

Nonlinear Faraday Rotation of Light Polarization in Time-Reversal-Invariant Materials

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We investigate the propagation of electromagnetic waves through materials displaying a nonlinear Hall effect. The coupled Maxwell-Boltzmann equations for traveling waves can be mapped onto ordinary differential equations that resemble those for the motion of a pendulum. In the weakly nonlinear regime relevant for most experiments, we find that the polarization of light displays a Faraday-like rotation when it propagates inside a metal above its plasma frequency. In contrast to the conventional Faraday effect, however, the polarization direction swings back and forth around the direction of Berry dipole vector as the light beam traverses the material. This occurs concomitantly with an oscillation of its degree of polarization, with a characteristic frequency that increases linearly with the intensity of the traveling wave. These effects could be observed by measuring thickness-dependent Faraday rotations in materials displaying the nonlinear Hall effect.

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Introduction—The Berry phases of electrons in Bloch bands can have important roles in a variety of nonlinear optoelectronic phenomena [1–6], that could be exploited to develop new functionalities and devices [7–14]. A phenomenon that has attracted interest among these is the nonlinear Hall effect, whereby a Hall-like current proportional to the square of the electric field can appear in metals with time-reversal symmetry controlled by the Berry curvature dipole tensor [15–19].

However, investigations of the nonlinear Hall effect have focused on viewing the electric field that drives it as being a predetermined nondynamical variable. This Letter will precisely fill in this gap by demonstrating that a beautiful and tractable set of nonlinear differential equations describe the traveling electromagnetic waves in quasi-two-dimensional metals with a nonzero Berry curvature dipole. When the electric field is viewed as the coordinate of a particle, the equations resemble those of a two-dimensional pendulum subjected to an effective gravitational force pointing in a direction determined by the Berry dipole vector. As we will demonstrate, when light has a frequency above the plasma scale so that it can propagate inside the metal, the nonlinear Hall effect gives rise to a rocking oscillatory motion of the angle of the polarization axis, $\theta(z)$, of an incident light beam with an angle θ_0 , as it travels a distance z inside the material as follows:

$$\theta(z) \approx \theta_0 \cos(Qkz). \quad (1)$$

The dimensionless parameter Q determines the change in the polarization angle of light per traveled wavelength $\lambda = 2\pi/k$. We estimate this parameter to be $Q = (I/I_0)\omega_p^2/6(ck)^2$, where I is the light intensity, ω_p is the metal's plasma scale, and $I_0 = (1/4\alpha h)(D/d)^2$ is an intensity scale determined by the Drude weight D and the Berry dipole d of the material ($\alpha \approx 1/137$ and h are the fine structure and Planck's constants, respectively). The scale I_0 distinguishes the weakly and strongly nonlinear regimes discussed below, although typical experiments are in the weakly nonlinear regime, $I \ll I_0$.

The above effect contrasts sharply with the standard Faraday effect, where the polarization angle rotates uniformly as the wave propagates, $\theta(z) \propto z$. It occurs in time-reversal-invariant but inversion-breaking metals, while the ordinary Faraday effect requires breaking of time-reversal symmetry. The effect is distinct from other nonlinear versions of the Faraday rotation, such as the Faraday kinetic effect (also known as current-induced optical activity) discussed in Refs. [20–23], where the polarization angle increases at a constant rate as light traverses the sample (as in the usual Faraday effect) proportional to an additionally superimposed dc current [20–23]. Moreover, in contrast to the Faraday kinetic effect, the above effect does not require an electric current to be injected into the material and, therefore, can be observed in contactless optical experiments. Our effect is also distinct from the nonlinear response of gases in which the light-induced ionization of the medium along with an external magnetic field can cause a modulation of the polarization [24].

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Electrodynamics of the nonlinear Hall plasma—We will focus on a 3D model of an infinite stack of 2D metallic layers with normal along $\hat{\mathbf{z}}$, as an approximation to layered quasi-two-dimensional materials where a large nonlinear Hall effect has been observed [25–27]. For electromagnetic waves traveling along $\hat{\mathbf{z}}$, electric currents $\mathbf{j}$ and fields $\mathbf{E}$ have components only along x, y [28] and can be described by the following set of coupled Maxwell and Boltzmann equations ($\hbar = 1$):

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{\mu}{w} \frac{\partial \mathbf{j}}{\partial t}, \quad (2)$$

$$\mathbf{j}(z, t) = e \int \frac{d^2 k}{(2\pi)^2} f(\mathbf{k}, z, t) [\partial_{\mathbf{k}} \epsilon(\mathbf{k}) - e \mathbf{E} \times \boldsymbol{\Omega}], \quad (3)$$

where $\mathbf{E}$ and $\mathbf{j}$ depend on only (z, t) , w is the interlayer distance, and we are taking a model of a simple single-band metal with carries of charge e , dispersion $\epsilon(\mathbf{k})$, and Berry curvature of the form $\boldsymbol{\Omega} = \Omega(\mathbf{k})\hat{\mathbf{z}}$, which is justified when the radiation has a frequency below the optical gap for interband transitions. We will be focusing on the collisionless regime relevant for plasmalike oscillations, which is well justified in typical metallic systems where the plasma scale is larger than the electron scattering rate (see the discussion section). Let us further specialize to a metal with time-reversal symmetry but no inversion symmetry, with a diagonal Drude weight within the 2D metallic planes $D_{lm} = e^2 \int (d^2 k / (2\pi)^2) f_0 \partial_{k_l} \partial_{k_m} \epsilon(\mathbf{k}) = \delta_{lm} D$ and a Berry dipole vector [17] taken along the x axis, $d = e^3 \int (d^2 k / (2\pi)^2) f_0 \partial_{k_x} \Omega$. Combining Eqs. (2) and (3) and using the Boltzmann equation $\partial_t f = -e \mathbf{E} \cdot \partial_{\mathbf{k}} f$, we find (see Appendix A)

$$\partial_z^2 \mathbf{E} - \frac{1}{c^2} \partial_t^2 \mathbf{E} - \frac{\omega_p^2}{c^2} \mathbf{E} = \frac{\omega_p}{c^2} \partial_t (\tilde{M} \hat{\mathbf{z}} \times \mathbf{E}), \quad (4)$$

$$\partial_t \tilde{M} = \omega_p E_x / E_0 \quad (5)$$

with the dimensionless average Berry curvature $\tilde{M} = (e^2 \omega_p / D) \int [d^2 k / (2\pi)^2] f(\mathbf{k}, z, t) \Omega(\mathbf{k})$. We note that, in the collisionless regime of interest in this Letter, the Berry dipole is expected to give rise to a circular photogalvanic effect, namely, a rectified dc current, only in response to light with a nonzero degree of circular polarization [11,17,29].

Traveling waves—We first focus on traveling-wave solutions, where the field depends on only $(v_\phi t - z)$ with v_ϕ the phase velocity. By keeping terms up to second order in electric fields, the above equations reduce to a set of three ordinary differential equations, given by (see Appendix A)

$$\ddot{X} = -X + XY + M\dot{Y}, \quad (6)$$

$$\dot{Y} = -Y - X^2 - M\dot{X}, \quad (7)$$

$$\dot{M} = X, \quad (8)$$

where $\dot{F} = dF/d\xi$ is the derivative with respect to the dimensionless comoving coordinate of the traveling wave, defined as

$$\xi \equiv \frac{\omega_p (v_\phi t - z)}{(v_\phi^2 - c^2)^{1/2}}. \quad (9)$$

Here, $X = E_x(z, t)/E_0$ and $Y = E_y(z, t)/E_0$ are dimensionless electric fields along the x, y directions with $E_0 = D/d$, $M = \tilde{M} v_\phi / (v_\phi^2 - c^2)^{1/2}$, and $\omega_p = (D/w\epsilon)^{1/2}$ is the plasma frequency. The above equations are obtained assuming a phase velocity larger than the speed of light in the material, $v_\phi > c = 1/(\mu\epsilon)^{1/2}$, which is natural in a metal, and, in particular, it is satisfied in the limit of usual plasma waves $\omega = [\omega_p^2 + (ck)^2]^{1/2}$ in the absence of a Berry curvature. It is remarkable that Eqs. (6)–(8) have no dimensionless parameter, and, therefore, their behavior is entirely controlled by the dimensionless initial conditions $\{X(0), Y(0), M(0), \dot{X}(0), \dot{Y}(0)\}$ (for the role of v_ϕ , see Appendix B). For example, when the amplitudes are small, the nonlinear terms can be neglected and Eqs. (6) and (7) describe a harmonic oscillator, which would physically correspond to ordinary plasma waves in a metal. In the opposite, strongly nonlinear regime (initial conditions much larger than one), we will see that, interestingly, Eqs. (6)–(8) can be mapped exactly to those of a pendulum for certain initial conditions.

Traveling waves in the strongly nonlinear regime—In the following, we solve Eqs. (6)–(8) for the case that the initial conditions do not break time-reversal symmetry, $\dot{X}(0) = \dot{Y}(0) = M(0) = 0$, relevant for time-reversal-invariant systems. In order to illustrate more clearly the impact of the Berry curvature dipole on the dynamics, we first set the Drude weight to zero:

$$\ddot{X} = XY + M\dot{Y}, \quad (10)$$

$$\dot{Y} = -X^2 - M\dot{X}, \quad (11)$$

$$\dot{M} = X. \quad (12)$$

Equations (11) and (12) lead to $d(MX)/d\xi = -\dot{Y}$, which yields $MX = -\dot{Y}$ after integration. Similarly, we obtain $MY = \dot{X}$ from Eqs. (10) and (12). If we denote the dimensionless electric field vector by $\mathbf{E} = [X, Y]$, these two relations lead to $\mathbf{E} \cdot \dot{\mathbf{E}} = X\dot{X} + Y\dot{Y} = 0$, implying that the electric field amplitude $E = \sqrt{X^2 + Y^2}$ is time independent. In polar coordinates, $Y = -E \cos \phi$, $X = E \sin \phi$, Eq. (11) can be written as

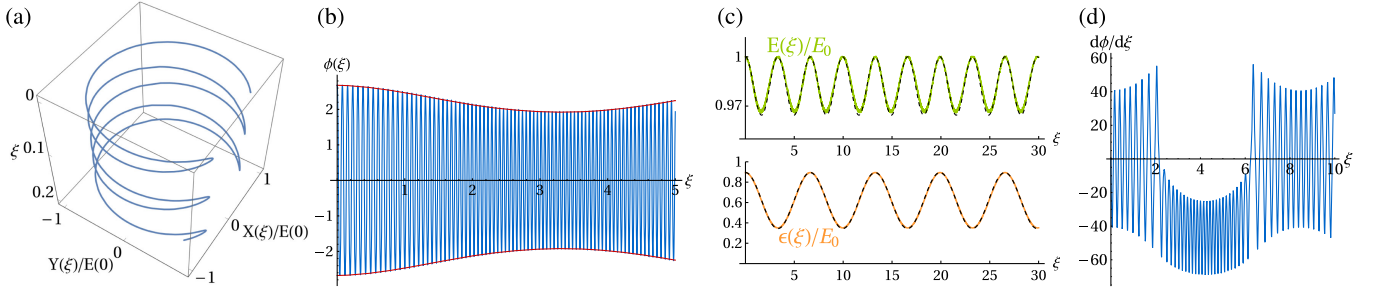


FIG. 1. Numerical solution of Eqs. (6)–(8) with initial conditions in strongly nonlinear regime: $\mathbf{E}(0) = [X(0), Y(0)] = (2, 1) \times 10^4$. (a) Electric field of the traveling wave in space. (b) The polar angle $\phi(\xi)$ exhibits rapid oscillations with period $\Delta\xi \simeq 0.08$. The envelope varies on much longer timescales of the order of one. The time dependence of the envelope is accurately captured by analytical expression for $\arccos[-\mathcal{E}(\xi)/E(0)]$ found in Supplemental Material, Sec. I [30] (red lines). (c) Electric field magnitude E and energy $\mathcal{E}$ of the pendulum (solid lines) together with the analytical expressions given in Supplemental Material, Sec. I [30] (black dashed lines). The only fitting parameter is $\omega_E \simeq 0.947$, which is very close to one, as expected. All other parameters are determined from the initial conditions. (d) Angular velocity $\dot{\phi}$ for initial conditions $Y(0) = -200$ and $X(0) = 1000$. The pendulum is flipping in the time interval $2 < \xi < 6$, where $\dot{\phi}$ has no zero crossing.

$$\ddot{\phi} + E \sin \phi = 0, \quad (13)$$

which is the equation of motion for a pendulum with a length $\sim 1/E$. Hence, if the electric field was affected only by the terms originating from the Berry dipole, it would follow a pendulum motion with gravity along the $-y$ direction. The pendulum motion is illustrated by a real space representation of the traveling wave in Fig. 1(a). The pendulum solution turns out to be unphysical, as it allows for an average electric field E_y . This is a consequence of ignoring the restoring force represented by the Drude weight. We can account for the restoring force by considering Eqs. (6)–(8) in the strongly nonlinear limit, $E \gg 1$. In this limit, the Berry dipole terms dominate and the electric field follows a pendulum motion on short timescales $\xi \sim O(1/\sqrt{E})$. The dynamics on longer timescales can be understood by considering the time average $\langle Y \rangle$ over a pendulum period T , and, using Eq. (7), we obtain

$$\langle \ddot{Y} \rangle + \langle Y \rangle \simeq \int_T d\xi (-X + \dot{M})X = 0, \quad (14)$$

where we have used partial integration and Eq. (8). Hence, the average height of the pendulum evolves like a harmonic oscillator, and we have $\langle Y \rangle \simeq Y_0 \cos(\omega_E \xi)$, where $\omega_E \simeq 1$ and Y_0 is a constant. Hence, even a small Drude weight is enough to ensure that the electric field averages to zero over the plasma timescale. In the language of the pendulum, the linear terms in Eqs. (6) and (7) lead to a periodic modulation of the pendulum energy $\mathcal{E} = \dot{\phi}^2/2 + Y$. An analytical treatment of the strongly nonlinear case (see Supplemental Material, Sec. I [30]) shows that the length of the pendulum is also modulated periodically with twice the period. The numerical solution is shown along with the analytical estimate in Figs. 1(b) and 1(c). Interestingly, the vanishing of the average field implies that in some cases

the pendulum may flip for a finite number of consecutive cycles, namely, when the pendulum is initially sufficiently close to the minimum [see Fig. 1(d)]. Such solutions are nevertheless time-reversal invariant as the rotation direction alternates.

Traveling waves in the weakly nonlinear regime—For typical materials, the light intensity scale I_0 that separates the weakly and strongly nonlinear regimes is very large (i.e., $I_0 \sim 10^{13} \text{ W/m}^2$; see the discussion section), and, thus, it is more realistic to consider the opposite regime of weak nonlinearity, where it is convenient to express Eqs. (6)–(8) in vector notation:

$$\ddot{\mathbf{E}} + \mathbf{E} = -X\hat{z} \times \mathbf{E} + \frac{\dot{X}}{1-Y}\hat{z} \times \dot{\mathbf{E}}. \quad (15)$$

Here, M has been eliminated using $M = \dot{X}/(Y-1)$, which follows from substituting Eq. (8) into Eq. (6) and then integrating. We now have a different separation of timescales, where on short timescales the left-hand side dominates and $\mathbf{E}$ evolves like a harmonic oscillator, which we associate with standard plasma oscillations. On longer timescales, the anharmonic terms on the right-hand side originating from the Berry curvature dipole become important. This can be understood by considering suitable constants of motion of the 2D harmonic oscillator, which can change only due to the nonlinear terms. We consider its energy $\epsilon = (E^2 + \dot{\mathbf{E}}^2)/2$, angular momentum $L = E^2 \dot{\phi}$, and the Laplace-Runge-Lenz vector (see Ref. [31])

$$\mathbf{A} = s \frac{\dot{\mathbf{E}} \times \hat{z} L + \mathbf{E}(\sqrt{\epsilon^2 - L^2} - \epsilon)}{\sqrt{E^2(\epsilon^2 - L^2)^{1/2} - E^2 \epsilon + L^2}}, \quad (16)$$

where $s = \pm 1$ is chosen such that $A_y > 0$. For an elliptic trajectory with semimajor and -minor axes a and b , we have $\epsilon = (a^2 + b^2)/2$, $|L| = ab$, and $\mathbf{A}$ points along the

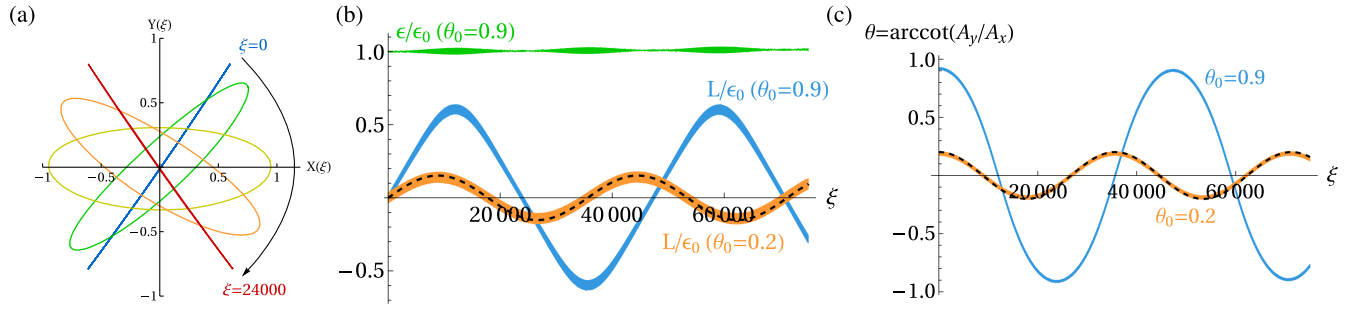


FIG. 2. Numerical solution of Eqs. (6)–(8) with initial conditions in the weakly nonlinear regime: $y(0) = a_0 \cos \theta_0$, $x(0) = a_0 \sin \theta_0$, and $a_0 = 0.04$. (a) Electric field during one period $\xi \rightarrow \xi + 2\pi$ at different points in time. Evolution of the semimajor axis is indicated by the arrow. The times are $\xi = \{0, 6, 12, 16, 24\} \times 10^3$ from blue to red along the arrow. At longer times, the trajectory is traced backward until the initial direction is recovered at $\xi \simeq 48000$. (b) Effective energy and angular momentum normalized by initial energy $\epsilon_0 = a_0^2/2$. (c) Angle of semimajor axis determined from Laplace-Runge-Lenz vector. The energy is almost constant in agreement with our analytical results. The black dashed lines in (b) and (c) indicate the analytical result for small initial angles given by $L = a_0 b$ and Eqs. (18) and (19). The apparent thickness of curves in (b) and (c) arises because of short time oscillations, whose amplitude decreases as $E \rightarrow 0$ (the short timescale $\delta\xi = 2\pi$ is not resolved on the scale of the plots).

semiminor axis. Linear polarization corresponds to $L = 0$ and circular to $|L| = \epsilon$. The sign of L indicates the sense of rotation.

Figure 2(a) illustrates one of our central findings; namely, it shows the trajectories traced by the electric field during one cycle, $\Delta\xi = 2\pi$, of the harmonic oscillator (i.e., the plasma timescale) at different points during the long time evolution. The light is initially linearly polarized and then changes to elliptic polarization and back to linear. The direction of the semimajor axis rotates in the plane until it reaches the mirror image of its initial direction with respect to the x axis, where the light is again linearly polarized. Subsequently, this path is traced backward until the initial condition is reached again. This rocking motion of the polarization can be viewed as taking place either in time at some fixed plane or as a function of position traveled by the wave and can be used to convert linear to elliptical polarization or rotate the polarization axis. Complex nonlinear dynamical evolutions of the light polarization have also been proposed to arise for intense light pulses in gases under large magnetic fields [24].

The energy of the harmonic oscillator, which corresponds to the intensity of light, is a constant even on long timescales as shown in Fig. 2(b) (ignoring the weak fluctuations, we find no appreciable change of ϵ). This is a reflection of the fact that the Berry curvature dipole contribution to the response is nondissipative. The change between linear and elliptic polarization is captured by the angular momentum also shown in Fig. 2(b), which oscillates around zero on a timescale $\xi \sim 1/E^2$. We see that the elliptic polarization has opposite rotation direction of the electric field vector during the first and second half of the period. Moreover, we have $|L| < \epsilon$ even at the maximum, meaning that circular polarization is never reached. The polarization direction, i.e., the angle of the semimajor axis $\theta = \text{arccot} A_y/A_x$, is shown in Fig. 2(c). The

polarization direction swings around the x axis between the initial point and its mirror image, somewhat similarly to the pendulum motion of the electric field discussed in the strongly nonlinear case; however, now the direction of the Berry dipole acts as the direction of effective gravity.

We can derive an approximate analytical solution to Eq. (15) in the limit of small amplitudes, $E \ll 1$, and small initial deviations of the polarization direction, $\theta_0 \ll 1$, from the x axis around which it performs its rocking motion. To lowest order in the electric field, the solution reads

$$X_0 + iY_0 = [a_0 \cos \xi + ib(\xi) \sin \xi] e^{i\theta(\xi)} \quad (17)$$

with $X_0(0) + iY_0(0) = a_0 e^{i\theta_0}$. The polarization vector describes an ellipse, whose semimajor axis direction has a time-dependent polar angle $\theta(\xi)$ and constant length a_0 , while the length of its semiminor axis, $b(\xi)$, changes in time. Notice that the elliptic motion is traced on short timescales of the order of $\xi \sim 1$, whereas $\theta(\xi)$ and $b(\xi)$ vary on much longer timescales $\xi \sim 1/E^2$. To obtain the long-time dynamics of these quantities, we expand the general solution to second order in E as $X \approx X_0 + (\dot{X}_0 \dot{Y}_0 - X_0 Y_0)/3$ and $Y \approx Y_0 + (X_0^2 - \dot{X}_0^2)/3$. We can use this expression to calculate the time derivative of the constants of motion of the 2D harmonic oscillator, like the angular momentum or the energy of the motion along y , $\epsilon_y = (Y^2 + \dot{Y}^2)/2$ (see Supplemental Material, Sec. II [30]), and we obtain an oscillatory motion of the semiminor axis and the angle:

$$b(\xi) = \frac{a_0 \theta_0}{\sqrt{7}} \sin(Q_\xi \xi), \quad (18)$$

$$\theta(\xi) = \theta_0 \cos(Q_\xi \xi) \quad (19)$$

on long timescales with the frequency $Q_\xi = a_0^2 \sqrt{7}/24$ and $a_0 \simeq E(0)$ the dimensionless magnitude of the electric field.

As illustrated in Fig. 2, the observables calculated from these analytical expressions are in excellent quantitative agreement with the numerical results at small angles and amplitudes. For larger initial angles, there is still qualitative agreement, although the period is somewhat longer and the time dependence deviates from a single harmonic.

Transmission through a sample—Let us now consider the experimentally relevant situation of light entering the material, creating a time-independent linear polarization at the interface, $\mathbf{E}(z=0) = \mathbf{E} \cos(\omega t)$. We are looking for stationary solutions of Eq. (4) inside the material ($z > 0$), whose polarization depends on space but not on time. We concentrate on the regime of small amplitudes and use a similar expansion as for traveling waves. We can define an angular momentum $L = \hat{z} \cdot (\mathbf{E} \times \partial_z \mathbf{E})$ and a Laplace-Runge-Lenz vector akin to Eq. (16) but with the dimensionless coordinate ξ replaced by the position z . Using a parametrization as in Eq. (17) with the substitution $\xi \rightarrow kz - \omega t$ with $\omega^2 = c^2 k^2 + \omega_p^2$ while θ and b depend only on z , we can calculate the time-averaged spatial derivatives of θ and L (see Supplemental Material, Sec. III [30]) for small θ and b :

$$\langle \partial_z \theta \rangle \simeq \frac{2\omega_p^2}{3k^2 c^2} \langle L \rangle, \quad (20)$$

$$\langle \partial_z L \rangle \simeq -\frac{(E/E_0)^4 \omega_p^2}{24c^2} \langle \theta \rangle. \quad (21)$$

We find $\partial_z^2 \langle \theta \rangle = -Q^2 k^2 \langle \theta \rangle$ with the dimensionless wave vector $Q = (I/I_0) \omega_p^2 / 6(ck)^2 \ll 1$, and, hence, the polarization direction oscillates around the x axis as given by Eq. (1). The oscillations in L result in the periodic switching between linear and elliptic polarization (for incident elliptic instead of linear polarization, the light follows the same oscillatory behavior). The wave vector of the oscillations has a $Q \sim 1/k^2$ dependence in contrast to the scale Q_ξ obtained from the traveling-wave ansatz. This difference arises because the traveling-wave solution mixes different frequencies, unlike the stationary solution.

We can understand the difference in wave vectors in more detail by considering a simpler example: the conventional Faraday effect due to a magnetization, which is governed by Eq. (4) if we set $\tilde{M} = \text{const}$. In this case, the stationary solution for small $\tilde{M}$ reads $\mathbf{E} = e^{i(kz - \omega t)} [\cos(qkz) \hat{x} + \sin(qkz) \hat{y}]$ with $\omega^2 = c^2 k^2 + \omega_p^2$, which describes a linearly polarized wave slowly rotating in space with a dimensionless wave vector $q = \omega_p \tilde{M} / 2(ck)^2$. In contrast, a traveling-wave solution of the conventional Faraday effect also finds a slowly rotating polarization but with a dimensionless wave vector $(c^2 k^2 / \omega_p^2) q$ (see Supplemental Material, Sec. IV [30]).

The ratio of wave vectors is identical (up to the numerical prefactor) to the one found for the nonlinear Faraday effect above, $Q_\xi = (\sqrt{7}/4)(c^2 k^2 / \omega_p^2) Q$, which suggests that our results for the stationary and traveling-wave approaches are consistent.

Discussion and outlook for experimental detection—We have demonstrated that when light travels in the direction perpendicular to a stack of 2D metals with a nonlinear Hall effect, its polarization direction swings back and forth like a pendulum. There is a variety of van der Waals layered materials where strong nonlinear Hall effects has been detected, such as WTe₂ [25,26], MoTe₂ [27], and TaIrTe₄ [10], which are prime candidates to observe this nonlinear Faraday rotation. As shown in Eq. (1), the polarization rotation per traveled wavelength is controlled by the dimensionless parameter $Q = (I/I_0) \omega_p^2 / 6(ck)^2$, where $I_0 = E_0^2 e^2 / (4ah)$ depends on a material-dependent electric field scale $E_0 = \sigma / \sigma_{\text{NLH}}$ determined by the ratio of the linear Drude conductivity σ and the nonlinear Hall conductivity σ_{NLH} . This parameter can be easily measured in a Hall bar geometry [25,27]. For example, measurements in a MoTe₂ bilayer [27] yield a scale $E_0 \approx 10^8$ V/m, which corresponds to a light intensity $I_0 \approx 3 \times 10^{13}$ W/m². The plasma energy is $\hbar \omega_p = [(e^2 / \epsilon w)(D / e^2)]^{1/2}$, where the 2D Drude weight is typically comparable to the Fermi energy, which we take as $D / e^2 \sim 0.2$ eV [27], and, using an interlayer distance of $w \sim 0.7$ nm [32] and $\epsilon \sim 15$ [33], one gets $\hbar \omega_p \sim 0.53$ eV. Notice that this scale is much larger than typically subpicosecond electron scattering rates ($\hbar / \tau \sim 10$ meV), justifying the collisionless analysis we have employed. Thus, for radiation with intensity $I \sim 10^8$ W/m² and frequency $\hbar \omega \sim 0.6$ eV, one gets a rotation rate of the plane of polarization of about $Q \sim 2 \times 10^{-6}$ rad per traveled wavelength, which increases linearly with the intensity of radiation. Therefore, the nonlinear Faraday effect appears to be comfortably within the detection limits of modern interferometers, which can have resolutions of 10^{-9} rad (see, e.g., Ref. [34]). Notice that the largest rotation of the angle occurs for incident elliptically polarized light with polarization parallel to the Berry curvature dipole ($\theta = 0$). The rocking of the polarization direction as a function of distance could be detected by measuring samples of different thicknesses. Measuring such a nonlinear Faraday effect in time-reversal-invariant metals would offer a new purely optical technique to detect the breaking of inversion symmetry and probe the Berry curvature around the Fermi surface.

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End Matter

Appendix A: Derivation of the equations of motion for traveling waves—Our starting point is the Maxwell equation (2), where the derivative of the current can be written as

$$\frac{\partial \mathbf{j}}{\partial t} = e \int \frac{d^2 k}{(2\pi)^2} [(\partial_t f)(\partial_{\mathbf{k}} \epsilon(\mathbf{k}) - e \mathbf{E} \times \boldsymbol{\Omega}) - e f(\partial_t \mathbf{E}) \times \boldsymbol{\Omega}]. \quad (\text{A1})$$

Substituting $\partial_t f = -e \mathbf{E} \cdot \partial_{\mathbf{k}} f$ yields after a partial integration of the first term

$$\frac{\partial \mathbf{j}}{\partial t} = \mathbf{E} D - \mathbf{E} \times \hat{\mathbf{z}} E_x d - (D/\omega_p)(\partial_t \mathbf{E}) \times \tilde{\mathbf{M}}, \quad (\text{A2})$$

where we have introduced the Drude weight D and the Berry dipole vector d along x (both defined in the main text) as well as the dimensionless average Berry curvature $\tilde{\mathbf{M}} = (e^2 \omega_p / D) \int [d^2 k / (2\pi)^2] f(\mathbf{k}, z, t) \boldsymbol{\Omega}(\mathbf{k})$ and we have assumed $\boldsymbol{\Omega} = \Omega \hat{\mathbf{z}}$. The time evolution of $\tilde{\mathbf{M}} = \tilde{M} \hat{\mathbf{z}}$

is determined by

$$\partial_t \tilde{M} = \frac{e^2 \omega_p}{D} \int \frac{d^2 k}{(2\pi)^2} (-e \mathbf{E} \cdot \partial_{\mathbf{k}} f) \Omega(\mathbf{k}) = \frac{\omega_p d}{D} E_x. \quad (\text{A3})$$

Taken together, Eqs. (3), (A2), and (A3) yield Eqs. (4) and (5). We now assume traveling waves, where fields depend on only the parameter $\xi = \omega_p(v_\phi t - z)/(v_\phi^2 - c^2)^{1/2}$. We can replace the derivatives in Eqs. (2), (A2), and (A3) by $\partial_\xi^2 - (1/c^2)\partial_t^2 = -(\omega_p^2/c^2)\partial_\xi^2$ and $\partial_t = [\omega_p v_\phi / (v_\phi^2 - c^2)^{1/2}] \partial_\xi$, and together we obtain

$$-w e \omega_p^2 \partial_\xi^2 \mathbf{E} = \mathbf{E} D - \mathbf{E} \times \hat{\mathbf{z}} E_x d - \frac{D v_\phi}{(v_\phi^2 - c^2)^{1/2}} (\partial_\xi \mathbf{E}) \times \tilde{\mathbf{M}}, \quad (\text{A4})$$

$$\frac{D v_\phi}{(v_\phi^2 - c^2)^{1/2}} \partial_\xi \tilde{M} = E_x d. \quad (\text{A5})$$

Dividing the two equations by $E_0 D$ and $E_0 d$, respectively, we arrive at Eqs. (6)–(8).

Appendix B: Scaling transformations and finite wave-vector traveling-wave solutions—The explicit disappearance of the phase velocity v_ϕ in Eqs. (6)–(8) also implies a kind of generalized dispersion relation between different solutions. To see this, consider a specific solution of Eqs. (6)–(8), with a characteristic dimensionless timescale ξ_0 , which could be a period but it could also be some other timescale, even if the solution is not perfectly periodic. If we choose $v_\phi \rightarrow \infty$ in Eq. (9), we would have spatially uniform fields with zero wave vector and characteristic frequency scale $\omega_0 = 2\pi\omega_p/\xi_0$. The timescales of these uniform solutions can be viewed as modified nonlinear plasma scales. By choosing a finite v_ϕ in Eq. (9), instead, we can construct a new solution with a finite wave vector from the original spatially uniform solution. In general, its timescales will depend on the characteristic wave vector k and frequency $\omega_k = v_\phi k$, which are related to

the frequency of the $k = 0$ solution as follows:

$$\omega_k = \sqrt{\omega_0^2 + (ck)^2}. \quad (\text{B1})$$

The key difference with respect to the linear case, however, is that there is no unique nonlinear frequency scale ω_0 associated with the solutions of Eqs. (6)–(8). In general, its timescales will depend on initial conditions, analogously to how the period of a pendulum depends on its initial energy.

In the limit of large amplitudes, when the linear terms in Eqs. (6)–(8) can be neglected, there is an additional transformation that relates different solutions of Eqs. (6)–(8) with different characteristic times ξ_0 and amplitudes. Namely, given a solution $[X(\xi), Y(\xi), M(\xi)]$, the following rescaling leads to a new solution: $[X'(\xi), Y'(\xi), M'(\xi)] = [\kappa X(\sqrt{\kappa}\xi), \kappa Y(\sqrt{\kappa}\xi), \sqrt{\kappa}M(\sqrt{\kappa}\xi)]$. Therefore, this generates a new set of solutions with rescaled dimensionless time $\xi_\kappa = \xi_0/\sqrt{\kappa}$, from a given solution with timescale ξ_0 .