell-Lucas numbers [33], which can be evaluated by $[(1 - \sqrt{2})^w + (1 + \sqrt{2})^w]/2$. The resulting large size of the space of functions can be dramatically reduced by exploiting theoretical constraints on the nonsinglet anomalous dimensions. In particular, γ_{ns} is expressible through a universal evolution kernel γ_u as [34–36]

$$\gamma_{\text{ns}}(N) = \gamma_u[N - \gamma_{\text{ns}}(N) - \beta(a_s)/a_s], \quad (5)$$

which is reciprocity-respecting (RR), i.e., invariant under the replacement $N \rightarrow -1 - N$. Here $\beta(a_s)$ is the standard β function in QCD, known to five loops [37–40]. The perturbative expansion of Eq. (5) separates the RR part from non-RR terms, so we have schematically

$$\gamma_{\text{ns}}^{(n)} = \gamma_{\text{u}}^{(n)} + \text{terms fixed by lower orders.} \quad (6)$$

A suitable space of RR functions for $\gamma_u^{(n)}$ is provided by powers of $\eta = 1/[N(N+1)]$ —with η^n contributing n to the overall weight—and the binomial sums [32]

$$\mathbb{S}_{m_1, \dots, m_d}(N) = \sum_{j=1}^N (-1)^{N+j} \binom{N}{j} \binom{N+j}{j} s_{m_1, \dots, m_d}(j). \quad (7)$$

Here all m_i are positive; the number of sums of weight w is 2^{w-1} . At four loops, $2^{w+1} - 1$ functions with up to an overall weight $w \leq 7$ can contribute to the most complicated (non- ζ) parts of γ_u , but considerable fewer enter its n_f contributions. This total size of the space of basis functions is much smaller than that for $\gamma_{\text{ns}}^{(3)}$, which is given by $a(w+2)$ of sequence [41], with, e.g., 127 and 255 instead of 407 and 984 functions up to $w = 6$ and $w = 7$.

A considerable number of coefficients of the basis functions (which are integer in a suitable normalization) is further fixed by information on the behavior of γ_{ns} for $N \rightarrow 0$ and $N \rightarrow \infty$. In x space, these correspond to the expansions of the splitting functions at small x in terms of double logarithms, $P_{\text{ns}}^{(n),\pm}(x) \propto \ln^{2n-k} x$, where

$0 \leq k \leq 2n - 1$ [42–44], and at large x , where $P_{\text{ns}}^{(n),\pm}(x) = A_q^{(n+1)}/(1-x)_+ + B_q^{(n+1)}\delta(1-x) + \dots$ in terms of the cusp anomalous dimension $A_q^{(\ell)}$, known to $\ell = 4$ [45,46], and the virtual anomalous dimension B_q .

The color structure of the n_f part of $\gamma_{\text{ns}}^{(3),\pm}$ is given by the quadratic (quartic) Casimir invariants $C_F(d_{FF}^{(4)})$ of the fundamental representation and the quadratic adjoint one C_A . We confirm the $n_f C_F^3$ part computed in [13]. The contributions with $n_f C_A C_F^2$, $n_f C_A^2 C_F$, and $n_f d_{FF}^{(4)}$ are new.

Only two of the latter are independent, as the large- n_c limit of $\gamma_{\text{ns}}^{(3),\pm}$ is completely known [9]. We choose to reconstruct the all- N form for $n_f C_A C_F^2$ from the moments $N \leq 32$ and that for $n_f d_{FF}^{(4)}$ from $N \leq 24$. Following [47] and subsequent works [9,11,15,48–50], we have determined the coefficients of a suitable ansatz by solving a system of Diophantine equations using the LLL algorithm [17] as implemented in the `fp111` package [18].

The $n_f d_{FF}^{(4)}$ contribution to $\gamma_{\text{ns}}^{(3),\pm}$ is RR by itself since this color factor does not enter any lower-order terms:

$$\begin{aligned} \gamma_{\text{ns}}^{(3)\sigma}(N) \Big|_{n_f d_{FF}^{(4)}} &= \frac{32}{3} [4\mathbb{S}_{1,2,1,2} - 4\mathbb{S}_{1,3,1,1} - 4\mathbb{S}_{2,1,1,2} + 4\mathbb{S}_{2,2,1,1} + 12(\mathbb{S}_{1,1,2,1} - \mathbb{S}_{1,2,1,1})\eta(1-\eta) - 6\mathbb{S}_{2,2,1} \\ &\quad - 4\mathbb{S}_{2,1,2}\eta + 2\mathbb{S}_{3,1,1}(3+2\eta) + 3\mathbb{S}_{3,1}(1-4\eta) - 12\mathbb{S}_{1,2,1}\eta(1-\eta^2) + 2\mathbb{S}_{2,1,1}(5+6\eta-6\eta^3) \\ &\quad - \mathbb{S}_{2,2}(13-12\eta) + 2\mathbb{S}_{1,2}(6-2\eta+\eta^2+2\eta^3) - \mathbb{S}_{2,1}(5-39\eta) - \mathbb{S}_2(59-28\eta+5\eta^2+6\eta^3) \\ &\quad + \mathbb{S}_{1,1}\eta(24-39\eta) + \mathbb{S}_1\eta(43-85\eta) + 18+47\eta-62\eta^2 + \sigma(2\mathbb{S}_3\eta(3+\eta+2\eta^2) \\ &\quad + 3\mathbb{S}_2\eta(1-5\eta-2\eta^2) - \eta(59-62\eta))] + \zeta_3, \zeta_5\text{-terms}, \end{aligned} \quad (8)$$

where the arguments N of the $\mathbb{S}$ sums have been omitted. Remarkably, the few terms of overall weight $w = 7$, such as $\eta^3 \mathbb{S}_{2,1,1}$, in Eq. (8) are exactly those that already occurred in the all- N expression in the large- n_c limit [9]. The complete ansatz for the LLL-reconstruction of the $n_f d_{FF}^{(4)}$ part had 148 coefficients, of which as many as 106 turned out to be zero and 25 differed by powers of 2. As expected, the highest-weight functions, with overall weight $w \geq 5$, have particularly simple coefficients. Together, these observations provide a powerful check of the correctness of the all- N result in Eq. (8).

The complete expressions for the n_f parts of $\gamma_{\text{ns}}^{(3),\pm}(N)$ can be found in Supplemental Material [51], together with the corresponding splitting functions $P_{\text{ns}}^{(3),\pm}(x)$ expressed in terms of harmonic polylogarithms [55,56]. The small- x limit of $P_{\text{ns}}^{(3),+}(x)$ in terms of $\ln^n x$, $n = 1, \dots, 5$ provides more checks of our new exact expressions for the $n_f d_{FF}^{(4)}$, $n_f C_A C_F^2$ and $n_f C_A^2 C_F$ contributions since none of the corresponding coefficients derived in [43,44] were used as constraints in the determinations of the result [Eq. (8)] and its $n_f C_A C_F^2$ counterpart.

The n_f parts of $P_{\text{ns}}^{(3),+}$ and $P_{\text{ns}}^{(3),-}$ are shown, in a normalization very close to an expansion in α_s , in Figs. 2(a) and 2(b), together with their common large- n_c (Lnc) limit and the error bands of [9], respectively, based on the first eight even- N and odd- N moments [Eq. (2)]. The latter provide very good approximations of $P_{\text{ns}}^{(3),\pm}$ at $x \gtrsim 10^{-2}$, but they cannot sufficiently constrain these functions at much smaller values of x . The Lnc limit is very close to both QCD results for $x \gtrsim 0.3$ and is a good guide to $x \lesssim 10^{-3}$.

Figure 2(c) shows the cumulative effect of the leading small- x log (LL), $\ln^5 x$, the next-to-leading log (NLL), etc., for $P_{\text{ns}}^{(3),+}$; the pattern for $P_{\text{ns}}^{(3),-}$ is similar. The LL, NLL, ... small- x expansions by themselves are thus of no phenomenological use at any small- x values relevant to measurements, a pattern that has been observed before, e.g., in Refs. [1,44]. As shown in Fig. 2(d), the n_f^1 parts of $P_{\text{ns}}^{(3),\sigma}$ exhibit a small- x rise only below $x \approx 10^{-6}$ for $\sigma = 1$ and below $x \approx 10^{-7}$ for $\sigma = -1$. Furthermore, the comparison of the results for an SU(2) gauge group with those for QCD ($n_c = 3$) and the Lnc limit of the SU(n_c) illustrates both the size and quality of convergence of the $1/n_c$ corrections as n_c grows. Clearly, only exact results in full QCD, like the ones presented here, can reliably provide $P_{\text{ns}}^{(n),\pm}(x)$ down to values of x relevant to DIS off ultrahigh energy neutrinos; see, e.g., [57,58].

The quark virtual anomalous dimension B_q can be read off from the $\delta(1-x)$ terms of $P_{\text{ns}}^{\pm}(x)$. We find for QCD

$$\begin{aligned} B_q^{(4)} &= 23393.79 \pm 0.02 - 5550.0446n_f \\ &\quad + 193.85504n_f^2 + 3.0149820n_f^3, \end{aligned} \quad (9)$$

where exact results have been rounded to eight significant digits. The complete n_f part of $B_q^{(4)}$ at four loops is a new result; the full color results are given in Supplemental Material. The corresponding four-loop gluon virtual anomalous dimension $B_g^{(4)}$ reads as [see also [59]]

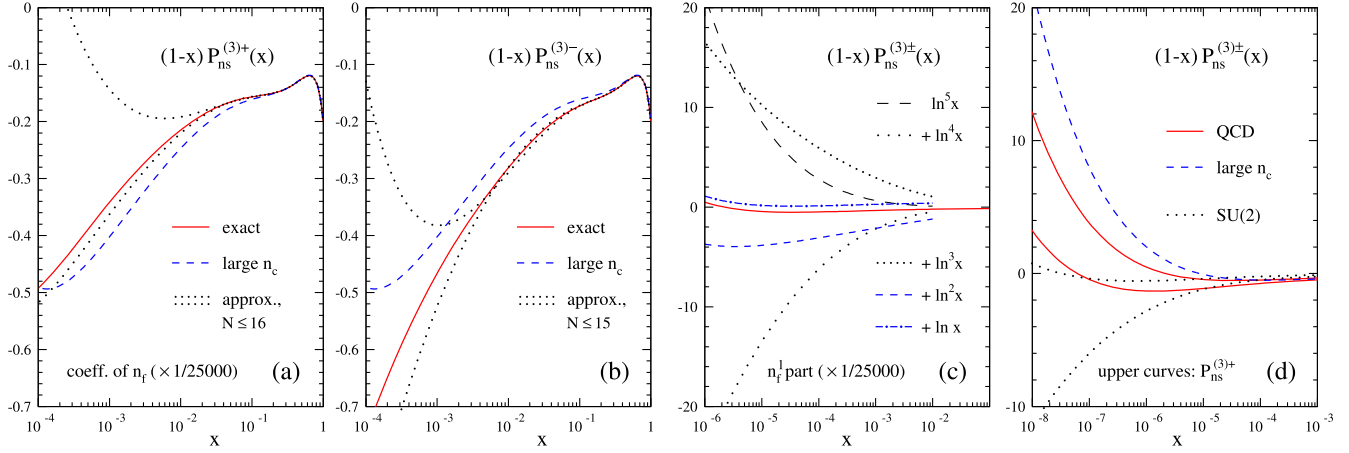


FIG. 2. Exact n_f terms of (a) $P_{\text{ns}}^{(3)+}(x)$ and (b) $P_{\text{ns}}^{(3)-}(x)$ (always in solid red) compared with their large- n_c limits (dashed blue) and the previous error bands (dotted black) presented in [9]. The exact results down to very small values of x , compared (c) to the successive “approximations” by the leading, next-next-to-leading, ... logarithms, and (d) to the large- n_c limit (dashed blue) and the results for the gauge group SU(2) instead of SU(3) (dotted black).

$$B_g^{(4)} = 68587.64 \pm 0.3 - 18143.982n_f + 423.81135n_f^2 + 0.90672154n_f^3. \quad (10)$$

This result has been derived with the help of the four-loop quark and gluon form factors in QCD [60,61]: their infrared poles are related to the respective virtual anomalous dimensions, B_q and B_g , and eikonal anomalous dimensions, f_q and f_g , corresponding to color-charged Wilson lines [20]. The latter exhibit the same maximal non-Abelian color structure as the cusp anomalous dimensions; thus f_q and f_g are related by (generalized [62]) Casimir scaling. The non- n_f parts of $B_q^{(4)}$ and $B_g^{(4)}$ are therefore also related and share one color coefficient which is not known analytically so far. In view of the known large- n_c limit, there is some freedom in the parametrization of this remaining unknown. We choose the coefficient of the quartic Casimir invariant $d_{FA}^{(4)}$ in $B_q^{(4)}$, for which a precise numerical result is available [16,59], $b_{4,FA}^q = -998.02 \pm 0.02$. This value has been used in Eqs. (9) and (10) and the equations below.

The corresponding numerical QCD results for the four-loop eikonal anomalous dimensions read as

$$f_q^{(4)} = -22284.01 \pm 0.04 + 3242.924n_f + 26.33248n_f^2 + 0.4858066n_f^3, \quad (11)$$

$$f_g^{(4)} = -52384.87 \pm 0.6 + 7210.446n_f + 59.24807n_f^2 + 1.093065n_f^3. \quad (12)$$

The full results are given in Supplemental Material. Other factorizations of soft and collinear parton dynamics, such as Soft-Collinear Effective Theory (SCET), introduce partonic jet functions; see, e.g., [22,63,64]. The anomalous

dimensions $\gamma_{J,I}$ of quark and gluon jet functions in SCET are given by $\gamma_{J,I} = -B_I - f_I$ for $I = q, g$ and can be read off from Eqs. (9)–(12), where the numerical uncertainties are anticorrelated in the differences.

Inclusive cross sections for processes like SIA, DIS, DY, or ggF are evaluated via partonic coefficient functions, calculable in perturbative QCD. They exhibit threshold enhanced double logarithms proportional to $\mathcal{D}_k = [\ln^k(1-x)/(1-x)]_+$ in terms of a momentum space variable x related to Feynman (or Bjorken) x in the case of SIA (DIS), or the ratio $x = M^2/S$ of the boson’s invariant mass M and the center-of-mass energy $\sqrt{S}$ for DY and ggF. At large x , the dominant channels read as

$$c_P(x) = \sum_{n=0} a_s^n c_P^{(n)}(x) \simeq \delta(1-x) + \sum_{n=1} \sum_{k=0}^{2n-1} a_s^n c_P^{(n,k)} \mathcal{D}_k \quad (13)$$

for the projections $P = T, q$ for SIA (transverse fragmentation function), $P = 2, q$ for DIS (F_2 structure function), as well as for quark-antiquark annihilation ($P = q\bar{q}$) in DY and ggF Higgs-boson production ($P = gg$).

The all-order resummation of the threshold logarithms in Eq. (13) is currently established to the fourth logarithmic order (N⁴LL), cf. for resummations in SIA [21,22], DIS [23], DY, and ggF [24,25]. The above virtual and eikonal anomalous dimensions provide the input for the determination of the coefficients of $\mathcal{D}_0$ in Eq. (13) at four loops. Eqs. (9)–(12) lead to new results:

$$c_{T,q}^{\text{SIA},(4,0)} = 140708.73 \pm 0.02 - 30298.49n_f + 1131.851n_f^2 - 12.08488n_f^3 + 47.55183n_f l_{11}, \quad (14)$$

$$c_{2,q}^{\text{DIS},(4,0)} = 38840.45 \pm 0.02 - 34964.90n_f + 2062.715n_f^2 - 12.08488n_f^3 + 47.55183n_f fl_{11}, \quad (15)$$

$$c_{q\bar{q}}^{\text{DY},(4,0)} = -1279330.4 \pm 0.1 - 73197.45n_f + 10025.71n_f^2 - 135.1441n_f^3, \quad (16)$$

$$c_{gg}^{\text{ggF},(4,0)} = -14987411.0 \pm 1.1 - 595317.8n_f + 33907.65n_f^2 - 304.0743n_f^3, \quad (17)$$

where, in terms of the fractional quark charge e_q , $fl_{11} = 3\langle e \rangle = (3/n_f) \sum_q e_q$ for flavor nonsinglet and $fl_{11} = (1/n_f) \langle e \rangle^2 / \langle e^2 \rangle = (1/n_f) (\sum_q e_q)^2 / (\sum_q e_q^2)$ for singlet coefficient functions [65]. For charged-current DIS one has $fl_{11} = 0$. The complete results with full color information are given in Supplemental Material [51].

The new $\mathcal{D}_0$ terms in Eqs. (14)–(17) also readily allow for an analytic determination, up to the same one color coefficient $b_{4,FA}^q$, of the process-dependent resummation coefficients governing soft-gluon exponentiation at $N^4\text{LL}$ accuracy for SIA, DIS, DY, or ggF, cf. [22–24,64] for the relevant formulae. In particular, for Higgs boson production in ggF, this leads to an improved perturbative stability of the QCD predictions, characterized by a small residual scale dependence of less than $\pm 3\%$ at approximate $N^4\text{LO}$ accuracy for variations of the renormalization scale μ_R in the range $\mu_R \in [m_H/4, 2m_H]$ relative to the Higgs boson mass m_H , and an apparent convergence of the perturbative expansion at the point of minimal sensitivity $\mu_R \sim m_H/2$ [24]. This underscores the importance of the above results, which are largely attributable to the universal nature of soft and collinear parton dynamics.

Exact full-QCD results for the n_f^0 parts of the four-loop nonsinglet splitting functions will require considerable further progress, either on their computation via differential equations [12,13] or via the determination of more fixed- N moments for the LLL-based reconstructions, supplemented by improved insights into their analytical structure. The complete determination of the flavor-singlet cases is even more challenging. Already the determination of their $x^{-1} \ln x$ contributions would suffice, though, in combination with the moments of [5–8], to reduce their presently remaining numerical uncertainties at small x to a phenomenologically irrelevant level.

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- [1] S. Moch, J. A. M. Vermaseren, and A. Vogt, *Nucl. Phys.* **B688**, 101 (2004).
 - [2] A. Vogt, S. Moch, and J. A. M. Vermaseren, *Nucl. Phys.* **B691**, 129 (2004).
 - [3] *CERN Yellow Report*, edited by A. Dainese *et al.* (CERN, Geneva, Switzerland, 2019), Report No. CERN-2019-007, Vol. 7, 10.23731/CYRM-2019-007.
 - [4] R. Abdul Khalek *et al.*, *Nucl. Phys.* **A1026**, 122447 (2022).
 - [5] G. Falcioni, F. Herzog, S. Moch, and A. Vogt, *Phys. Lett. B* **842**, 137944 (2023).
 - [6] G. Falcioni, F. Herzog, S. Moch, and A. Vogt, *Phys. Lett. B* **846**, 138215 (2023).
 - [7] G. Falcioni, F. Herzog, S. Moch, A. Pelloni, and A. Vogt, *Phys. Lett. B* **856**, 138906 (2024).
 - [8] G. Falcioni, F. Herzog, S. Moch, A. Pelloni, and A. Vogt, *Phys. Lett. B* **860**, 139194 (2025).
 - [9] S. Moch, B. Ruijl, T. Ueda, J. A. M. Vermaseren, and A. Vogt, *J. High Energy Phys.* **10** (2017) 041.
 - [10] J. A. Gracey, *Phys. Lett. B* **322**, 141 (1994).
 - [11] J. Davies, A. Vogt, B. Ruijl, T. Ueda, and J. Vermaseren, *Nucl. Phys.* **B915**, 335 (2017).
 - [12] A. Basdev-Sharma, A. Pelloni, F. Herzog, and A. Vogt, *J. High Energy Phys.* **03** (2023) 183.
 - [13] T. Gehrmann, A. von Manteuffel, V. Sotnikov, and T. Yang, *Phys. Lett. B* **849**, 138427 (2024).
 - [14] J. Davies and A. Vogt, *Phys. Lett. B* **776**, 189 (2018).
 - [15] B. A. Kniehl and V. N. Velizhanin, *Phys. Rev. Lett.* **134**, 131901 (2025).
 - [16] S. Moch, B. Ruijl, T. Ueda, J. A. M. Vermaseren, and A. Vogt, preprint Report No. DESY-25-047, LTH 1398 (2025).
 - [17] A. K. Lenstra, H. W. Lenstra, and L. Lovász, *Math. Ann.* **261**, 515 (1982).
 - [18] The `fp1ll` development team, <https://github.com/fp1ll/fp1ll>.
 - [19] V. Ravindran, *Nucl. Phys.* **B752**, 173 (2006).
 - [20] L. J. Dixon, L. Magnea, and G. F. Sterman, *J. High Energy Phys.* **08** (2008) 022.
 - [21] S. Moch and A. Vogt, *Phys. Lett. B* **680**, 239 (2009).
 - [22] Z. Xu and H. X. Zhu, *J. High Energy Phys.* **06** (2025) 176.
 - [23] G. Das, S. Moch, and A. Vogt, *J. High Energy Phys.* **03** (2020) 116.
 - [24] G. Das, S. Moch, and A. Vogt, *Phys. Lett. B* **807**, 135546 (2020).
 - [25] T. Ahmed, A. H. Ajjath, P. Mukherjee, V. Ravindran, and A. Sankar, *Eur. Phys. J. C* **81**, 943 (2021).
 - [26] P. Nogueira, *J. Comput. Phys.* **105**, 279 (1993).
 - [27] J. A. M. Vermaseren, [arXiv:math-ph/0010025](https://arxiv.org/abs/math-ph/0010025).
 - [28] J. Kuipers, T. Ueda, J. A. M. Vermaseren, and J. Vollinga, *Comput. Phys. Commun.* **184**, 1453 (2013).
 - [29] B. Ruijl, T. Ueda, and J. Vermaseren, [arXiv:1707.06453](https://arxiv.org/abs/1707.06453).
 - [30] T. van Ritbergen, A. N. Schellekens, and J. Vermaseren, *Int. J. Mod. Phys. A* **14**, 41 (1999).

- [31] B. Ruijl, T. Ueda, and J. A. M. Vermaseren, *Comput. Phys. Commun.* **253**, 107198 (2020).
- [32] J. A. M. Vermaseren, *Int. J. Mod. Phys. A* **14**, 2037 (1999).
- [33] The On-Line Encyclopedia of Integer Sequences, <https://oeis.org/A001333> entry A001333.
- [34] Y. L. Dokshitzer, G. Marchesini, and G. P. Salam, *Phys. Lett. B* **634**, 504 (2006).
- [35] B. Basso and G. P. Korchemsky, *Nucl. Phys. B* **775**, 1 (2007).
- [36] Y. L. Dokshitzer and G. Marchesini, *Phys. Lett. B* **646**, 189 (2007).
- [37] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, *Phys. Rev. Lett.* **118**, 082002 (2017).
- [38] F. Herzog, B. Ruijl, T. Ueda, J. A. M. Vermaseren, and A. Vogt, *J. High Energy Phys.* **02** (2017) 090.
- [39] T. Luthe, A. Maier, P. Marquard, and Y. Schröder, *J. High Energy Phys.* **10** (2017) 166.
- [40] K. G. Chetyrkin, G. Falcioni, F. Herzog, and J. Vermaseren, *J. High Energy Phys.* **10** (2017) 179.
- [41] The On-Line Encyclopedia of Integer Sequences, <https://oeis.org/A005409> entry A005409.
- [42] A. Vogt *et al.*, *Proc. Sci.*, LL2012 (2013) 004 [arXiv:1212.2932].
- [43] V. N. Velizhanin, *Mod. Phys. Lett. A* **32**, 1750213 (2017).
- [44] J. Davies, C. H. Kom, S. Moch, and A. Vogt, *J. High Energy Phys.* **08** (2022) 135.
- [45] J. M. Henn, G. P. Korchemsky, and B. Mistlberger, *J. High Energy Phys.* **04** (2020) 018.
- [46] A. von Manteuffel, E. Panzer, and R. M. Schabinger, *Phys. Rev. Lett.* **124**, 162001 (2020).
- [47] V. N. Velizhanin, *J. High Energy Phys.* **11** (2010) 129.
- [48] V. N. Velizhanin, *Nucl. Phys.* **B864**, 113 (2012).
- [49] S. Moch, J. A. M. Vermaseren, and A. Vogt, *Nucl. Phys.* **B889**, 351 (2014).
- [50] G. Falcioni, F. Herzog, S. Moch, J. Vermaseren, and A. Vogt, *Phys. Lett. B* **848**, 138351 (2024).
- [51] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/hkg5-88hr> for full analytic expressions of $\gamma_{\text{ns}}^{(3),\pm}$, $P_{\text{ns}}^{(3),\pm}$, $B_1^{(4)}$, and $f_1^{(4)}$ for $I = q, g$, as well as the coefficient functions $c_{2,q}^{\text{DIS},(4,0)}$, $c_{q\bar{q}}^{\text{DY},(4,0)}$, and $c_{\text{gg}}^{\text{ggF},(4,0)}$. It also includes Refs. [52–54].
- [52] T. Ahmed, M. Mahakhud, N. Rana, and V. Ravindran, *Phys. Rev. Lett.* **113**, 112002 (2014).
- [53] S. Moch and A. Vogt, *Phys. Lett. B* **631**, 48 (2005).
- [54] C. Anastasiou, C. Duhr, F. Dulat, E. Furlan, T. Gehrmann, F. Herzog, and B. Mistlberger, *Phys. Lett. B* **737**, 325 (2014).
- [55] E. Remiddi and J. A. M. Vermaseren, *Int. J. Mod. Phys. A* **15**, 725 (2000).
- [56] T. Gehrmann and E. Remiddi, *Comput. Phys. Commun.* **141**, 296 (2001).
- [57] R. Gauld, *Phys. Rev. D* **100**, 091301(R) (2019).
- [58] M. H. Reno, *Annu. Rev. Nucl. Part. Sci.* **73**, 181 (2023).
- [59] S. Moch, B. Ruijl, T. Ueda, J. Vermaseren, and A. Vogt, *Phys. Lett. B* **849**, 138468 (2024).
- [60] B. Agarwal, A. von Manteuffel, E. Panzer, and R. M. Schabinger, *Phys. Lett. B* **820**, 136503 (2021).
- [61] R. N. Lee, A. von Manteuffel, R. M. Schabinger, A. V. Smirnov, V. A. Smirnov, and M. Steinhauser, *Phys. Rev. Lett.* **128**, 212002 (2022).
- [62] S. Moch, B. Ruijl, T. Ueda, J. Vermaseren, and A. Vogt, *Phys. Lett. B* **782**, 627 (2018).
- [63] P. Banerjee, P. K. Dhani, and V. Ravindran, *Phys. Rev. D* **98**, 094016 (2018).
- [64] C. Duhr, B. Mistlberger, and G. Vita, *J. High Energy Phys.* **09** (2022) 155.
- [65] J. A. M. Vermaseren, A. Vogt, and S. Moch, *Nucl. Phys.* **B724**, 3 (2005).