

Emergent Photons and Confinement: A Numerical Study on $\mathbb{Z}_N$ Lattice Gauge Theory

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We numerically study $\mathbb{Z}_N$ lattice gauge theories in 4D as prototypical models of systems with $\mathbb{Z}_N$ 1-form symmetry. For $N \geq 3$, we provide evidence that such systems exhibit not only the expected phases with spontaneously broken/restored symmetry but also a third photon phase. When present, the 1-form symmetry provides a precise notion of confinement, and it is commonly believed that confinement ensues due to the proliferation of extended, stringlike objects known as *center vortices*, which carry a $\mathbb{Z}_N$ flux. However, this picture is challenged by the three-phase scenario investigated here. We show that both the confined and the photon phases are associated with the proliferation of center vortices and that the key difference between them lies in whether or not vortex junctions—the *monopoles*—proliferate.

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Introduction—At center stage of the standard model of particle physics, two theories play a distinguished role: the theory of quantum electrodynamics (QED), describing light and chemistry, and the theory of quantum chromodynamics (QCD), describing the formation of nuclei. While the former is a theory of massless photons that enables us to see far into our universe, the latter is a system with a tiny correlation length of ~ 1 fm and is responsible for most of the atomic mass.

Yet both of these theories and regimes are linked to an exotic symmetry called the 1-form symmetry that has been of particular interest in the past decade, mostly starting with the works [1–4]. On one hand, in QCD, quark confinement is associated with the would-be $\mathbb{Z}_N$ 1-form symmetry—a symmetry of the pure glue sector—while the massless photons of QED are seen as Goldstone bosons of a $U(1)$ 1-form symmetry [2,6]. Therefore, both the massless photon regime and the nuclear confined regime can potentially be understood as phases of matter within the Ginzburg-Landau paradigm for 1-form symmetries (see also Ref. [7]).

In [8], it was argued that theories with $\mathbb{Z}_N$ 1-form symmetry in $3 + 1$ D have a generic photon phase for any N . This phase is seen as an enhancement of the 1-form symmetry, $\mathbb{Z}_N \rightarrow U(1)$, and consequent spontaneous breaking of it, leading to a massless Goldstone boson: the photon.

In this Letter, we will consider $\mathbb{Z}_N$ lattice gauge theory (LGT), a model of classical Ising-like spins on the links of the lattice. These theories have been known to exhibit three

phases when $N \geq 5$ [9,10]. Here, we show numerically that this is also true for $N = 3, 4$.

In parallel, the $3 + 1$ D $\mathbb{Z}_N$ LGT is a toy model of pure $SU(N)$ Yang-Mills theory, having both confined and deconfined phases. This model offers a deceptively simple picture of confinement as originating from the proliferation of stringlike objects analogous to center vortices of $SU(N)$ Yang-Mills theory (see Refs. [11,12] for a review). However, the existence of the photon phase requires this picture to be revised by also considering monopole particles [8,13].

The occurrence of a robust exactly massless Goldstone phase without an exact continuous symmetry may be surprising from the point of view of 0-form symmetry intuition (see SM [14] for a review). This robustness in $3 + 1$ D is what makes the Millennium Problem of whether $SU(N)$ Yang-Mills theory has a mass gap [22] nontrivial. Indeed, based on the $\mathbb{Z}_N$ 1-form global symmetry group, we may be erroneously tempted to conclude that a gapless phase is untenable. That a robust photon phase exists invalidates this lore.

Incidentally, in $2 + 1$ D the lore is true, as the photon phase is generically unstable [8], resulting in the necessity of a mass gap under all generic symmetry-preserving deformations (see, e.g., Refs. [23–26]).

The $\mathbb{Z}_N$ Lattice Gauge Theory (LGT)—The model we consider here is a $\mathbb{Z}_N$ lattice gauge theory, which is to $\mathbb{Z}_N$ 1-form symmetry what the Ising model is to an ordinary $\mathbb{Z}_2$ symmetry. It is a theory with discrete N -state spins $s_\ell = 0, 1, 2, \dots, N-1$ on lattice links ℓ . The spins s_ℓ naturally form a $\mathbb{Z}_N$ phase $U_\ell = e^{(2\pi i s_\ell / N)}$. An action that dictates the dynamics of these spins can be constructed by defining $W_p = \prod_{\ell \in \partial p} U_\ell$ —a plaquette-sized Wilson loop—where ∂p is the set of boundary links of a plaquette p , inheriting the orientation of the plaquette (with the convention $U_\ell = U_{-\ell}^*$). The standard Wilson action and the partition function are then given by

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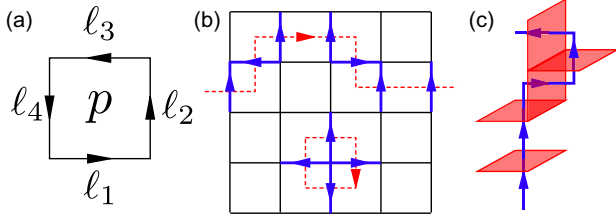


FIG. 1. (a) The definition of the plaquette Wilson loop $W_p = U_{\ell_1} U_{\ell_2} U_{\ell_3} U_{\ell_4}$. (b) The illustration of the $\mathbb{Z}_N$ 1-form symmetry transformation in 2D. (c) The center vortex as an excitation of the plaquette fluxes.

$$S = -\beta \sum_p \text{Re} W_p, \quad Z = \sum_{\{s\}} e^{-S}, \quad (1)$$

where the sum is over all spin configurations $\{s_\ell\}$.

A crucial property of this action in general dimension d is that it is invariant under the $\mathbb{Z}_N$ 1-form symmetry [2]. To understand this symmetry applied to (1), consider an oriented $(d-1)$ -dimensional hypersurface H constructed out of $(d-1)$ cells of the dual lattice [an example for $d=2$ is given in Fig. 1(b)]. This will intersect a set of lattice links, which we collect into a set L [marked blue in Fig. 1(b)] and orient compatibly relative to H . A $\mathbb{Z}_N$ 1-form symmetry is generated by shifting [27] $s_\ell \rightarrow s_\ell + 1 \bmod N$ whenever $\ell \in L$; this is clearly an invariance of the action.

It is easily checked that for H contractible, the resulting invariance of action is the conventional gauge transformations, while noncontractible choices of H correspond to a global 1-form symmetry. Thus, the transformations associated with homotopic choices of H are gauge equivalent, and so the 1-form symmetry is valued in the codimension 1 homology of the space-time manifold with coefficients in $\mathbb{Z}_N$.

The operators charged under the 1-form symmetry are Wilson loops $W[C] = \prod_{\ell \in C} U_\ell$, where C is a contour made out of the links of the lattice. It is easy to see that under the 1-form symmetry transformation for a hypersurface H the Wilson loop transforms as $W[C] \rightarrow W[C] \exp\{i[2\pi I(H, C)/N]\}$, where $I(H, C)$ is the net intersection number between H and C . The fact that in the continuum the charged operator is a Wilson loop, which is an integral over the 1-form gauge field A_μ is what gives the 1-form symmetry its name.

The naive Landau paradigm would then indicate at least two phases: the $\mathbb{Z}_N$ restored (confined) and the spontaneously symmetry broken (SSB) phase, which can be seen respectively in the limits $\beta \rightarrow 0, \infty$. When $0 < \beta \ll 1$, the contractible Wilson loops $W[C]$ get a leading contribution at order β^A , where A is the area of a minimal surface S whose boundary is C . By cluster decomposition, this implies that the noncontractible Wilson loops (i.e., Polyakov loops) must have zero expectation value; i.e., the $\mathbb{Z}_N$ 1-form symmetry is restored. In the limit $\beta \gg 1$, the dominant

configurations obey $W_p = 1$, indicating that noncontractible Wilson loops get a nonzero expectation value.

Further, in this regime, excitations for which $W_p \neq 1$ [see Fig. 1(c)] can be thought of as heavy stringlike excitations in $3+1$ D describing a 2D surface on the dual lattice. These stringlike objects are analogous to center vortices studied in the QCD community, and this is the name we will adopt. One of the most important features of the center vortex is that whenever it links with a Wilson loop, it is weighted by a $\mathbb{Z}_N$ phase, giving a compelling picture of confinement that can be summarized as follows. As one lowers β , the center vortex tension goes down, and increasingly larger vortices become favored. At some value of β , the vortices proliferate, causing dramatic interference with the Wilson loop and eventual area law behavior. This is the essence of the so-called center vortex mechanism of confinement [28,29].

It has been known since 1979 that for $N \geq 5$, the $\mathbb{Z}_N$ LGT in 4D has a robust phase of emergent photons for intermediate β [9,10], making it clear that center vortex proliferation alone cannot possibly explain both the photon and the confined phase. Therefore, a new ingredient is needed here [8]: the worldline representing a junction of N center vortices—the monopole junction.

In fact, a picture of the photon and the confined phase was first briefly touched on in Appendix B of the little-known paper by Stone [13] and recently in [8], which we aim to confirm here. In this framework, both the photon and confined phases involve proliferating center vortices, but differ in the behavior of monopole junctions: these remain heavy in the photon phase, while they proliferate in the confined phase.

We highlight two important observations: (i) the three phases known to exist for $N \geq 5$ are indeed associated with the (non)proliferation of the center vortices and monopoles in the way described above and in [8]; and (ii) the theories for $N = 3, 4$ have a deformation which allows for three phases.

Conspicuously, we will not discuss the theory with $N = 2$ in this Letter because it has no charge-conjugation symmetry and hence no natural choice of monopole junction. The question of whether theories with (only) $\mathbb{Z}_2$ 1-form symmetry have a photon phase is left for future explorations.

Numerical simulations and results—In this section, we numerically study the two different classes of theories: $N \geq 5$ and $2 < N < 5$. We choose $N = 7$ to represent the $N \geq 5$ regime in which three phases are well established [9,10]. The goal is to associate these phases with (non)proliferating center vortices and monopole-junctions. To this end, we begin by formally defining these objects on the lattice.

The center vortex is easy to define by considering a flux through a plaquette p given by $W_p = \exp[i(2\pi f_p/N)]$, where f_p is an integer well-defined modulo N , for which we choose the unique representative in the range:

$$-\frac{N}{2} < f_p \leq \frac{N}{2}. \quad (2)$$

We have $f_p = (ds)_p + Nm_p$, where m_p is some integer defined on plaquettes (a 2-cochain). Further, since $d^2 = 0$ (see, e.g., Refs. [30,31]), we have that $(df)_c = \sum_{p \in \partial c} f_p \equiv 0 \pmod{N}$, with c denoting a cube of the lattice.

In Fig. 1(c), we give an example of a center vortex configuration by shading the plaquettes for which $f_p \equiv 1 \pmod{N}$. If $(df)_c = 0$ were to hold, then the center vortex would form closed string world sheet configurations on the dual lattice. However, because $(df)_c \equiv 0 \pmod{N}$, there can be configurations for which $(df)_c = N(dm)_c$, so $(dm)_c \neq 0$ (a loop on the dual lattice) is our definition of a monopole junction—a worldline where N center-vortex world sheets meet.

Consider now the $\mathbb{Z}_7$ LGT with the usual Wilson action (1). To probe this model, we ran Monte Carlo simulations for 200 values of $\beta \in [0.5, 2.7]$, and performed 2×10^4 measurements each, with 20 decorrelation sweeps between measurements [see Supplemental Material (SM) [14] (see also Refs. [15–21] therein) for more details on the simulations]. Furthermore, we chose $L = 10$ and estimated the error via bootstrap. We present the results in Fig. 2. Note that the errors are too small to be seen.

This theory has three phases, which we can distinguish by the Polyakov loop (i.e., the Wilson loop winding along one of the cycles of the torus). Figure 2 shows the Polyakov loop expectation value, which shows a clear first order

jump at two values of $\beta_1 \approx 1.157 \pm 0.006$ and $\beta_2 \approx 2.108 \pm 0.006$. The confined phase appears for small values where the Polyakov loop expectation value is (nearly) zero, but the middle (the photon) and the large β (the SSB) phases both have a finite Polyakov loop VEV. To distinguish the two, we measure the following. Let $x = (t, x_1, x_2, x_3)$ parametrize the 4D position on a torus (i.e., $t \sim t + L$ and $x_i \sim x_i + L$). Let $\mathcal{L}_x$ be the sequence of all links pointing in the temporal direction at the spatial point $x = (x_1, x_2, x_3)$. Now, define the temporal Polyakov loop to be $P_x = \prod_{\ell \in \mathcal{L}_x} U_\ell$. To each configuration we assign a spatially smeared Polyakov loop $\bar{P} = (1/L^3) \sum_x P_x$, and plot its distribution across the Monte Carlo measurements for the typical representatives of the three phases in Fig. 2, as well as 3D slices of the corresponding configurations.

In the SSB phase $\beta > \beta_2$, the smeared Polyakov loop shows a $\mathbb{Z}_7$ symmetric distribution reflecting a spontaneously broken $\mathbb{Z}_7$ 1-form symmetry. In this phase, the center vortices correspond to short loops. As β is lowered to $\beta_1 \lesssim \beta \lesssim \beta_2$, the system transitions into the photon phase, the center vortices proliferate, and $\bar{P}$ develops a circular distribution, indicating the emergence of the U(1) 1-form symmetry. Notably, the monopoles—depicted by green and yellow cubes in the figure—are uncommon, indicating that monopole worldlines are short and nonproliferating. As β is reduced further to $0 < \beta < \beta_1$, the confined phase sets in, whose hallmark is a distribution of $\bar{P}$ unsurprisingly sharply peaked around zero, and a proliferation of both center vortices as well as monopoles, consistent with the proposal of [8].

This behavior is expected for any $N \geq 5$. However, in [32–34], a $\mathbb{Z}_4$ theory was studied, with the following modified action:

$$S = -\beta \sum_p \text{Re} W_p - \tilde{\beta} \sum_p \text{Re} W_p^2. \quad (3)$$

They argued that for a particular choice of β and $\tilde{\beta}$, a massless photon phase emerges. Indeed in Fig. 3 we present the Monte Carlo time series histogram of the smeared Polyakov loops in Fig. 3. As can be observed, we see the same pattern of circle formation indicating a photon phase

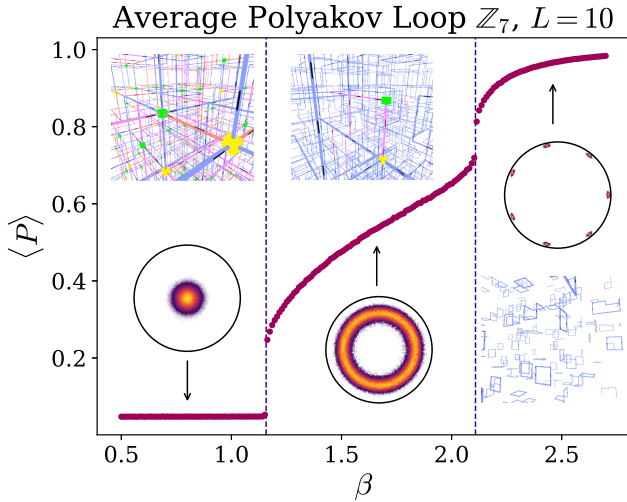


FIG. 2. Phase structure of $\mathbb{Z}_7$ Lattice Gauge Theory. Plots show the average Polyakov loop and the corresponding distributions of the smeared Polyakov loop $\bar{P}$. The histograms of $\bar{P}$ indicate confinement (left), emergent U(1) symmetry (middle), and spontaneous $\mathbb{Z}_7$ 1-form symmetry breaking (right). Insets: typical 3D Monte Carlo configurations showing center vortices (colored by charge $\pm 1, \pm 2, \pm 3$, arrows indicating the sign) and monopoles/antimonopoles (green/yellow boxes).

Smeared Polyakov loops for extended $\mathbb{Z}_4$ gauge theories, $L = 6$

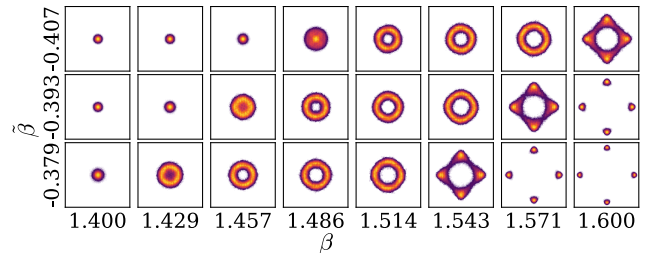


FIG. 3. Histogram of the smeared Polyakov loop as a function of β and $\tilde{\beta}$ [see Eq. (3)] of the $\mathbb{Z}_4$ lattice gauge theory.

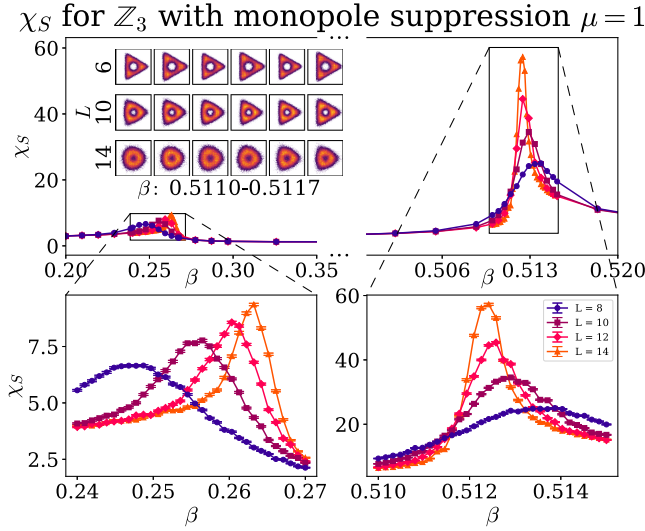


FIG. 4. Susceptibility χ_S versus β for the $\mathbb{Z}_3$ model with monopole suppression ($\mu = 1$) for lattices $L = 8, 10, 12$ and 14 . Two prominent peaks indicate phase transitions. Bottom panels show a close-up of the low- β (confined $\rightarrow$ photon) and high- β (photon $\rightarrow$ SSB) transition regions. Inset: smeared Polyakov loop histograms near the high- β transition for $L = 6, 10$, and 14 .

in between the confined and the SSB phase, confirming the photon phase.

Now we turn to the $\mathbb{Z}_3$ model. In this model with the standard Wilson action (1), there are only two phases. The picture of this model is that as β decreases from $\beta \gg 1$ to $\beta \ll 1$, both center vortices and monopole junctions become abundant simultaneously. To open a photon phase, we therefore add a monopole suppression term to the action:

$$S = -\beta \sum_p \text{Re} W_p + \mu \sum_c (dm)_c^2. \quad (4)$$

To identify phase transitions, we look at the action susceptibility (“heat capacity”) defined as

$$\chi_S = \frac{1}{L^4} (\langle S^2 \rangle - \langle S \rangle^2). \quad (5)$$

In Fig. 4 we plot numerical Monte Carlo results for χ_S as a function of β at $\mu = 1$. We ran Monte Carlo simulations for $L = 8, 10, 12$, and 14 for various values of β (for simulation details see SM [14]). The results for χ_S can be seen in Fig. 4. As is clear, the susceptibility shows two peaks, indicating a three-phase scenario, with phase transitions at $\beta_1 \approx 0.265$ and $\beta_2 \approx 0.512$. The inset shows the smeared Polyakov loop $\tilde{P}$ for values of β close to the second transition, showing the circles forming in the photon phase.

Correlator ratios with $L = 16$

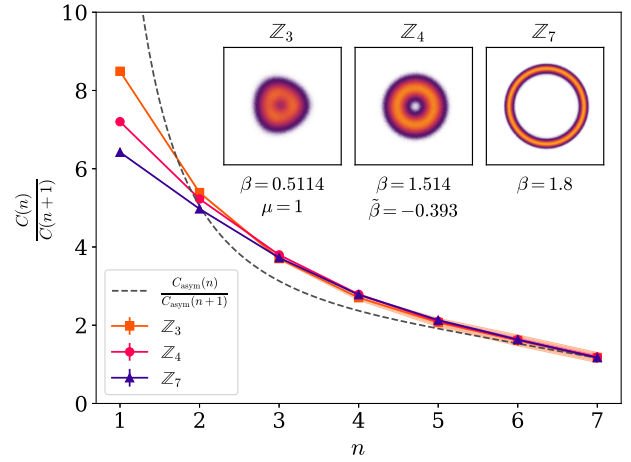


FIG. 5. The correlator ratio $C(n)/C(n+1)$ in the photon phases of $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_7$ on an $L = 16$ lattice compared to the long-distance expectation [see Eq. (8)]. Insets: Corresponding smeared Polyakov loops.

Finally, we also consider the connected plaquette-plaquette correlators (PPC) defined as (no sum over μ, ν)

$$C_{x-y;\mu\nu}^{\pm} = \left\langle W_{p_{x,\mu\nu}}^{\pm 1} W_{p_{y,\mu\nu}} \right\rangle - \left\langle W_{p_{x,\mu\nu}}^{\pm 1} \right\rangle \left\langle W_{p_{y,\mu\nu}} \right\rangle, \quad (6)$$

where $W_p^{-1} = 1/W_p$, and where $p_{x,\mu\nu}$ indicates a plaquette with a corner at lattice site x , expanding in directions $\mu, \nu = 0, 1, 2, 3$, and averages are over ensemble and all spatial translations. At long distances in the photon phase, the theory flows to a pure U(1) gauge theory with the action $S = \frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu}$, which describes free photons. All gauge invariant local operators of the U(1) gauge theory must be built from the field-strength operator $F_{\mu\nu}$, so we expect $C_{x-y;\mu\nu} \propto \langle F_{\mu\nu}(x) F_{\mu\nu}(y) \rangle$ for large $|x - y|$. Further, we have that

$$\langle F_{\mu\nu}(x) F_{\rho\sigma}(y) \rangle \propto \frac{D_{\mu\nu;\rho\sigma}}{|x - y|^4}, \quad (7)$$

where $D_{\mu\nu;\rho\sigma}$ is independent of the modulus $|x - y|$ and is easily computed [35], but is not relevant for our discussion. Let us define a combination [36]

$$C(n) = \sum_{\rho, \mu, \nu} \sum_{\mu > \nu \neq \rho} C_{n\hat{\rho}, \mu\nu}^- + \sum_{\rho, \mu} \sum_{\rho \neq \mu} C_{n\hat{\rho}, \rho\mu}^+ \approx \underbrace{\frac{K}{n^4} + \frac{K}{(L-n)^4}}_{C_{\text{asym}}(n)}, \quad (8)$$

where $\hat{\rho}$ is a lattice unit vector in the direction ρ , and K is a constant, and where we took into account the corrections due to the finiteness of the periodic lattice.

Since in the ratio $C(n)/C(n+1)$ the constant K cancels, all correlators in a Coulomb phase should asymptote to a universal function. In Fig. 5 we show numerical results of this ratio for $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_7$ in the Coulomb phase and find excellent asymptotic agreement, confirming the photon phase of the model. In SM [14], we also show independent correlators $C^\pm(n)$ and find that they perfectly fit the expansion into powers of $1/n^2$.

Conclusions—We numerically showed that $\mathbb{Z}_N$ lattice gauge theories have three phases for all $N \geq 3$, supplementing the spontaneously broken and restored $\mathbb{Z}_N$ 1-form symmetry with a phase of emergent photons where the symmetry enhances to $U(1)$.

In addition, we demonstrated a direct link between the photon phase and the proliferation of stringlike center vortices, while the confined phase was linked to proliferation of both vortices and vortex junctions (monopoles) as proposed in [8]. This suggests that center vortices are not directly responsible for confinement, as commonly believed for $SU(N)$ gauge theory [37].

Our Letter indicates that the Landau paradigm for $\mathbb{Z}_N$ 1-form symmetries must include a massless phase when $N \geq 3$, distinguishing it from its 0-form counterpart.

The case of $\mathbb{Z}_2$ lattice gauge theory is more subtle as it does not have a global charge conjugation symmetry C . [43] While it was proposed in [8] to have a photon phase when deformed, such deformations would introduce the C symmetry. A similar issue occurs in dihedral group (D_N) lattice gauge theories, which have no C symmetry and only at most a $\mathbb{Z}_2$ 1-form symmetry. Yet for large enough N , D_N LGT should be indistinguishable from $O(2)$ LGT, which certainly has a photon phase, albeit with a noninvertible emergent 1-form symmetry [44]. The Landau paradigm for $\mathbb{Z}_2$ symmetry therefore likely involves a massless phase as suggested in [8], although the details are more subtle and are left for future explorations.

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Data availability—The data needed to reproduce the figures in this article will be made available in [46] upon publication.

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 - [37] Defining monopoles and center vortices requires gauge fixing in $SU(N)$ gauge theories (see, e.g., Refs. [11,12,38]), although there are some efforts to identify such objects without it [39]. Further, a novel proposal of how to discretize $SU(N)$ gauge theories [40,41], may be related to a gauge invariant definition of vortices and monopoles [42].
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