

Sum of Entanglement and Subsystem Coherence Is Invariant under Quantum Reference Frame Transformations

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Recent work on quantum reference frames (QRFs) has demonstrated that superposition and entanglement are properties that change under QRF transformations. Given their utility in quantum information processing, it is important to understand how a mere change of perspective can produce or reduce these resources. Here we prove the existence of a QRF invariant which can be decomposed such that it captures the trade-off between entanglement and subsystem coherence: we demonstrate the invariance of the sum of entanglement and subsystem coherence for two pairs of resource quantifiers. Moreover, we find a weaker trade-off that holds for any possible pair of measures. Finally, we discuss the implications of this interplay for violations of Bell's inequalities, clarifying that for any choice of QRF, there is a quantum resource responsible for the violation. These findings contribute to a better understanding of the quantum information theoretic aspects of QRFs, offering a foundation for future exploration in both quantum theory and quantum gravity.

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Introduction—The principle of covariance, which states that physical laws remain unchanged under a reference frame transformation, forms our fundamental understanding of the physical world. However, a variety of measurable physical quantities, such as energy, or magnetic and electric fields, are frame dependent. Similar considerations arise when investigating the behavior of quantum systems relative to a *quantum reference frame* (QRF). For example, a system that is well localized in space in one QRF may be in a superposition of different locations in a different QRF [1]. The same applies to quantum entanglement, making it a frame-dependent notion in quantum theory. Since both coherence and entanglement are recognized as useful resources in quantum information science, the question naturally arises, *how can a mere change of perspective from one frame of reference to another create a useful resource?*

In this Letter we give a natural answer to this question by showing that, under QRF transformations, there exists an invariant which can be decomposed such that it captures a trade-off between the entanglement and subsystem coherence, in the basis associated with the symmetry group of the transformation. We show that, for certain entanglement and coherence quantifiers, their sum is preserved under QRF changes, for any pure bipartite state. Thus, any entanglement increase under a QRF transformation comes at the expense of a decrease in coherence, and vice versa. Furthermore, we show that this property does not hold for all quantifier pairs: fixing an entanglement (or coherence) quantifier, there is at most one coherence (or entanglement) quantifier that yields an invariant sum.

The apparent “creation” of coherence or entanglement under a QRF change can then be explained by there existing either one of these resources in the original frame, and the change of frame transforming one into the other. This is analogous to the case of electrodynamics, where a magnetic field can appear when applying a Lorentz boost, as a consequence of an electric field being present in the rest frame. As Maxwell demonstrated, they are two sides of the same coin, just described in different frames: they are different components of the electromagnetic tensor. Here we show that coherence and entanglement exhibit a similar behavior under QRF transformations. Moreover, just like in

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electromagnetism, where there is a quantity combining both magnetic and electric fields that is left invariant under Lorentz boosts (the free-field Lagrangian density), here we identify meaningful quantities that are left invariant under QRF changes.

To demonstrate the significance of the trade-off between coherence and entanglement in the field of quantum information, we discuss violations of Bell's inequalities and clarify the potential misunderstandings regarding "production" or "destruction" of resources through QRF transformations. Different QRFs induce different factorizations of the Hilbert space, both for the states and for the observables [2–6], such that, for example, entangled states measured by local operations (as in the Bell experiment) transform into product states and nonseparable operations. Thus the resource of entanglement is transferred from the state to the observables.

The study of various notions of reference frames in quantum theory has been pursued through different approaches within the contexts of superselection rules and quantum information protocols [7–10], quantum gravity [11–13], and symmetries [14,15]. During a more recent wave of research, several approaches to QRFs have been formulated [1,3–5,16–19] and applied at the interface of quantum physics and gravity [6,20–28].

One particular question we address in this Letter is relevant to most of these approaches. *If both superposition and entanglement are each frame-dependent notions on their own, is there any physical quantity involving the two that is invariant under quantum frame changes?* A first step to answering this question was taken by Savi and Angelo [29], who found that the so-called "total quantumness" of the system is conserved under QRF changes. However, the conservation of "total quantumness" was shown using only the unitarity of QRF transformations, and not their specific form; the same result thus holds for any possible unitary transformation. In the present Letter, we show that the specific form of (ideal) QRF transformations [1,16] leaves the sum of entanglement and subsystem coherence invariant. Despite different motivations, at the technical level our result is closely related to quantum resource theory findings in Refs. [30,31]. These results can be recovered by restricting to finite cyclic QRF transformations and vanishing initial entanglement, suggesting that the coherence-entanglement link identified in Refs. [30,31] admits an interpretation in terms of invariance under QRF transformations. See also Refs. [32,33] for related studies on trade-offs between single-system and bipartite properties in multiqubits systems.

The Letter is structured as follows. Sec. I introduces the essential features of QRFs with a simple qubit example. Sec. II presents the general definition of QRF transformations, and states our main results: a conservation theorem for two quantifier pairs, along with a weaker trade-off for any pair. Proof ideas are outlined, with details in

Supplemental Material [34]. Sec. III discusses Bell inequality violations in QRFs, and Sec. IV concludes with open questions.

A simple example—Let us start with a simple example to build some intuition for changes of classical and quantum reference frame (cf. [16], Ex. 1)). Consider three systems A , B , and C , which can, for now, be in two states, $\uparrow$ or $\downarrow$. We can conventionally choose that each system considers themselves to be in the state $\uparrow$, similarly to how an astronaut freely floating in space would consider the direction "up" to be aligned from their toes to their head. Take the classical systems to be in the state $\uparrow_A^{(A)} \uparrow_B^{(A)} \downarrow_C^{(A)}$ relative to A . Then, the corresponding state relative to B is $\uparrow_B^{(B)} \uparrow_A^{(B)} \downarrow_C^{(B)}$, while that relative to C is $\uparrow_C^{(C)} \downarrow_A^{(C)} \downarrow_B^{(C)}$. Note how the *relative* configurations between the systems are left invariant under these classical changes of frame.

Let us now extend this to the case of *quantum* reference frames (cf. [16], Ex. 2)). For this, take A , B , and C to be quantum systems with Hilbert space $\mathbb{C}^2$, and basis states $|\uparrow\rangle$ and $|\downarrow\rangle$. Correspondingly, if the state relative to A is $|\psi\rangle_{ABC}^{(A)} = |\uparrow\rangle_A^{(A)} |\uparrow\rangle_B^{(A)} |\downarrow\rangle_C^{(A)}$, then the state relative to B is $|\psi\rangle_{ABC}^{(B)} = |\uparrow\rangle_B^{(B)} |\uparrow\rangle_A^{(B)} |\downarrow\rangle_C^{(B)}$ and that relative to C is $|\psi\rangle_{ABC}^{(C)} = |\uparrow\rangle_C^{(C)} |\downarrow\rangle_A^{(C)} |\downarrow\rangle_B^{(C)}$. Thus, A , B , and C can be understood as qubits described *relative* to one another, meaning that $|\psi\rangle_{ABC}^{(C)}$ encodes the relative degrees of freedom of A and B with respect to C [1].

Take now the state

$$|\psi\rangle_{ABC}^{(C)} = |\uparrow\rangle_C^{(C)} \frac{|\uparrow\rangle_A^{(C)} + |\downarrow\rangle_A^{(C)}}{\sqrt{2}} |\uparrow\rangle_B^{(C)}. \quad (1)$$

We can infer the corresponding state $|\psi\rangle_{ABC}^{(A)}$ by applying the principle of *coherent change of reference frame*, namely in each branch we apply the classical reference frame transformation described above. The result is that the state relative to A is

$$|\psi\rangle_{ABC}^{(A)} = |\uparrow\rangle_A^{(A)} \frac{|\uparrow\rangle_B^{(A)} |\uparrow\rangle_C^{(A)} + |\downarrow\rangle_B^{(A)} |\downarrow\rangle_C^{(A)}}{\sqrt{2}}. \quad (2)$$

The transformation between Eqs. (1) and (2) is a QRF transformation and is illustrated in Fig. 1.

Note, furthermore, that the conventional state of the reference frame with respect to itself can be omitted, as it does not contain any information, being always $|\uparrow\rangle$. We can then represent states of two systems relative to the third in the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$. Furthermore, for future convenience, we identify the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ with the computational basis $\{|0\rangle, |1\rangle\}$, since they are isomorphic. In this simplified and more abstract notation, we can write,

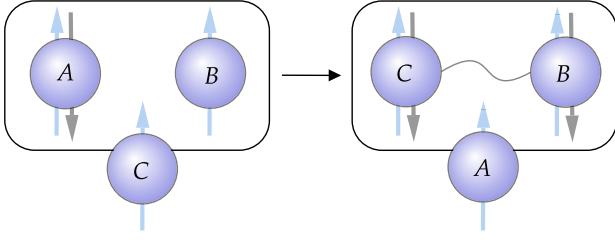


FIG. 1. Three qubits A , B , and C are described relative to one another. On the left, systems A and B are described relative to system C . While C is in the state $|\uparrow\rangle$ relative to itself, A is in a coherent superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$ and B is in the state $|\uparrow\rangle$. Coherently changing reference frame to that of A , we find that B and C are in an entangled state. In general, superposition and entanglement depend on the QRF.

$$|\psi\rangle_{AB}^{(C)} = \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}} |0\rangle_B \quad (3)$$

and correspondingly

$$|\psi\rangle_{BC}^{(A)} = \frac{|0\rangle_B |0\rangle_C + |1\rangle_B |1\rangle_C}{\sqrt{2}}. \quad (4)$$

The transformation between the two states can be implemented with the unitary

$$S^{(C) \rightarrow (A)} = |0\rangle_{CA} \langle 0| \otimes \mathbb{1}_B + |1\rangle_{CA} \langle 1| \otimes \sigma_B^x, \quad (5)$$

meaning that the QRF transformation for these systems amounts to a quantum controlled flip, or CNOT, and can thus be seen as a refactorization of the Hilbert space. Note, however, that while it maps to a specific perspective relative to a single qubit, all measurements and operations are still performed in the laboratory frame.

While the state of A relative to C in Eq. (3) has some coherence (in the computational basis), the total state is factorized, that is, there is no entanglement between A and B relative to C . On the contrary, the state in the perspective of A in Eq. (4) is maximally entangled, and as such each subsystem is maximally mixed, meaning it has no coherence. Thus, the QRF transformation maps a factorized state with subsystem coherence to an entangled state with zero subsystem coherence. This is an instance of a generic feature of QRF transformations, which we prove as our main result: We show that the sum of entanglement and subsystem coherence is invariant under QRF transformations, for specific choices of entanglement and coherence quantifiers. Additionally, if there exists a frame in which either the coherence or the entanglement vanishes, a trade-off holds for any possible pair of measures.

Main results—Let us consider the most general ideal QRF transformation: Given a locally compact group G and a Hilbert space $\mathcal{H} = L^2(G) \otimes L^2(G)$, the most general QRF transformation can be written as [16]

$$S^{(C) \rightarrow (A)} = \int_G dg |g^{-1}\rangle_{CA} \langle g| \otimes U_B^\dagger(g), \quad (6)$$

with $\{|g\rangle\}_{g \in G}$ being the basis of $L^2(G)$, i.e., $\langle g|g'\rangle = \delta(g^{-1} \circ g')$, and $U(g)$ the left regular representation of G , such that $U(g)|g'\rangle = |g \circ g'\rangle$. In the case of discrete groups the integrals can be replaced by sums and the Hilbert space $L^2(G)$ with $\mathbb{C}[G]$. For instance, in the illustrative example above, the group is $G = \mathbb{Z}_2$ and $\mathcal{H} = \mathbb{C}[\mathbb{Z}_2] \cong \mathbb{C}^2$.

Given a generic bipartite pure state in the perspective of C , namely,

$$|\psi\rangle_{AB}^{(C)} = \int_G dg dg' \psi_{AB}^{(C)}(g, g') |g\rangle_A |g'\rangle_B, \quad (7)$$

we can calculate what the state in the perspective of A is, namely,

$$\begin{aligned} |\psi\rangle_{BC}^{(A)} &= S^{(C) \rightarrow (A)} |\psi\rangle_{AB}^{(C)} \\ &= \int_G dg dg' \psi_{BC}^{(A)}(g, g') |g\rangle_A |g'\rangle_B, \end{aligned} \quad (8)$$

with $\psi_{BC}^{(A)}(g, g') = \psi_{AB}^{(C)}(g^{-1}, g^{-1} \circ g')$.

Upon direct application of the definition, QRF transformations can be shown to be incoherent operations (see Supplemental Material A [34]); i.e., they cannot increase the coherence of a bipartite state—and in particular they preserve it. Nevertheless, when studying a subsystem only, QRF transformations can increase or decrease its coherence, as pointed out in the previous example. Hence, it is meaningful to characterize the amount of subsystem coherence before and after the QRF change, as well as the entanglement of the bipartite state, which can change as well.

To begin, we study entanglement entropy and relative entropy of coherence as our resource measures. The entanglement entropy is defined as $\mathcal{E}_e[|\psi\rangle_{AB}] = S[\rho_A]$, where $\rho_A = \text{Tr}_B[|\psi\rangle_{AB} \langle \psi|]$ is the reduced density matrix of the subsystem A and $S[\rho] = -\text{Tr}[\rho \log \rho]$ is the von Neumann entropy. The relative entropy of coherence is $\mathcal{C}_e[\rho] = S[\rho_d] - S[\rho]$, where ρ_d is the state obtained from ρ by keeping only the diagonal terms in the group element basis $\{|g\rangle\}_{g \in G}$. Intuitively, $\mathcal{C}$ quantifies how large the off-diagonal elements of ρ are in the given basis.

Given the entanglement in the perspective of C , i.e., $\mathcal{E}_e^{(C)} = \mathcal{E}_e[|\psi\rangle_{AB}^{(C)}]$, the coherence of subsystem A in the perspective of C , i.e., $\mathcal{C}_e^{(C)} = \mathcal{C}_e[\rho_A^{(C)}]$, where $\rho_A^{(C)} = \text{Tr}_B[\rho_{AB}^{(C)}]$, and analogous definitions for the perspective of A , namely $\mathcal{E}_e^{(A)} = \mathcal{E}_e[|\psi\rangle_{BC}^{(A)}]$ and $\mathcal{C}_e^{(A)} = \mathcal{C}_e[\rho_C^{(A)}]$, the following holds:

$$\mathcal{C}_e^{(C)} + \mathcal{E}_e^{(C)} = \mathcal{C}_e^{(A)} + \mathcal{E}_e^{(A)}. \quad (9)$$

The proof relies on showing that $\rho_A^{(C)}$ has the same diagonal elements as $\rho_C^{(A)}$ up to a permutation (see Supplemental Material B [34]), and hence the entropy of the diagonal states is the same before and after the transformation. Another consequence of the above is an additional conservation law, with a different pair of coherence and entanglement quantifiers: given the l^2 -norm of coherence [46] $\mathcal{C}_l[\rho] = \sum_{g \neq h} |\rho_{g,h}|^2$ and the linear entropy of the subsystem $\mathcal{E}_l[|\psi_{AB}\rangle] = 1 - \text{Tr}[\rho_A^2]$, it holds that

$$\mathcal{C}_l^{(C)} + \mathcal{E}_l^{(C)} = \mathcal{C}_l^{(A)} + \mathcal{E}_l^{(A)}. \quad (10)$$

The proofs of Eqs. (9) and (10) are detailed in Supplemental Material C1 and C2 [34]. We formulate our main result in the form of a theorem.

Theorem 1—The sum of entanglement and subsystem coherence $\mathcal{E} + \mathcal{C}$ is invariant under QRF transformations of pure bipartite states, for $(\mathcal{E}, \mathcal{C}) = (\mathcal{E}_e, \mathcal{C}_e)$ or $(\mathcal{E}_l, \mathcal{C}_l)$.

Our theorem demonstrates the invariance of the sum of entanglement and subsystem coherence under QRF changes, for any pure bipartite state. Moreover, in Supplemental Material C3 [34], we prove that the pairs of quantifiers in Theorem 1 cannot be chosen arbitrarily: for each entanglement (or coherence) quantifier, there is at most one coherence (or entanglement) quantifier such that their sum is conserved. This provides insight into which quantifiers can form a QRF invariant.

Theorem 1 entails a trade-off between the two quantities: a QRF transformation that increases entanglement only does so by consuming the existing coherence, and vice versa. In other words, it follows from Eqs. (9) and (10) that for both choices of quantifiers, when coherence increases under a change of frame, entanglement decreases, and vice versa, i.e., $\Delta \mathcal{C}_e \Delta \mathcal{E}_e \leq 0$ and $\Delta \mathcal{C}_l \Delta \mathcal{E}_l \leq 0$, with $\Delta \mathcal{C}_e = \mathcal{C}_e^{(A)} - \mathcal{C}_e^{(C)}$, and similarly for the others. Note, however, that even this weaker trade-off does not hold for any pair of coherence and entanglement quantifiers; a numerical example where $\Delta \mathcal{C}_l \Delta \mathcal{E}_e \geq 0$ is given in Supplemental Material C4 [34]. This means that there is a pair of measures such that coherence and entanglement can both increase under a QRF change. This is connected to the property of different entanglement and coherence measures not being monotones with respect to each other (see Refs. [52,53]): they do not order states such that, given two different measures $\mathcal{C}_1$ and $\mathcal{C}_2$, for any two pairs of states such that $\Delta \mathcal{C}_1 < 0$, it is also true that $\Delta \mathcal{C}_2 < 0$.

Nevertheless, it is still possible to find a trade-off for any possible pair of measures, for some specific set of states. This can be formulated as follows.

Theorem 2—For any choice of entanglement measure $\mathcal{E}$ and coherence measure $\mathcal{C}$, if there is a QRF where a pure bipartite state has $\mathcal{E} = 0$ or $\mathcal{C} = 0$, under a QRF transformation it holds that $\Delta \mathcal{E} \Delta \mathcal{C} \leq 0$.

The proof is analogous to that of Theorem 1 and is detailed in Supplemental Material C4 [34].

In the more general case of imperfect QRFs, where different frame orientations are assigned nonorthogonal states, both theorems fail, as detailed in Supplemental Material D [34].

Bell tests in QRFs—Nonlocal quantum states violating Bell’s inequalities are a key quantum information resource [54], including applications in quantum cryptography [55–59], randomness generation [60–63], and communication [64–67]. As discussed above, a change of QRF can lead to a decrease in entanglement and hence in the degree of nonlocality of quantum states, at the expense of a corresponding increase in coherence. However, since nonlocality is independent of single-particle coherence, it may appear lost or gained under QRF changes. If this hypothesis is indeed correct, it would suggest that those QRFs in which states manifest nonlocality are “preferred,” which would challenge the view that all QRFs are equally legitimate perspectives.

In the following we will show that the aforementioned hypothesis is incorrect. A set of probabilities that violates Bell’s inequalities in one QRF will violate them in any other QRF, since probabilities are invariant under QRF changes. While the resource for this violation may be entanglement of states in one QRF, it can be the nonseparability of operations in another QRF. In this sense the QRF change can transfer the “nonlocality of states” in one QRF into the “nonlocality of operations” in another QRF.

Bell’s local causality is a condition on joint probabilities

$$p(a, b|x, y) = \sum_{\lambda} p(\lambda) p(a|x, \lambda) p(b|y, \lambda), \quad (11)$$

where $p(\lambda) \geq 0$, $\sum_{\lambda} p(\lambda) = 1$ is a probability distribution of the “hidden variable” and a and b are outcomes in Alice’s and Bob’s laboratories when they choose settings x and y , respectively. The probabilities that satisfy Bell’s local causality satisfy Bell’s inequalities.

In quantum mechanics, Bell’s inequalities can be violated, which demonstrates the incompatibility of Bell’s local causality and quantum mechanics. To demonstrate the violation, Alice and Bob may share a pair of particles in an entangled quantum state and perform local operations on them. We refer to this as the “nonlocality of states”.

We now proceed to an analysis of how the resources required for a violation of Bell’s inequalities change under QRF transformations. We begin our analysis in QRF C. Suppose that in this frame the quantum state $\rho_{AB}^{(C)}$ of the composite system A and B is nonlocal in the sense that there are local POVM elements $A_{a|x}^{(C)}$, $B_{b|y}^{(C)}$ such that

$$\text{Tr} \left[\rho_{AB}^{(C)} \left(A_{a|x}^{(C)} \otimes B_{b|y}^{(C)} \right) \right] \neq \sum_{\lambda} p(\lambda) p(a|x, \lambda) p(b|y, \lambda). \quad (12)$$

The change of QRF results in a refactorization of the Hilbert space from $\mathcal{H}_{AB}$ to $\mathcal{H}_{CB}$, encompassing both the states and the observables. In general, the initial entangled state in QRF C can be mapped to a separable one in QRF A while products of POVMs are in general mapped into nonseparable POVMs,

$$S^{(C) \rightarrow (A)} \rho_{AB}^{(C)} (S^{(C) \rightarrow (A)})^\dagger = \rho_C^{(A)} \otimes \rho_B^{(A)} \quad (13)$$

$$(S^{(C) \rightarrow (A)})^\dagger \left(A_{a|x}^{(C)} \otimes B_{b|y}^{(C)} \right) S^{(C) \rightarrow (A)} = D_{a|x, b|y}^{(A)}, \quad (14)$$

where, in general, $D_{a|x, b|y}^{(A)} \neq \sum_j q_j C_{j, a|x}^{(A)} \otimes B_{j, b|y}^{(A)}$, with $q_j \geq 0$ and $\sum_j q_j = 1$.

We note that as the change of QRF is a unitary transformation, the probabilities for measurement outcomes remain the same before and after the change. Hence, although the state in QRF A is a local one, the probabilities still display nonlocality since

$$\text{Tr} \left[\left(\rho_C^{(A)} \otimes \rho_B^{(A)} \right) D_{a|x, b|y}^{(A)} \right] \neq \sum_\lambda p(\lambda) p(a|x, \lambda) p(b|y, \lambda). \quad (15)$$

This is because the operations in QRF A are in general not in a separable form. In fact, if a state in QRF C violates a Bell's inequality for some local POVMs, then the whole set of transformed POVMs in QRF A cannot be separable, because otherwise one would not violate Bell's inequalities in that frame. We refer to this violation as resulting from “nonlocality of operations”. We conclude that the change from one to another QRF results in a change from “nonlocality of states” to “nonlocality of operations”.

We illustrate our analysis with an example. Consider the test of Clauser-Horne-Shimony-Holt (CHSH) inequality [54,68]:

$$\sum_{x,y=0}^1 p(a \oplus b = xy|x, y) \leq 3, \quad (16)$$

where $p(a \oplus b = xy|x, y)$ is the probability to obtain outcomes $a \oplus b$ to be equal to the product xy , if the choice of the settings is the pair x, y . It is known that if Alice and Bob share an entangled state of qubits A and B , as described by a reference qubit C ,

$$|\psi\rangle_{AB}^{(C)} = \frac{|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B}{\sqrt{2}}, \quad (17)$$

and measure eigenprojectors of a product of spins $\sigma_{\vec{m}} \otimes \sigma_{\vec{n}}$ along directions $\vec{m}$ and $\vec{n}$, they can maximally violate the CHSH inequality by $2 + \sqrt{2}$.

Let us now analyze the CHSH test from the perspective of QRF A . Note that performing a QRF change to the frame

of A , the state of B and C becomes

$$|\psi\rangle_{BC}^{(A)} = |0\rangle_B \frac{|0\rangle_C + |1\rangle_C}{\sqrt{2}}. \quad (18)$$

As we can expect from our results, the entanglement in the perspective of C has been transformed into coherence in the perspective of A , and now Alice and Bob share a separable state. However, the observables also change. As an example, the observable $\sigma_x \otimes [(\sigma_x + \sigma_z)/\sqrt{2}]$, which is factorized in the perspective of C , here becomes nonseparable, i.e., $(S^{(C) \rightarrow (A)})^\dagger \sigma_x \otimes [(\sigma_x + \sigma_z)/\sqrt{2}] S^{(C) \rightarrow (A)} = 1/\sqrt{2}(\sigma_x \otimes \mathbb{1} - \sigma_y \otimes \sigma_y)$, in agreement with the general analysis given above.

Conclusion—Quantum coherence and entanglement are fundamentally intertwined. In this Letter, we explored this connection in the framework of QRFs, focusing on bipartite pure states. We proved the existence of a QRF invariant and showed that it can be decomposed in a way that captures the connection between coherence and entanglement. This was explicitly demonstrated for two pairs of quantifiers, showing that the sum of entanglement and subsystem coherence is conserved under QRF changes, implying that an increase of coherence under QRF change must come at the expense of a decrease in entanglement, and vice versa. Furthermore, we showed that a weaker trade-off holds for any possible pair of measures when there exists a QRF in which either the coherence or the entanglement vanishes.

Finally, we discussed violations of Bell's inequalities in different QRFs and highlighted the role of different quantum resources for the violation in different QRFs. As an example, we show that the entangled states and separable observables that violate the CHSH inequality are mapped to factorized states and nonseparable observables that violate the same inequality.

In Supplementary Material [34], we prove all statements and theorems from the main text and investigate imperfect QRFs.

Future work could extend the present results to multipartite entanglement involving any number of parties. Moreover, generalizing our results beyond the case of locally compact groups to the group of diffeomorphisms holds significant potential for insights into scenarios involving quantum gravity. Our findings suggest that while entanglement and subsystem coherence individually remain invariant under a classical diffeomorphism, their combined sum is preserved under *quantum* diffeomorphisms. This extension would imply that $\mathcal{C} + \mathcal{E}$ is an observable within the framework of general relativity, opening up exciting avenues for exploring the implications of this insight in the realm of quantum gravity.

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