

Violation of the Third Law of Black Hole Mechanics in Vacuum Gravity

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We demonstrate numerically the existence of solutions of five-dimensional vacuum gravity describing the formation, in finite time, of an extremal rotating black hole from a preexisting Schwarzschild black hole. This is the first example of a violation of the third law of black hole mechanics in vacuum gravity and demonstrates that the third law is false independently of any matter model. We also demonstrate the existence of solutions describing the formation, in finite time, of an extremal rotating black hole from vacuum initial data that do not contain a black hole.

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Introduction—In classical general relativity with physically reasonable matter, the third law of black hole mechanics asserts that it is impossible for a nonextremal black hole to become extremal in finite time [1,2]. There exist counterexamples to this statement involving thin shells of charged matter [3]. However, it was argued that the distributional nature of such matter is unphysical and that the third law holds for sufficiently smooth matter fields [2]. Recently, Kehle and Unger have identified a mistake in this argument and constructed counterexamples of arbitrarily high differentiability [4]. For Einstein-Maxwell theory coupled to a massless charged scalar field (or massless charged Vlasov matter [5]), their counterexamples describe spherically symmetric gravitational collapse to form an exactly extremal Reissner-Nordström black hole in finite time, with an intermediate period in which the black hole is exactly Schwarzschild at the horizon.

These counterexamples involve matter fields with very large charge to mass ratios. It is known that similar counterexamples cannot exist if the charge to mass ratio is suitably bounded [6]. This raises the possibility that the third law might be true for some types of matter. However, Kehle and Unger have conjectured that there exist third-law-violating solutions of *vacuum* gravity in which an

extremal Kerr black hole forms in finite time [4,7]. If correct, this implies that the third law is false independently of any choice of matter model. But such solutions would be cohomogeneity-4 (i.e., depend on four coordinates), and so, constructing them is much more difficult than for the cohomogeneity-2 solutions of [4].

We will show that third-law-violating solutions do exist in vacuum gravity in *five* spacetime dimensions. The reason for considering 5D is that there exist rotating black holes that are more symmetrical than a Kerr black hole. These are Myers-Perry black holes [8] with two equal rotation parameters. They are cohomogeneity-1 with isometry group $\mathbb{R} \times \text{SU}(2) \times \text{U}(1)$. By combining the approach of [4] with numerical methods, we construct cohomogeneity-2 vacuum solutions that describe either (i) a Schwarzschild black hole absorbing gravitational waves to become an extremal Myers-Perry (EMP) black hole in finite time or (ii) the formation of an EMP black hole in finite time from non-black-hole initial data (i.e., gravitational waves). Our solutions of type (i) demonstrate that the third law is violated in 5D vacuum gravity.

The ansatz—There exist dynamical vacuum solutions with $\text{SU}(2) \times \text{U}(1)$ symmetry [9,10], but conservation of the Komar angular momentum defined by the $\text{U}(1)$ generator implies that they are not type (i) or (ii). We consider spacetimes in which $\text{U}(1)$ is broken to $\mathbb{Z}_4$. Consider the group manifold of $\text{SU}(2)$, which is topologically S^3 . $\text{SU}(2)$ acts on this group manifold via either a left action or a right action. We introduce the standard set of 1-forms σ_i invariant under the left action. These are given in terms of Euler angles (θ, ϕ, ψ) in the Supplemental Material (SM) [11]. Now consider $\mathbb{Z}_4 \subset \text{SU}(2)$, which acts via *right* action as $\psi \rightarrow \psi + \pi$, so $\sigma_1 \rightarrow -\sigma_1$, $\sigma_2 \rightarrow -\sigma_2$, $\sigma_3 \rightarrow \sigma_3$. We now consider a 5D spacetime with coordinates $(U, V, \theta, \phi, \psi)$ and an $\text{SU}(2) \times \mathbb{Z}_4$ invariant metric,

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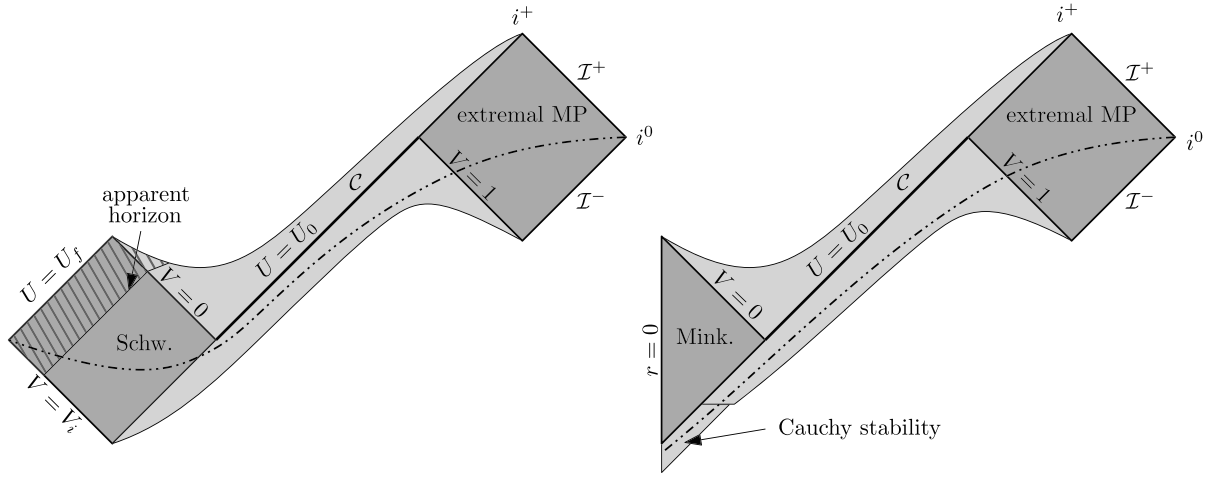


FIG. 1. Penrose diagrams for gluing constructions of type (i) (left) and (ii) (right). The dark regions are exactly isometric to subsets of the Schwarzschild, EMP, and Minkowski spacetimes. Light-shaded regions are local solutions obtained by solving characteristic initial or final value problems. The dash-dotted curves are sample Cauchy surfaces. In the left panel, the Schwarzschild region occupies $[U_0, U_f] \times [V_i, 0]$; we do not extend it to the singularity at $r = 0$. The striped pattern denotes the trapped region; its boundary is the apparent horizon. In the right panel, Cauchy stability ensures the solution in the light-shaded region can be extended to the center of symmetry $r = 0$ using an argument in [4].

$$ds^2 = -\Omega^2 dU dV + \frac{r^2}{4} e^{\mathcal{B}} \left[\sqrt{1 + |\Phi|^2} \sigma_+ \sigma_- + \frac{\bar{\Phi}}{2} \sigma_+^2 + \frac{\Phi}{2} \sigma_-^2 + e^{-3\mathcal{B}} (\sigma_3 + 2\mathcal{A})^2 \right], \quad (1)$$

where (U, V) are double null coordinates and $\sigma_{\pm} = \sigma_1 \pm i\sigma_2$. In this ansatz, Ω , r , and $\mathcal{B}$ are real functions of (U, V) ; Φ is a complex function of (U, V) ; and $\mathcal{A} = \mathcal{A}_U(U, V)dU + \mathcal{A}_V(U, V)dV$ is a real 1-form. A diffeomorphism $\psi \rightarrow \psi + 2\Lambda(U, V)$ gives $\Phi \rightarrow e^{-4i\Lambda}\Phi$ and $\mathcal{A} \rightarrow \mathcal{A} + d\Lambda$, so Φ transforms as a charged scalar field and $\mathcal{A}$ as a $U(1)$ gauge field. Henceforth, we use the gauge choice $\mathcal{A}_V = 0$. The equations of motion resulting from this ansatz are given in the SM [11], where we also discuss the form of the MP solution.

Quasilocal mass and angular momentum—We define the Hawking mass as [10]

$$m_H(U, V) = \frac{3\pi}{8G_5} r^2 \left(1 + 4 \frac{\partial_U r \partial_V r}{\Omega^2} \right), \quad (2)$$

where G_5 is the 5D Newton constant. This satisfies $\partial_U m_H \leq 0$, $\partial_V m_H \geq 0$ if $\partial_V r \geq 0$, $\partial_U r \leq 0$ (see SM [11]). For $\mathbf{m} = \partial/\partial\psi$ and S a sphere of constant U, V , we define the angular momentum $\mathcal{J}(U, V) = (1/16\pi G_5) \int_S \star \mathbf{d}\mathbf{m}^b = (\pi/4G_5)J$, where $J = (1/2)r^5 e^{-2\mathcal{B}} \Omega^{-2} \partial_V \mathcal{A}_U$. The EMP solution has $m_H = (3\pi/8G_5)r_+^2$ for a horizon cross section, and $J = J_{\text{EMP}} \equiv r_+^3$ everywhere.

Characteristic gluing—We use the method of characteristic gluing [16] as in [4, 17]. We prescribe data on a null hypersurface $\mathcal{C}$ with equation $U = U_0$ that for $V \leq 0$

coincides exactly with the data of an outgoing spherically symmetric null hypersurface outside the apparent horizon in a Schwarzschild spacetime and, for $V \geq 1$, coincides exactly with the data on the event horizon of EMP. See Fig. 1, left panel. Given such data, we consider a spacetime that is exactly Schwarzschild in a region $U > U_0$, $V < 0$. The characteristic initial value problem defined by the data on the surfaces $\{U = U_0, V \geq 0\}$ and $\{V = 0, U \geq U_0\}$ admits a local solution [18], so our spacetime can be extended, as a solution of the Einstein equation, into a region to the future of $\mathcal{C}$, shown in light gray. We next assume that the spacetime is exactly EMP in the region $U \leq U_0$, $V \geq 1$. The spacetime can be extended into the light gray region to the past of $\mathcal{C}$ by solving the characteristic final value problem defined by the data on the surfaces $\{U = U_0, V \leq 1\}$ and $\{V = 1, U \leq U_0\}$. The problem is to choose the data for $V \in (0, 1)$ on $\mathcal{C}$ such that the resulting spacetime is suitably smooth across $\mathcal{C}$. If it is, then we have constructed a solution of type (i). For type (ii), we replace the early time Schwarzschild data with data on an outgoing spherically symmetric null cone in Minkowski spacetime, as shown in Fig. 1, right panel.

We aim to construct a solution with a metric that is C^k on $\mathcal{C}$. The equations of motion then imply that J is C^k and r is C^{k+1} . We have the gauge freedom

$$U' = f(U) \quad V' = g(V) \quad \Omega'^2 = \frac{\Omega^2}{f'(U)g'(V)}. \quad (3)$$

We use this to set $\Omega(U_0, V) \equiv 1$ and $\partial_U^j \Omega(U_0, 0) = 0$ ($1 \leq j \leq k$). The free data are the apparent horizon radius r_i of the initial Schwarzschild solution, the horizon radius

r_+ of the final EMP solution, and the fields $\mathcal{B}$ and Φ on $\mathcal{C}$. From these, r is determined on $\mathcal{C}$ by solving Eq. (9a) of the SM [11] backward starting with $r = r_+$ and $\partial_V r = 0$ at $V = 1$. [Gluing at $V = 1$ removes the residual freedom $g'(V) = 1/f'(U_0)$ in Eq. (3).] The data must be chosen such that $r(0) > 0$. Then, r_i is chosen such that $r(0) > r_i > 0$. Equation (9c) is used to propagate J along $\mathcal{C}$. The data must be chosen so that J matches J_{EMP} at $V = 1$. The initial Schwarzschild hole has $m_H \equiv 3\pi r_i^2/(8G_5)$, so Eq. (2) determines $\partial_U r(0)$; then, Eq. (9f) can be used to propagate $\partial_U r$ along $\mathcal{C}$. We require that $\partial_U r < 0$ along $\mathcal{C}$ to ensure that the gluing does not occur inside a white hole. To check smoothness on $\mathcal{C}$, note that transverse derivatives $\partial_U^j \Omega$, $\partial_U^j r$, $\partial_U^j \Phi$, $\partial_U^j \mathcal{B}$, $\partial_U^j \mathcal{A}_U$ on $\mathcal{C}$ satisfy first-order transport equations on $\mathcal{C}$ obtained by taking U derivatives of equations of motion. Hence, these quantities are uniquely defined by their values at either $V = 0$ or $V = 1$. We need these two choices to give the same result, i.e., when transported along $\mathcal{C}$ starting at $V = 0$, these quantities must match their values at $V = 1$. We can apply Eq. (3) with $g'(V) = 1/f'(U_0)$ to the EMP solution [in $\Omega(U_0, V) \equiv 1$ gauge], and by choosing $f'(U_0)$ and $f^{(j+1)}(U_0)$ appropriately, this can be used to ensure that the EMP values of $\partial_U r$ and $\partial_U^j \Omega$ ($1 \leq j \leq k$) match our solution at $V = 1$. Similarly, the residual gauge symmetry $\psi \rightarrow \psi + 2\Lambda(U)$ can be used to ensure that the EMP values of $\partial_U^j \mathcal{A}_U$ ($0 \leq j \leq k$) match our solution at $V = 1$. Constraints on the free data arise from matching of $\partial_U^j \Phi$ and $\partial_U^j \mathcal{B}$ ($j \leq k$). At $V = 1$, we need $\partial_U^j \Phi$ to match its EMP value (zero). To match $\partial_U^j \mathcal{B}$, we define $L \equiv \Omega^{-2} \partial_U$. The quantities $\mathcal{B}^j \equiv [(L^j \mathcal{B})/(Lr)^j]$ are gauge invariant under Eq. (3), and so, at $V = 1$, need to match their EMP values given in the SM [11]. Equation (9b) then implies that $\partial_U^j r$ ($2 \leq j \leq k+1$) must also match its EMP value. (We use this as a check on our numerical solutions below, finding a relative error less than 10^{-21} .) In summary, the problem is to choose the free data to achieve the matching of J , $\mathcal{B}^j$, and $\partial_U^j \Phi$ ($0 \leq j \leq k$) while ensuring $r > 0$ and $\partial_U r < 0$ for $V \in [0, 1]$.

Solutions of type (i) constructed this way are C^k everywhere (not just on $\mathcal{C}$). Solutions of type (ii) extend to $r = 0$, where the equations are singular. As in [4], there is a loss of regularity at $r = 0$. In the SM [11] we show that, away from $r = 0$, our solutions will be piecewise C^{k+1} with bounded $(k+1)$ th derivatives and that this implies that they will be C^{k-2} at $r = 0$. So, achieving a type (ii) solution that is everywhere C^2 , requires $k = 4$.

Numerical approach—Our ansatz for the gluing functions on $[0, 1]$ is

$$\Phi(U_0, V) = \varphi(V)P(V)e^{i\omega(V)V} \quad (4)$$

$$\mathcal{B}(U_0, V) = \beta(V)P(V) - \frac{1}{3} \log 2Q(V), \quad (5)$$

where P , Q are explicit real functions (see SM [11]) that are C^4 , piecewise C^5 , with $P(0) = P(1) = Q(0) = 0$, $Q(1) = 1$, and the first four derivatives of P , Q vanish at $V = 0, 1$. Thus, φ , ω , and β are real functions, each taking the form of a single hidden layer neural network with activation function $\tanh$:

$$\chi_0 + \sum_{i=1}^p \chi_i \tanh(\mu_i V + \kappa_i). \quad (6)$$

The values of p and the parameters $\chi_0, \chi_i, \mu_i, \kappa_i$ may differ for φ, ω , and β . We denote the set of all parameters as θ . We perform the gluing procedure numerically. A weighted sum of the squared residuals of the gluing conditions is used as an objective function $L(\theta)$. A set of parameters θ provides a solution when $L(\theta) = 0$, $r > 0$, and $\partial_U r < 0$. At initialization, χ_i and μ_i are randomly sampled from $\mathcal{N}(0, 1)$, $\kappa_i = -\frac{1}{2}\mu_i$, and χ_0 is set to zero. We repeatedly sample χ_i and μ_i , choosing the set with the least $L(\theta)$ as the seed. Gradient descent with the Adam optimizer [19] is used to update the parameters, where the gradient is approximated by a fourth-order central difference formula:

$$(\nabla_{\theta} L)_i \approx \frac{1}{12\epsilon} [L(\theta - 2\epsilon e_i) - 8L(\theta - \epsilon e_i) + 8L(\theta + \epsilon e_i) - L(\theta + 2\epsilon e_i)] + \mathcal{O}(\epsilon^4). \quad (7)$$

In this stage, penalty terms are added to $L(\theta)$ to prevent r and $\partial_U r$ from vanishing. When $L(\theta)$ becomes small, the penalty terms are removed and the BFGS algorithm [20] is used to reach a local minimum. If this does not exhibit convergence to zero, the p values are increased. Typically, convergence to a solution from a random seed is faster with a large number of parameters. Solutions with fewer parameters are then found by using a fewer-parameter approximation to the higher-parameter solution as a seed. Once a good candidate solution has been identified, we switch to a traditional quasi-Newton method implemented in extended precision. In this final stage, the quasi-Newton search is applied to only a subset of the parameters, chosen so that its dimension matches the number of gluing conditions to be imposed. Specifically, we search for solutions for which $\mathcal{B}^j$, $\partial_U^j \Phi$ (with $1 \leq j \leq k$), and J are fixed to their EMP values, while $r(0)$ and $\partial_U r(1)$ are fixed to the values obtained in the final iteration of the BFGS algorithm. This yields $3k+3$ conditions, which are satisfied by tuning $3k+3$ parameters in the final Newton solve, initialized from the last BFGS iterate. We present extensive convergence tests in the SM.

Results—We have obtained $k = 2$ solutions, choosing $p = (2, 2, 3)$ for φ, ω , and β , a total of 24 parameters in our ansatz. With so many parameters, we do not expect solutions to be unique [17]. Figure 2 shows the behavior of Φ , $\mathcal{B}$, r , J , and m_H on $\mathcal{C}$ for a typical solution of type

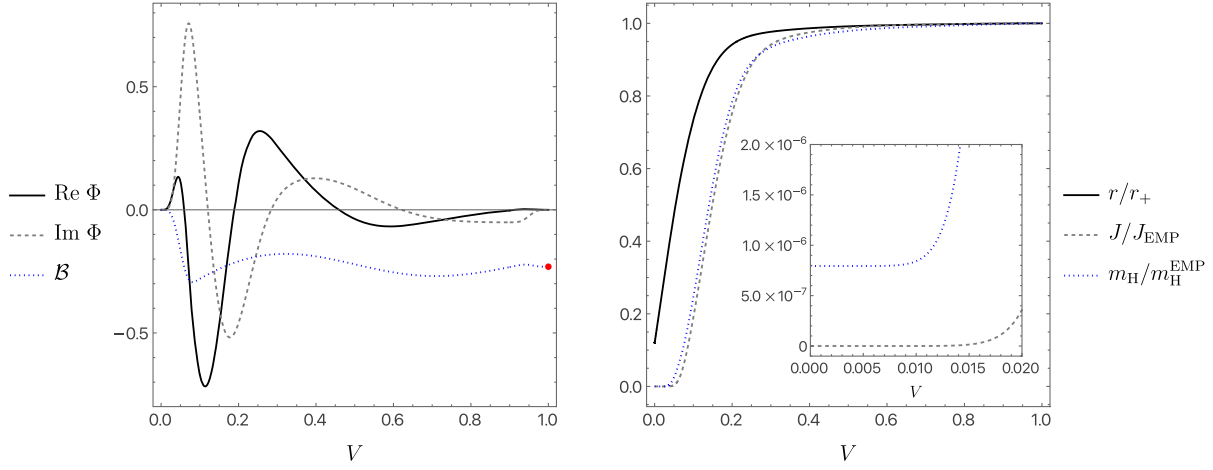


FIG. 2. A $k = 2$ gluing solution of type (i). Left: Φ and $\mathcal{B}$, with the red dot marking the EMP limit of $\mathcal{B}$. Right: r , J , and m_H . The inset plot on the right panel shows that $J = 0$, but $m_H > 0$ at $V = 0$.

(i) with $k = 2$. This solution has $r_i/r_+ \approx 0.0008909$ and $r(U_0, 0)/r_+ \approx 0.1194095$, so the initial event horizon lies well outside the Schwarzschild radius. Φ and $\mathcal{B}$ are largest at early time on $\mathcal{C}$. This also occurs for gluing to extremal Reissner-Nordström, which can be understood as a consequence of Raychaudhuri's equation [17].

Type (ii) solutions with $k = 2$ are no harder to find numerically, but we need $k = 4$ for a type (ii) solution that is C^2 at $r = 0$. Finding $k = 4$ solutions numerically is considerably more challenging because there are many more equations to solve, $\mathcal{B}^j$ increases rapidly with j , and V -derivatives of quantities are larger. To find $k = 4$ solutions, we needed more parameters: $p = (4, 16, 8)$ for φ , ω , and β , a total of 87 parameters. Figure 3 shows such a solution. It has $r(U_0, 0)/r_+ \approx 0.0097801$, much smaller than for $k = 2$.

Plots of other quantities for our solutions are provided in the SM. Parameter values for the solutions are available in the GitHub repository [21].

Discussion—Our solutions of type (i) describe a tiny Schwarzschild black hole absorbing gravitational waves to become a large EMP black hole in finite time, a violation of the third law of black hole mechanics in vacuum. Our solutions of type (ii) describe the formation of an EMP black hole from gravitational collapse of gravitational waves. It should be possible to perform a two-stage gluing construction, as in [4], to combine types (i) and (ii) to construct a solution which starts from non-black-hole initial data and forms an EMP black hole with an intermediate Schwarzschild phase. The method can also be used to obtain solutions describing formation of a subextremal MP black hole with any parameter values.

Numerical methods could be used to study the properties of these solutions away from $\mathcal{C}$. For example, do type (ii) solutions contain trapped surfaces? If so, they cannot be critical solutions [5] (solutions at the threshold between black hole formation and dispersion). One could also try to construct critical solutions numerically by considering

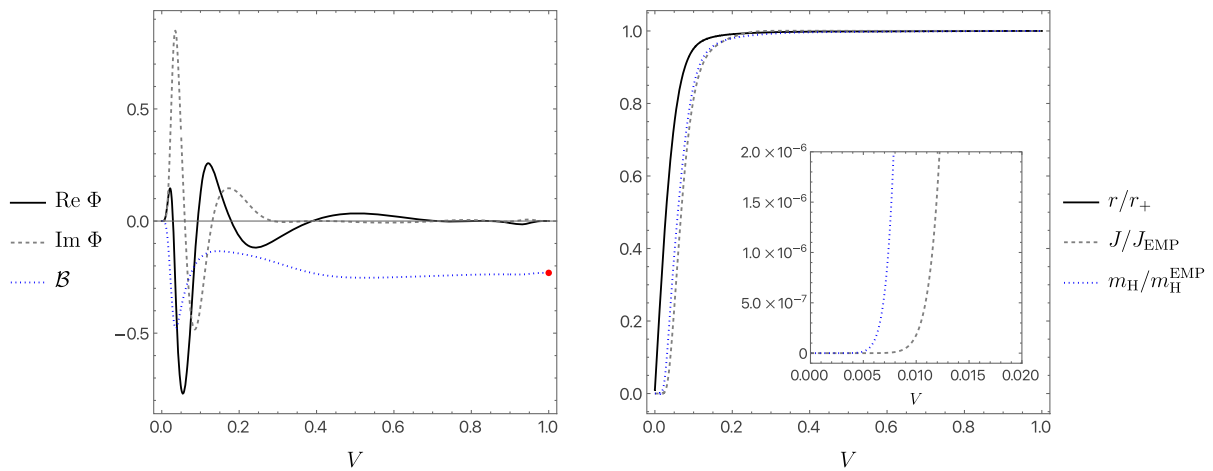


FIG. 3. A $k = 4$ gluing solution of type (ii).

Cauchy evolution in the class of $SU(2) \times \mathbb{Z}_4$ symmetric spacetimes [Eq. (1)].

A more ambitious goal is to perform the gluing construction for 4D vacuum gravity to construct solutions describing finite time formation of an extremal Kerr black hole.

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Data availability—The data that support the findings of this article are openly available [21].

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