From Weyl Anomaly to Universal Defect Casimir Energy and Rényi Entropy

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We establish universal relations between surface defect Weyl anomalies, the ground state energy, and entanglement structure in higher-dimensional supersymmetric quantum field theories. As a concrete example, we show that for surface defects in $6d$ superconformal field theories (SCFTs), the defect contribution to supersymmetric Casimir energy is determined by the combination of b and d_2, where b and d_2 are the surface defect Weyl anomalies. A similar closed-form expression is also derived for the defect contribution to supersymmetric Rényi entropy. The comprehensive method we develop here combines supersymmetric localization, anomaly polynomials, and holography, with applications to broader SCFTs with defects.

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Introduction—It is a well-established result that the ground state energy of a $2d$ conformal field theory (CFT) placed on a cylinder is universal and entirely determined by the conformal anomaly coefficient c, [1]

$$\mathsf{E}^{\mathrm{CFT}} = -\frac{c}{12R}, \tag{1}$$

where R denotes the radius of the spatial circle. This vacuum energy is commonly referred to as the Casimir energy, which can be equivalently extracted from the torus partition function Z on $\mathbb{S}^1_\beta \times \mathbb{S}^1_R$ by taking the low-temperature limit,

$$\log \mathsf{Z} \sim -\beta \mathsf{E}^{\mathrm{CFT}}, \quad \beta \to \infty. \tag{2}$$

The above relation can be generalized to higher-dimensional CFTs on $\mathbb{S}^1_\beta \times \mathbb{S}^{d-1}$ with d even. In higher dimensions, the Casimir energy can be scheme dependent. Its supersymmetric (SUSY) counterpart, however, is a scheme-independent quantity that is completely fixed by the Weyl anomaly [2]. Moreover, the supersymmetric Casimir energy (SCE) admits a natural formulation in terms of an equivariant integral of the anomaly polynomial, a topological quantity that encodes the anomaly coefficients of the theory [3].

The Weyl anomaly also governs the universal piece of the entanglement structure. For example, it appears in the Rényi entropy of a $2d$ CFT,

$$\mathsf{S}_n^{\mathrm{CFT}} = \frac{c}{6}\left(\frac{1}{n}+1\right)\log\frac{\ell}{\epsilon}, \tag{3}$$

with n the Rényi index. Analogous to the case of the Casimir energy, in higher-dimensional theories, Weyl anomalies also control the SUSY version of Rényi entropy, the supersymmetric Rényi entropy (SRE) [4,5]. Furthermore, the SCE and the SRE are related due to the geometric relation of the backgrounds that they are defined on [5].

Defects—such as Wilson lines and Wilson surfaces in gauge theories—act as external probes coupled to the theory and provide a powerful tool for exploring its nonperturbative properties. In general, the insertion of defects gives rise to nontrivial contributions to the ground state energy as well as the entanglement structure. Therefore, a central problem in higher-dimensional superconformal field theories (SCFTs) is whether the universal relations among Weyl anomalies, the ground state energy, and entanglement measures persist in the presence of defects. In particular, for even-dimensional defects that admit defect Weyl anomalies, are the defect contributions to the Casimir energy and the Rényi entropy universally controlled by these anomalies?

In this Letter, we address this question by studying the half-Bogomol'nyi-Prasad-Sommerfield (half-BPS) surface defects in higher-dimensional SCFTs. Although the explicit expressions of defect contribution to the Casimir energy and the Rényi entropy differ among SCFTs, we find that they share similar structures and are determined by the

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defect Weyl anomalies or the associated defect 't Hooft anomalies.

As a representative example, we focus on half-BPS surface defects in $6d$ $\mathcal{N}=(2,0)$ theories [6]. These theories are strongly coupled and are regarded as the *mother theory* of lower-dimensional quantum field theories via dimensional reduction, thus providing a vital playground for understanding the nonperturbative properties with the defect insertion. For theories placed on $\mathbb{S}^1_\beta \times \mathbb{S}^5$, we compute the partition function in the presence of defect, and show that the surface defect contribution to SUSY Casimir energy (dSCE) is governed by the defect Weyl anomalies b and d_2,

$$E[\mathfrak{g},\mathcal{R}] = -\frac{1}{\omega_1}\left[\frac{d_2-b}{6}\omega_2\omega_3 + \frac{2b-d_2}{24}\sigma_1\sigma_2\right], \quad (4)$$

where $\mathcal{R}$ is the representation labeling the defect, $\omega_{1,2,3}$ are the chemical potentials for SO(6) angular momentum $J_{1,2,3}$, and $\sigma_{1,2}$ are the chemical potentials for SO(5) R charges $R_{1,2}$ [7]. We further show that dSCE can be expressed as the equivariant integral of the defect-localized anomaly polynomial (16). This result illustrates a more general underlying phenomenon that the defect contribution to twisted Casimir energy is fixed by anomalies, which appears to be independent of the spacetime dimension and the amount of supersymmetry.

Beyond the universal relations between defect anomalies and ground state energy, an outstanding open problem is to compute entanglement structure in the presence of defects. While substantial progress has been made in understanding defect contributions to the entanglement entropy (EE) [8–13], a corresponding formulation for the Rényi entropy has remained elusive. Using the relation between the SCE and the SRE, we derive a closed-form expression for the defect contribution to the supersymmetric Rényi entropy (dSRE). Remarkably, it exhibits a structure closely analogous to that of a $2d$ CFT, cf. Eq. (3), with the overall coefficient governed by a linear combination of defect Weyl anomaly coefficients, $2b-d_2$,

$$S_n[\mathfrak{g},\mathcal{R}] = \frac{2b-d_2}{6}\left[\frac{r_1 r_2}{2}\left(\frac{1}{n}-1\right)+1\right]\log\frac{\ell}{\epsilon}, \quad (5)$$

where $r_{1,2}$ are the weights of the chemical potentials (9) associated with $R_{1,2}$, satisfying $r_1+r_2=1$. The validity of our results is firmly established through the consistency of three independent approaches: supersymmetric localization, anomaly polynomials, and holography. Together, these methods provide a comprehensive and fully nonperturbative framework for studying defects in higher-dimensional SCFTs.

Importantly, the approach developed in this Letter is not limited to $6d$ (2,0) theories. One can apply the same framework to defects in other settings, including surface defects in $6d$ (1,0) SCFTs as well as Gukov-Witten defects in $4d$ $\mathcal{N}=4$ super-Yang-Mills theory. In these examples, both the dSCE and the dSRE can be computed explicitly [14]. Remarkably, their structures closely parallel the six-dimensional results of this Letter, lending further support to the underlying universality.

Defect Weyl anomalies: Even-dimensional defects are subject to Weyl anomalies: the trace of the defect contribution to the stress-energy tensor is nonvanishing. For a surface defect embedded in a higher-dimensional CFT, this trace anomaly is governed by defect Weyl anomaly coefficients [15–18],

$$\langle T^\mu_\mu\rangle_{\rm def} = -\frac{1}{24\pi}\left[bR^\Sigma + d_1\tilde{\Pi}^\mu_{ab}\tilde{\Pi}^{ab}_\mu - d_2 W^{ab}_{ab}\right], \quad (6)$$

where R^Σ is the intrinsic Ricci scalar of the defect submanifold Σ, $\tilde{\Pi}^\mu_{ab}$ is the traceless part of the second fundamental form, W^{ab}_{ab} is the pullback of the bulk Weyl tensor to Σ, and b, d_1, d_2 are the defect Weyl anomaly coefficients. The anomaly coefficient b determines the free energy of a spherical surface defect, $F=-\log\langle D[\mathbb{S}^2]\rangle = -(b/3)\log\frac{\ell}{\epsilon}$, while d_2 controls the one-point function of the bulk stress-energy tensor in the presence of the defect [19–22]. For the half-BPS Wilson surface in $6d$ A-type $\mathcal{N}=(2,0)$ SCFTs, the defect Weyl anomaly coefficients b and d_2 are given by [11,12]

$$b = 24(\Lambda,\rho)+3(\Lambda,\Lambda), \qquad d_2 = 24(\Lambda,\rho)+6(\Lambda,\Lambda), \quad (7)$$

where Λ is the highest weight of the representation $\mathcal{R}$, ρ is the Weyl vector, and $(\cdot,\cdot)$ denotes the standard invariant inner product on the weight space of $\mathfrak{su}(N)$. For later convenience, we list the explicit expressions for the rank-k symmetric representation (k) and the rank-k antisymmetric representation $[k]$,

$$\begin{aligned} b_{(k)} &= 12Nk+3k^2-12k-3k^2/N, \\ d_{2(k)} &= 12Nk+6k^2-12k-6k^2/N, \\ b_{[k]} &= 12Nk-12k^2+3k-3k^2/N, \\ d_{2[k]} &= 12Nk-12k^2+6k-6k^2/N. \end{aligned} \quad (8)$$

Supersymmetric Rényi entropy: Rényi entropy naturally arises in the computation of entanglement entropy via the replica trick, $\mathsf{S}_n \equiv (1/1-n)\log[\mathsf{Z}_n/(\mathsf{Z}_1)^n]$, where Z_n denotes the partition function on the n-replica manifold. For a d-dimensional CFT in flat spacetime, the replica geometry associated with a spherical entangling surface can be conformally mapped to the conic space $\mathbb{S}^1_n \times \mathbb{H}^{d-1}$ [23,24]. The case of $d=6$ is illustrated in Fig. 1. The SRE is a natural generalization of Rényi entropy in supersymmetric theories. When computing SRE, additional background fields for the R symmetry must be introduced

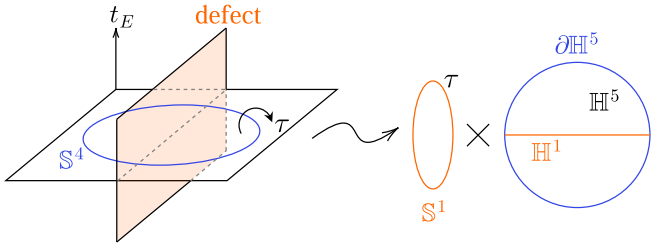


FIG. 1. A conformal transformation maps the bulk geometry $\mathbb{R}^6$ to $\mathbb{S}^1 \times \mathbb{H}^5$, the surface defect geometry $\mathbb{R}^2 \subset \mathbb{R}^6$ to $\mathbb{S}^1 \times \mathbb{H}^1 \subset \mathbb{S}^1 \times \mathbb{H}^5$, and the entangling surface $\mathbb{S}^4$ to the boundary of $\mathbb{H}^5$. After the replica trick, $\mathbb{S}^1$ becomes $\mathbb{S}^1_n$.

in order to preserve supersymmetry [4,25–27]. For the $6d$ (2,0) theories considered in this Letter, there are two independent background gauge fields (chemical potentials) that can be activated, weighted by parameters $r_{1,2}$ [27],

$$A^1 = (n-1)r_1, \quad A^2 = (n-1)r_2, \quad r_1 + r_2 = 1. \tag{9}$$

These chemical potentials are fixed by the Killing spinor equations on $\mathbb{S}^1_n \times \mathbb{H}^5$.

dSCE from localization—To elucidate the relation between the defect Weyl anomaly and dSCE, we present an explicit calculation for a half-BPS surface defect in $6d$ (2,0) theories. Despite the absence of a Lagrangian description, the partition function of $6d$ (2,0) theories on $\mathbb{S}^1_\beta \times \mathbb{S}^5$ can be captured by that of $5d$ maximally supersymmetric Yang-Mills (MSYM) theory on $\mathbb{S}^5$ [28–31]. Upon compactification on $\mathbb{S}^1_\beta$ with an appropriate supersymmetry-preserving twist, the $6d$ A_{N-1} (2,0) theory reduces to the $5d$ SU(N) MSYM theory on $\mathbb{S}^5$, with the coupling $g^2_{\mathrm{YM}} = 2\pi\beta$. In the large-$N$ limit, instanton contributions are suppressed. The perturbative $5d$ partition function can be computed via localization [3], and in the strong-coupling limit $g^2_{\mathrm{YM}} \to \infty$ it reduces to the following matrix integral:

$$\int \prod_{i=1}^{N} \mathrm{d}\nu_i \exp\left[\frac{2\pi}{\omega_1\omega_2\omega_3}\left(-\frac{\pi}{\beta}\sum_{i=1}^{N}\nu_i^2 + \frac{\sigma_1\sigma_2}{2}\sum_{j<i}(\nu_i - \nu_j)\right)\right], \tag{10}$$

where ν_i are the scalar eigenvalues, ordered such that $\nu_i < \nu_j$ for $i < j$.

Under the above dimensional reduction, a Wilson surface wrapping $\mathbb{S}^1_\beta \times \mathbb{S}^1_{(j)} \subset \mathbb{S}^1_\beta \times \mathbb{S}^5$ in the $6d$ theory reduces to a Wilson loop wrapping $\mathbb{S}^1_{(j)}$ in the $5d$ theory. Here $\mathbb{S}^1_{(j)}$ denotes a great circle of length $2\pi/\omega_j$. It was argued in Refs. [32,33] that the $5d$ partition function with line defects suffices to capture the $6d$ partition function with surface defects, and we adopt the same working assumption.

For a Wilson loop inserted along $\mathbb{S}^1_{(j)}$, localization enforces its contribution to the partition function (10) as $\mathrm{Tr}_{\mathcal{R}} \exp(2\pi\nu/\omega_j)$ [32]. Since the $5d$ strong-coupling limit $g^2_{\mathrm{YM}} \to \infty$ corresponds to the $6d$ low-temperature limit $\beta \to \infty$, evaluating the matrix integral (10) with the Wilson loop insertion computes the dSCE [34,35],

$$\log\langle D_{\mathcal{R}}\rangle \sim -\beta E_{\mathcal{R}}, \quad \beta \to \infty. \tag{11}$$

Without loss of generality, we set $j = 1$. The results for the rank-k symmetric representation (k) and the antisymmetric representation $[k]$ are

$$\begin{aligned}\beta E_{(k)} &= -\frac{\beta}{2\omega_1}(k(N-1)\sigma_1\sigma_2 + k^2\omega_2\omega_3),\\ \beta E_{[k]} &= -\frac{\beta}{2\omega_1}(k(N-k)\sigma_1\sigma_2 + k\omega_2\omega_3).\end{aligned} \tag{12}$$

Based on Eqs. (8) and (12), we obtain the dSCE (4), which is determined by combinations of the defect Weyl anomalies b and d_2. One may wonder whether the relation (4) holds only for A-type algebras in the large-rank limit (the large N limit). In fact, from the anomaly-polynomial derivation in the following section, we confirm that Eq. (4) holds for all ranks and all defect representations $\mathcal{R}$ in simply-laced Lie algebras of A, D, and E type.

dSCE and anomaly polynomial—It was conjectured in Ref. [3] that for an even d-dimensional SCFT on a space with topology $\mathcal{M} = \mathbb{S}^1 \times \mathbb{S}^{d-1}$, the SCE is given by the equivariant integral of the anomaly polynomial I_{d+2},

$$E_{\mathcal{M}} = -\frac{1}{(2\pi)^{d/2}}\int I_{d+2}(\mathcal{M}). \tag{13}$$

A surface defect wrapping $\Sigma \hookrightarrow \mathcal{M}$ deforms the anomaly polynomial by a defect-localized term,

$$I_{d+2}(\mathcal{M}) \to I_{d+2}(\mathcal{M}) + \delta_\Sigma I_4(\Sigma), \tag{14}$$

suggesting a defect analogue of the relation (13). In this section, we apply this idea to compute the dSCE from the equivariant integral of the defect anomaly polynomial.

The anomaly polynomial of a $2d$ $\mathcal{N} = (4,4)$ surface defect in the $6d$ $\mathcal{N} = (2,0)$ SCFT labeled by an ADE Lie algebra $\mathfrak{g}$ is given by [37,38]

$$\begin{aligned}I_4 = &\frac{1}{2}(\Lambda,\Lambda)(c_2(F_L) - c_2(F_R))\\ &+ (\Lambda,\rho)(c_2(F_I) - c_2(F_F)),\end{aligned} \tag{15}$$

where $F_{L,R,I,F}$ are background field strengths for the $2d$ $\mathcal{N} = (4,4)$ R-symmetry $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_F \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)_I$ on the defect submanifold, and $c_2(F)$ denotes the second Chern class of the corresponding SU(2)

bundle [35,39]. For a surface defect wrapping $\mathbb{S}^1_\beta \times \mathbb{S}^1_{(1)}$, the equivariant integral of the defect anomaly polynomial (15) is evaluated on $\mathbb{R}^2$ with equivariant parameter ω_1, yielding

$$-\frac{1}{4\omega_1}\left[\frac{1}{2}(\Lambda,\Lambda)(a_L^2-a_R^2)+(\Lambda,\rho)(a_I^2-a_F^2)\right], \tag{16}$$

where $a_{L,R,I,F}$ are the chemical potentials for the $2d$ $\mathcal{N}=(4,4)$ R-symmetry. Identifying the $2d$ R-symmetry Cartans $J_{L,R,I,F}$ with the bulk Cartans $J_{2,3}, R_{1,2}$ relates the chemical potentials as [35]

$$\begin{aligned} a_L &= \omega_2+\omega_3, \qquad a_R = \omega_2-\omega_3, \\ a_I &= -(\sigma_1+\sigma_2), \qquad a_F = -(\sigma_1-\sigma_2). \end{aligned} \tag{17}$$

Using Eqs. (7) and (17), the anomaly polynomial result (16) can be expressed in terms of the defect central charges b, d_2 and the $6d$ bulk chemical potentials, namely, Eq. (4). This agrees with the localization results (12). In the chiral-algebra limit $\omega_1=\omega_2=1$ and $\sigma_1\sigma_2=2\omega_3$ [40], the dSCE (4) reduces to $-\omega_3 d_2/12$, manifesting its proportionality to the defect central charge d_2 [41,42]. We emphasize that the dSCE (4) computed from the anomaly polynomial is exact for all N and all defect representations.

From dSCE to dSRE—As noted in Refs. [4,5], the SCE on the extremally squashed sphere controls the $n\to 0$ asymptotics of the SRE. This relation follows from two key facts: (i) the SRE manifold $\mathcal{M}_1 \equiv \mathbb{S}^1_n \times \mathbb{H}^5$ is conformally equivalent to $\mathcal{M}_2 \equiv \mathbb{S}^5_n \times \mathbb{H}^1$, which can be used to define the SCE as $\mathbb{H}^1$ can be viewed as an infinitely large circle; (ii) the background gauge fields on these two geometries coincide in the extremally squashed limit $n\to 0$. Together, (i) and (ii) relate the free energies on $\mathcal{M}_1$ and $\mathcal{M}_2$, establishing the relation between SRE and SCE in the extremally squashed limit.

This idea extends naturally to the case with defect insertions. The geometric relation remains valid at the level of the defect submanifold. More precisely, dSRE submanifold $\mathbb{S}^1_n \times \mathbb{H}^1 \subset \mathbb{S}^1_n \times \mathbb{H}^5$ (see Fig. 1) is conformally equivalent to $\mathbb{H}^1 \times \mathbb{S}^1_n \subset \mathbb{H}^1 \times \mathbb{S}^5_n$. The $n\to 0$ limit further relates the leading asymptotics of the free energies, as the R-symmetry background gauge fields coincide. Concretely, in the $n\to 0$ limit, the local chemical potentials σ_1 and σ_2 on the (d)SCE manifold $\mathcal{M}_2$ [4]

$$\sigma_1^2(n\to 0)=\frac{r_1^2}{n^2}, \quad \sigma_2^2(n\to 0)=\frac{r_2^2}{n^2} \tag{18}$$

match the setting (9) on the (d)SRE manifold $\mathcal{M}_1$.

Substituting Eq. (18) into Eq. (12), we derive the $n\to 0$ behavior of the squashed dSCE,

$$E|_{n\to 0} = -\frac{2b-d_2}{24}\frac{r_1 r_2}{n}, \tag{19}$$

which justifies the leading term of the dSRE (5) for small n, up to a volume factor $V_{\mathbb{H}^1}=2\log(\ell/\epsilon)$. Here we have identified the shape parameters as $\omega_1=1/n$ and $\omega_{2,3}=1$, since the defect wraps the squashed circle.

Defect SUSY entanglement entropy: The value of dSRE at $n=1$ is the surface defect contribution to the SUSY EE. Defining EE requires specifying boundary conditions at the entangling surface. For defect contributions, supersymmetric boundary conditions lead to results that differ from their nonsupersymmetric counterparts [44]. We assume that the SUSY result is likewise a linear combination of b and d_2, analogous to Eq. (24),

$$S_1 = \log\langle D\rangle + \alpha d_2 \log\frac{\ell}{\epsilon} = \left(\frac{b}{3}+\alpha d_2\right)\log\frac{\ell}{\epsilon}, \tag{20}$$

where the undetermined constant α is fixed to $-1/6$ by matching to the holographic result (22) at $n=1$. Consequently, the surface defect contribution to SUSY EE takes the form

$$S_1 = \frac{2b-d_2}{6}\log\frac{\ell}{\epsilon}. \tag{21}$$

Combining the defect SUSY EE (21) with the $n\to 0$ limit of dSRE yields the closed-form expression for the dSRE in Eq. (5).

Defects in the large N limit—Now we perform a holographic computation to verify the large-N behavior of the dSRE formula (5) derived above. The $6d$ A_{N-1} (2,0) SCFT is conjectured to be dual to M theory on $\mathrm{AdS}_7\times\mathbb{S}^4$ with N units of four-form flux through $\mathbb{S}^4$ [45]. In this dual description, the SRE can be computed holographically from the two-charge $7d$ topological black hole [4,46]. A surface defect (in fundamental representation) in the large-N limit is described by a probe M2-brane. Following the method of Refs. [4,47], we obtain the dSRE for a surface defect in the fundamental representation in the large-N limit [35]:

$$S_n = N\left[r_1 r_2\left(\frac{1}{n}-1\right)+2\right]\log\frac{\ell}{\epsilon}. \tag{22}$$

This result agrees with Eq. (5) precisely in the large-N limit.

There is an alternative way to verify the exact dSRE formula (5). As observed in Ref. [5], for general even-dimensional SCFTs the SRE admits a precise relation to an *extended* SCE, obtained by introducing an additional rotational chemical potential. Following this approach, one can extend the dSCE (4) with an additional chemical potential ω_4,

$$\tilde{E}[\mathfrak{g},\mathcal{R}] = -\frac{1}{\omega_1\omega_4}\left[\frac{d_2-b}{6}\omega_2\omega_3 + \frac{2b-d_2}{24}\sigma_1\sigma_2\right]. \quad (23)$$

With a suitable identification of chemical potentials, which are consistent with the theory without defects, one can check that the extended dSCE reproduces precisely the dSRE (5) [35]. This offers an independent and nontrivial consistency check of our exact formula.

Discussions—In this Letter, we proposed closed-form expressions for the contributions of half-BPS surface defects to the supersymmetric Casimir energy and the supersymmetric Rényi entropy in $6d$ (2,0) theories, expressed in terms of the defect Weyl anomalies b and d_2. These formulas are supported by strong evidence and have passed a variety of nontrivial consistency checks. Although our explicit results are presented for surface defects in $6d$ (2,0) theories, the structures of our formulas suggest a broader applicability.

Future directions include generalizing our approach to defects of other codimensions (e.g., codimension 2), as well as defects in bulk theories with different dimensions. Another intriguing future direction is to explore the high temperature universality, namely, the Cardy limit of the surface defect [48]. It would also be interesting to numerically verify some of our results by the recent numerical approach [49,50].

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Data availability—No data were created or analyzed in this study.

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