

Supertranslations from Scattering Amplitudes

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On shell methods have found a new application to local observables such as asymptotic radiation fields and gravitational waveforms. While these observables are invariant under small gauge transformations, they are known to depend on a choice of asymptotic gauge; in gravity on asymptotically Minkowski spacetimes, this is a choice of Bondi-Van der Burg-Metzner-Sachs frame. In this Letter, we provide a method for capturing these supertranslations, to all orders in perturbation theory, using the on shell framework of scattering amplitudes.

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Introduction—The literature on applications of scattering amplitudes to local asymptotic observables, such as the gravitational waveform, relies on a certain large-distance approximation [1–21]. Consider the perturbative mediator operator (here graviton with the convention $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, but also photon or gluon):

$$\begin{aligned} \mathbb{h}_{\mu\nu}(x) &= \sum_{\eta} \int d\Phi(k) e_{\mu\nu}^{\eta}(k) a_{\eta}(k) e^{-ik \cdot x} + \text{H.c.} \\ &\simeq \sum_{\eta} \frac{-i}{4\pi|\mathbf{x}|} \int_0^{\infty} d\omega e_{\mu\nu}^{\eta}(k) a_{\eta}(k) e^{-i\omega u} \Big|_{k=\omega n} \\ &\quad + \text{H.c.}, \end{aligned} \quad (1)$$

where $u = x^0 - |\mathbf{x}|$ is the retarded time and $n^{\mu} = (1, \hat{\mathbf{x}})$. The integration on the first line is over the on shell phase space of a massless mediator, so that

$$\begin{aligned} d\Phi(k) &\equiv d^4k \hat{\delta}_+(k^2), \quad \hat{dk} \equiv \frac{dk}{2\pi}, \quad \hat{\delta}(k) \equiv 2\pi \delta(k), \\ \delta_+(k^2) &\equiv \delta(k^2) \Theta(k^0). \end{aligned} \quad (2)$$

The approximation leading to the second line of Eq. (1) assumes that $\omega|\mathbf{x}| \gg 1$ in the exponent on the first line.

However, this assumption is certainly false in important situations even for very large $|\mathbf{x}|$.

Take, e.g., the gravitational waveform sourced when two pointlike objects scatter. Then the difference between the

metric in the far future and the far past is nonzero because the different initial and final velocities of the objects lead to different linearized Schwarzschild solutions. Correspondingly, the metric perturbation at fixed large distance $|\mathbf{x}|$ will contain modes of frequency of order $1/|\mathbf{x}|$ or smaller. A possible solution is to avoid the large-distance expansion and define creation operators at finite distance and time, as suggested in [22] in the development of a novel perspective on the Lehmann-Symanzik-Zimmermann (LSZ) reduction.

In this Letter we describe a method of capturing the contributions of these low-frequency modes in the spirit of the on shell approach to scattering amplitudes. As reemphasised in Ref. [23], a traditional way of incorporating the effects of low-frequency modes is to include the off shell Coulombic field of the initial particles. Absence of gauge invariance for off shell gravitons leads, however, to a complicated formalism. Here, following [14], we will instead deform the on shell condition of matter particles in a precise manner, which will be explained below. We will see that, in electromagnetism, the Coulombic modes do not modify the expectation value of the gauge field beyond leading order. In gravity, these modes lead to a Bondi-Van der Burg-Metzner-Sachs (BMS) supertranslation of the gravitational strain, which can be computed to all orders in perturbation theory.

Coherent background—An appropriate quantum state for a single scalar pointlike particle in the classical approximation is $|\psi\rangle = \int d\Phi(p) \varphi(p) |p\rangle$, where $\varphi(p)$ is a momentum-space wave packet, which is steeply peaked around some expected classical momentum. Since there is no suppression of (classical) emission of low-frequency mediators, emission from a charged particle effectively creates its classical Coulombic field.

Figure 1 captures schematically the low-frequency contributions to the final-state expectation value [1] of the

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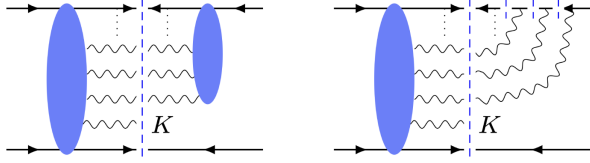


FIG. 1. Soft-mediator contribution to the asymptotic expression of the mediator field. The dashed vertical line crossing all horizontal lines represents the final state. The short dashed lines on the matter propagator in the second panel represent cuts as these lines are on shell.

mediator operator in Eq. (1). The right-hand side of the left diagram contains a $1 \rightarrow 1 + n$ amplitude, whose support is only on such modes. Soft-limit properties of amplitudes and $\hbar$ counting then imply that only the leading classically singular (superclassical) terms in this amplitude can contribute to the classical waveform. These are shown in the right diagram of Fig. 1. As a means to enhance the large-distance approximation, it is, thus, appropriate to resum their contributions.

A little work following Refs. [24,25] shows that these $1 \rightarrow 1 + n$ amplitudes resum into an exponential factor U ,

$$|\psi\rangle \rightarrow U|\psi\rangle = \int d\Phi(p) \varphi(p) e^{iQ_s} |p\rangle, \quad (3)$$

“dressing” the asymptotic state. The mechanism is very similar to that of Refs. [24,25], though now in Minkowski signature: only half-ladder diagrams contribute, they all have equal numerators, all matter propagators are linear, and the eikonal identity [26] sets them all on shell. The factorial factor needed for exponentiation is the symmetry factor for the identical soft mediators in the final state. The exponent is

$$Q_s \equiv \sum_{\eta} \int d\Phi(k) \hat{\delta}(2p \cdot k) \mathcal{A}(p+k|p, k_{\eta}) a_{\eta}^{\dagger}(k) + \text{H.c.}, \quad (4)$$

where the sum is over the helicity η of the mediator and $\mathcal{A}(a|b, c)$ is the $(a \rightarrow b, c)$ one-mediator amplitude. There is one such factor for each matter particle of momentum p . The delta functions and on shell conditions imply that the integrand has support only for vanishing mediator frequency. When forming observables using the state in Eq. (3), we will regularize the delta functions allowing low-frequency modes to possibly contribute. For reasons which will become clear, we will refer to Q_s as the “soft charge.”

The operator Q_s in Eqs. (3) and (4) is a displacement operator, generating a coherent state of mediators. To demonstrate its relation to the Coulombic background, we recall that the gravitational three-point amplitude is

$$\mathcal{A}(p+k|p, k_{\eta}) = -\kappa e_{\mu\nu}^{\eta*} p^{\mu} p^{\nu}, \quad (5)$$

where $\kappa^2 = 32\pi G$ so that

$$\kappa \langle \psi | e^{-iQ_s} \mathbb{h}^{\mu\nu}(x) e^{iQ_s} | \psi \rangle = -\frac{4G}{|\mathbf{x}|} \frac{\Pi^{\mu\nu, \rho\sigma} p_{\rho} p_{\sigma}}{p \cdot n}. \quad (6)$$

The object Π is the physical state projector [27]. Writing $p = mv$, we obtain the linearized metric $h_{\mu\nu}^{\text{Schw}}(x)$ of a Schwarzschild black hole of mass m and velocity vector v . Notice that we evaluated the metric operator in Eq. (6) after using the approximation of Eq. (1). In effect we are *defining* the metric operator by the use of this approximation as part of our prescription for incorporating Coulombic modes in an on shell approach. This is only possible because the field point x is taken to be far from massive sources. More generally, we have

$$e^{-iQ_s} \mathbb{h}_{\mu\nu}(x) e^{iQ_s} = \mathbb{h}_{\mu\nu}(x) + h_{\mu\nu}^{\text{Schw}}(x). \quad (7)$$

The hard charge—We will soon be interested in evolving our initial state with the full S matrix. As a step in this direction, let us examine the commutator $[S, Q_s]$ of the soft charge with the S matrix. To that end, consider

$$C \equiv \sum_{\eta} \int d\Phi(k) \hat{\delta}(2p \cdot k) \mathcal{A}(p+k|p, k_{\eta}) [S, a_{\eta}^{\dagger}(k)]. \quad (8)$$

Because of the small-frequency support, $k = \omega n$ and $\omega \rightarrow 0$, following from the on shell conditions, we evaluate the commutator $[S, a_{\eta}^{\dagger}(k)]$ in the soft limit. We work to leading order in ω , restricting to situations in the classical approximation, which involve scattering of some fixed number of stable and distinguishable massive particles. Self-consistently, we assume that the energies of the incoming particles are below the massive-particle pair-production threshold. The scattering may nevertheless involve any number of incoming or outgoing mediators.

The commutator $[S, a_{\eta}^{\dagger}(k)]$ is nonzero only for S matrix elements with an incoming soft mediator. Thus, we write $C = C_M + C_0$, where the two terms correspond to the mediator attaching to a massive and a massless line, respectively, and discuss the two terms separately. We use the β regulator of Ref. [14] to resolve scaleless integrals. Details are included in End Matter.

The matter contribution C_M to the commutator $[S, Q_s]$ can be written as

$$C_M = -[S, Q_{h,M}], \quad Q_{h,M} = \sum_{i=\text{matter}} Q_{h,i} a_i^{\dagger} a_i. \quad (9)$$

$Q_{h,i}$ is a combination of three-mass cut triangle integrals and their derivatives, and further is a sum over all the matter particles. We will not need its explicit expression.

In electromagnetism, mediators only interact by attaching to external massive (charged) particles, and thus

$[S, Q_s + Q_{h,M}] = 0$. We will discuss below consequences of this relation.

The mediator contribution C_0 to the commutator $[S, Q_s]$ can be written as $C_0 = -[S, Q_{h,0}]$, where

$$Q_{h,0} = 2G \sum_{\eta} \sum_{i=\text{matter}} \int d\Phi(K) p_i \cdot K \times \left[\log \left(\frac{p_i \cdot K}{mK^0} \right) + Y(\beta, \epsilon) \right] a_{\eta}^{\dagger}(K) a_{\eta}(K). \quad (10)$$

$Y(\beta, \epsilon)$ is a regulator-dependent and momentum-independent constant. Referring to $Q_h = Q_{h,0} + Q_{h,M}$ as the “hard charge,” it follows that its sum with the soft charge commutes with the S matrix in gravity:

$$[S, Q_s + Q_h] = 0. \quad (11)$$

It is also not difficult to see that the hard and soft charges commute with each other, $[Q_s, Q_h] = 0$.

Equation (11) can be interpreted as an equivalence between inclusion of a vanishingly soft mediator in an S -matrix element and a particular rescaling of the S -matrix element without that low-frequency particle. In the remainder of this Letter, we will discuss consequences for observables.

Supertranslations—Consider the expectation value of the metric operator in the future asymptotic state $|\psi\rangle_{\text{out}} = S \exp(iQ_s) |\psi\rangle$:

$$h_{\mu\nu}(x) \equiv \langle \psi | e^{-iQ_s} S^{\dagger} \mathbb{h}_{\mu\nu}(x) S e^{iQ_s} | \psi \rangle = \langle \psi | e^{-i(Q_s + Q_h)} S^{\dagger} \mathbb{h}_{\mu\nu}(x) S e^{i(Q_s + Q_h)} | \psi \rangle. \quad (12)$$

We introduced the hard charge above, noting that, in our setup, $Q_{h,0} |\psi\rangle = 0$. We expect that the contributions of the matter hard charge $Q_{h,M}$ cancels in the expectation value because its action on $|\psi\rangle$ introduces a phase that cancels in the expectation value as it is a function only of incoming kinematic invariants. Further, we have explicitly verified this for $\langle \mathbb{h}_{\mu\nu} \rangle$ in processes with two incoming and two outgoing matter particles. More generally, if the initial state includes a gravitational background, then inclusion of $Q_{h,0}$ in this way also introduces a phase that cancels in the expectation value for a similar reason. Since $[S, Q_s + Q_h] = 0$ and $[Q_s, Q_h] = 0$, we may equivalently rewrite Eq. (12), using also Eq. (7),

$$h_{\mu\nu}(x) = \langle \psi | S^{\dagger} e^{-iQ_h} \mathbb{h}_{\mu\nu}(x) e^{iQ_h} S | \psi \rangle + h_{\mu\nu}^{\text{Schw}}(x). \quad (13)$$

Thus, to all orders, the effect of the soft charge is to introduce the linearized Schwarzschild term in the metric.

The effect of the hard charge can also be made explicit in Eq. (13) to all orders. Indeed, using Eq. (10), it is not difficult to check that

$$e^{-iQ_h} a_{\eta}(k) e^{iQ_h} = e^{i\omega T(n)} a_{\eta}(k), \quad (14)$$

where $T(n) \equiv 2G(p \cdot n)[\log(v \cdot n) + Y(\beta, \epsilon)]$ for each matter particle. Here, we set $\omega = k^0$ and $n = k/\omega$ and made use of the relation between the momentum p and the velocity v . Notice that the ratio K^{μ}/K^0 appearing in the definition of the hard charge, Eq. (10), became n^{μ} due to the action of the number operator on $a_{\eta}(k)$.

In view of the u dependence of Eq. (1), the net result is a shift for each matter particle,

$$u \rightarrow u - 2Gmv \cdot n \log v \cdot n, \quad (15)$$

where we absorbed the regulator-dependent $Y(\beta, \epsilon)$ by an appropriate choice of the origin of retarded time. As $n = (1, \hat{x})$ after use of Eq. (1), the shift in Eq. (15) is an angle-dependent translation of the retarded time, i.e., a BMS supertranslation, which is in fact precisely that of Veneziano and Vilkovisky.

Classically, this supertranslation [28] relates the “intrinsic” and “canonical” BMS frames. In the former [28], the metric strain at $\mathcal{O}(G)$ is that of linearized Schwarzschild while in the latter the metric strain is zero [29]. Our result demonstrates that the hard charge generates a large change of coordinates. Thus, while rooted in perturbative (quantum) field theory, our approach makes direct contact with the classical physics of large gauge transformations.

Equation (15) also shows that the Veneziano-Vilkovisky transformation connects the intrinsic and canonical frames to all orders in the coupling. This situation is reminiscent of the Kerr-Schild coordinates, in which all higher-order corrections to the linearized metric are absorbed by a judicious change of coordinates. Although the phase (14) amounts to a simple coordinate shift in the waveform, it is essential, as noted in [11] and further confirmed in [14,15], for quantum field theory [4–7] and multipolar post-Minkowskian [30] results to agree for the one-loop waveform.

As another perspective, notice that the supertranslation of Eq. (15) could be introduced in correlation functions by supertranslating the quantum fields. This echoes the suggestion of Ref. [31] that large gauge effects can be connected to a modified LSZ prescription.

Angular momentum—The leading contribution of static fields, if present, enters the radiated angular momentum at $\mathcal{O}(G^2)$; see End Matter for a brief review. Here we discuss the contribution of these static fields using our approach to all orders in G .

Working in the intrinsic frame, we start by evaluating the expectation value of the angular momentum operator in the future dressed particle state $|\psi\rangle_{\text{out}}$:

$$L_{\mu\nu} \equiv \langle \psi | e^{-iQ_s} S^{\dagger} \mathbb{L}_{\mu\nu} S e^{iQ_s} | \psi \rangle, \quad (16)$$

where $\mathbb{L}_{\mu\nu} = -i \sum_{\eta} \int d\Phi(k) a_{\eta}^{\dagger}(k) k_{[\mu} \overleftrightarrow{\partial}_{\nu]} a_{\eta}(k)$ is the orbital angular momentum operator [32]. We may introduce the

hard charge Q_h as in Eq. (12) and use Eq. (11) to rewrite the orbital angular momentum in terms of the operator $e^{-i(Q_s+Q_{h,0})}a_\eta^\dagger(k)e^{i(Q_s+Q_{h,0})}$ and its conjugate. This form allows us to exploit the relation

$$e^{-i(Q_s+Q_{h,0})}a_\eta^\dagger(k)e^{i(Q_s+Q_{h,0})} = e^{-i\omega T(n)}a_\eta^\dagger(k) + \alpha_\eta^*(k)$$

and its conjugate, where $\alpha_\eta(k) \equiv \hat{\delta}(2p \cdot k)\mathcal{A}(p+k|p, k_\eta)$.

There are now three classes of terms in $L_{\mu\nu}$, tagged by powers of α . All terms with two powers of α vanish after integration over k because the observable angular momentum tensor must be antisymmetric in μ and ν but can only be built from the velocity vector (and the metric). The linear-in- α terms contain the $\mathcal{O}(G^2)$ contribution to the angular momentum flux noted in [2,33,34]. Note that the phase $i\omega T(n)$ drops out of these terms due to the delta function in the definition of α_η . We conclude that the hard charge does not contribute to these terms.

Turning to the α -independent terms, the hard charge yields an additional contribution to the orbital angular momentum flux since

$$e^{-i\omega T(n)}k_{[\mu}\overset{\leftrightarrow}{\partial}_{\nu]}e^{i\omega T(n)} = 2iGk_{[\mu}p_{\nu]}\log(v \cdot n).$$

Overall, we find that the hard charge contributes to the orbital angular momentum flux as

$$L_{\mu\nu}^{\text{hard}} = 2G \sum_{X,h} \int d\Phi(k, \tilde{p}_1, \tilde{p}_2, X) \langle \psi | S^\dagger a_\eta^\dagger(k) | \tilde{p}_1, \tilde{p}_2, X \rangle \times \langle \tilde{p}_1, \tilde{p}_2, X | a_\eta(k) S | \psi \rangle k_{[\mu}p_{\nu]} \log(v \cdot n), \quad (17)$$

where we have inserted a complete set of intermediate states. Noting that each of the matrix elements above contributes at $\mathcal{O}(\kappa^3)$, we see that the leading term of $L_{\mu\nu}^{\text{hard}}$ is $\mathcal{O}(G^4)$. Indeed, this contribution is known to arise from static fields [18] and is associated with the nonlinear memory effect [35–37].

Connection to classical treatment—The classical charges are expectation values of the quantum operators. To make the connection with the classical BMS transformation more transparent, we will reorganize the frequency integrals in Eqs. (4) and (10) as integrals over the retarded time and also expose the parameter $T(n)$ of the transformation.

Focusing on the most familiar case of gravity and using the amplitude (5), the soft charge (4) operator is

$$Q_s = - \sum_\eta \frac{1}{4} \int \hat{d}\omega \hat{d}^2\Omega \omega \hat{\delta}_+(\omega) \frac{\kappa e_\eta^{\mu\nu} p_\mu p_\nu}{n \cdot p} a_\eta(\omega n) + \text{H.c.},$$

where, as before, $n = k/\omega = [1, \mathbf{n}(\theta, \phi)]$ is a general null vector. We denote the polar angles θ and ϕ as n^A .

It is useful to choose a basis of polarization vectors [38] $\epsilon_A^\mu = \partial_A n^\mu = D_A n^\mu$ for $A = 1, 2$, where D_A is the covariant

derivative on the (unit) sphere. These vectors have the property that $\epsilon_A \cdot \epsilon_B = -\Omega_{AB}$, where Ω_{AB} is the unit metric on the two sphere. In this basis the graviton polarization tensor is $e_{\mu\nu}^\eta = e_{AB}^\eta \epsilon_\mu^A \epsilon_\nu^B$. (The A, B indices are raised and lowered using Ω .) Tracelessness of $e_{\mu\nu}^\eta$ implies that $e_{AB}^\eta \Omega^{AB} = 0$. The soft charge then becomes

$$Q_s = -\frac{1}{4} \int \hat{d}^2\Omega \kappa \frac{D_A(n \cdot p) D_B(n \cdot p)}{n \cdot p} \times \sum_\eta \int \hat{d}\omega \omega \hat{\delta}_+(\omega) a_\eta(\omega n) e_\eta^{AB} + \text{H.c.} \quad (18)$$

We proceed by invoking the equality

$$e_\eta^{AB} D_A D_B T(n) = 2G e_\eta^{AB} \frac{D_A(n \cdot p) D_B(n \cdot p)}{n \cdot p}, \quad (19)$$

which relies on $D_A D_B n = -\Omega_{AB} n$ and tracelessness of e_η^{AB} . This allows us to expose the parameter $T(n)$ in the first integral in Eq. (18). For the ω integral, we define a shear operator (of dimension length) by

$$\mathbb{f}_{AB}(u, \mathbf{n}) = |\mathbf{x}| \kappa \mathbb{h}_{\mu\nu}(x) \epsilon_A^\mu \epsilon_B^\nu, \quad (20)$$

with $\mathbb{h}_{\mu\nu}$ given by the stationary phase approximation in the second Eq. (1). Then, the soft charge becomes

$$Q_s = \frac{1}{8\pi G} \int du d^2\Omega \mathbb{f}^{AB} D_A D_B T(n). \quad (21)$$

To rewrite the hard charge in a similar manner, we note that

$$\sum_\eta \int_0^\infty \hat{d}\omega \omega^2 a_\eta^\dagger(\omega n) a_\eta(\omega n) = \frac{\pi}{4G} \int du : \mathbb{f}_{AB} \dot{\mathbb{f}}^{AB}, \quad (22)$$

using the normal-ordering symbol, and discarding terms involving $\int d\omega \theta(\omega) \theta(-\omega)$. The massless hard charge is

$$Q_{h,0} = \frac{1}{32\pi G} \int du d^2\Omega : \mathbb{f}_{AB} \dot{\mathbb{f}}^{AB} : T(n). \quad (23)$$

The total charge operator at null infinity is $Q_0 = Q_s + Q_{h,0}$. The classical expectation value of this operator in the out-state $S|\psi\rangle$, $Q_{\text{out}} \equiv \langle \psi | S^\dagger Q_0 S | \psi \rangle$ is

$$Q_{\text{out}} = \frac{1}{8\pi G} \int du d^2\Omega T(n) \left(D_A D_B \dot{f}^{AB} + \frac{1}{4} \dot{f}_{AB} \dot{f}^{AB} \right),$$

where the classical shear (in the future) is $f_{AB} \equiv \langle \psi | S^\dagger \mathbb{f}_{AB} S | \psi \rangle$. Here, we assumed that $S|\psi\rangle$ is a classical state so that expectation values of products of operators factorise; see Ref. [39] for an amplitudes-based discussion of these relations. Notice that the forward scattering ($S = S^\dagger = 1$) term in f_{AB} does not vanish but instead

contains the static $\mathcal{O}(G)$ contribution to the shear. However, the charge Q_{out} only depends on $\dot{f}_{AB}$ so it is insensitive to the forward scattering term. This allows us to identify

$$Q_{\text{out}} = \Delta Q_0 \equiv \langle \psi | S^\dagger Q_0 S | \psi \rangle - \langle \psi | Q_0 | \psi \rangle, \quad (24)$$

where we have subtracted the vanishing term $\langle \psi | Q_0 | \psi \rangle$.

To connect this to the classical BMS charges, consider the total variation of the classical supertranslation charge at null infinity

$$\Delta Q_{\text{cl}} \equiv \int du \dot{Q}_{\text{cl}} = - \int du \hat{d}^2 \Omega \dot{m} T(n). \quad (25)$$

Using the Bondi mass-loss formula, which, in our conventions, is

$$\dot{m} = -\frac{1}{8\pi G} \left(D_A D_B \dot{f}^{AB} + \frac{1}{4} \dot{f}_{AB} \dot{f}^{AB} \right), \quad (26)$$

it is easy to see that ΔQ_{cl} coincides with Q_{out} . We emphasise that this equality holds provided that there are no gravitons in the initial state $|\psi\rangle$.

Note that the classical charge in Eq. (25) is defined by a surface integral at null infinity. In particular, it does not include contributions $Q_{h,M}$ from massive particles located at future or past timelike infinity, i^+ or i^- , respectively. Such contributions are accounted for using the boundary-to-bulk propagators introduced in [40–42]. In general BMS conservation laws require that the variation of the total BMS charge vanishes, i.e., $\Delta Q_0 + \Delta Q_{h,M} = 0$, where $\Delta Q_{h,M}$ is the difference of the hard charges of the incoming and outgoing matter particles.

Discussion—The choice of BMS frame has observable consequences, as demonstrated by differences in physical predictions in the intrinsic and canonical frames. Starting with the observation that low-frequency mediators break assumptions built into the large-distance approximation for asymptotic observables, we defined new asymptotic states that resum their contribution. They turn out to be coherent states dressing the usual single-particle asymptotic states. We showed that the argument Q_s of the dressing phase operator can be completed to an operator $Q = Q_s + Q_h$ that commutes with the scattering operator to all orders in the coupling, at least in the classical (large mass) limit.

Using this conservation we found the exact contribution of the coherent state of low-frequency mediators to the asymptotic radiation field, observing the close relation between the linearized Schwarzschild background and a particular BMS supertranslation in the far future. We strengthened this connection by explicitly showing that the final-state expectation value of the conserved operator Q equals the (variation of the) classical BMS charge. This also makes contact with discussions of BMS symmetry of the S matrix from a general relativity perspective,

e.g., [43–48], and its connection to the Faddeev-Kulish formalism [49,50] while effectively solving the supertranslation Ward identity for final-state observables.

It would be interesting to understand how to include general BMS supertranslations in the on shell formalism, perhaps by extending the expression for the soft charge to a general function of angles and constructing the corresponding hard charge. Similarly, it remains an interesting problem for the future to understand whether analogous results hold for superrotations [51–53]. The corresponding Ward identities have been discussed in [54,55] and their connection to classical soft theorems in [56–60].

We have also evaluated the contribution of the low-frequency mediators to the orbital angular momentum radiated in a scattering process and extended fixed-order known results to all orders in the coupling.

While our main focus was on gravitationally interacting massive particles, our analysis carries over, with minimal modifications, to other types of mediators. It would be interesting to explore the scalar charge model [61,62], in which one of the massive scalars interacts with gravity through a massless scalar [63].

One aim of our work was to include low-frequency mediators, which spoil the large $\omega|x|$ assumption of the large-distance approximation. As mentioned, an alternative approach to this problem is to have the asymptotic states created and destroyed at finite distance as well as a finite IR cutoff for the mediator’s frequency. At large but finite distance (when $1/|x|^2$ terms in the metric are relevant), BMS transformations are no longer expected to be symmetries of the metric [29], and, thus, our analysis does not apply. Exploring the relation between this and our approach should shed light on the interpretation of BMS transformations away from null infinity.

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End Matter

The derivation of the hard-charge components—In this End Matter we describe the derivation of the matter and mediator hard charges, Eqs. (9) and (10).

Focusing first on C_M , our assumptions imply that the massive particle is present in both the initial and final state, and therefore there are two related amplitudes of interest, shown in Fig. 2. It is convenient to discuss them together. Because of the soft limit, the S matrix element associated with the left diagram in Fig. 2 is, to all orders in the coupling,

$$-i \sum_{\eta} \int d\Phi(k, P, P') \frac{H(P, P') \mathcal{A}(P, k_{\eta} | P+k)}{2P \cdot k + i\epsilon} a_{\eta}(k), \quad (\text{A1})$$

where $H(P, P')$ refers to the generic “hard” matrix element with all external particles other than the soft mediator, and we have explicitly written the annihilation operator of the soft particle. Notice that $\mathcal{A}_3$ appearing here is more precisely a soft factor; however, at the leading order in ω , the difference is negligible. Similarly, to this order, we have neglected the difference between $H(P+k, P')$ and $H(P, P')$ because k is subleading in frequency (and $\hbar$). In the same way, the S -matrix element associated with the right diagram is

$$i \sum_{\eta} \int d\Phi(k, P, P') \frac{H(P, P') \mathcal{A}(P'-k, k_{\eta} | P')}{2P' \cdot k - i\epsilon} a_{\eta}(k). \quad (\text{A2})$$

Thus, the two contributions to C_M contain the sums

$$\begin{aligned} X &= \sum_{\eta} \mathcal{A}(P, k_{\eta} | P+k) \mathcal{A}(p+k | p, k_{\eta}), \\ X' &= \sum_{\eta} \mathcal{A}(P'-k, k_{\eta} | P') \mathcal{A}(p+k | p, k_{\eta}), \end{aligned} \quad (\text{A3})$$

which are the same to leading order in the $\hbar$ expansion but differ at the next-to-leading order. Their common leading-order value contributes to Eq. (8) as

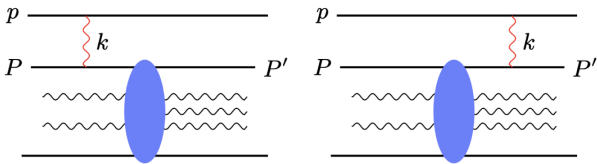


FIG. 2. A soft mediator (red) arising from dressing a charged particle of momentum p may attach to another particle of momentum P , leading to two Feynman diagrams of this general type. Several massive particles could be present, and, in principle, any number of incoming or outgoing mediators. Note that the case $p = P$ may arise.

$$\begin{aligned} C_M &= \int d\Phi(k, P', P) \hat{\delta}(2p \cdot k) X[iH(P, P')] \\ &\times \left(\frac{1}{2P' \cdot k - i\epsilon} - \frac{1}{2P \cdot k + i\epsilon} \right). \end{aligned} \quad (\text{A4})$$

To obtain a classical result we must expand to one subdominant order in $\hbar$. Using momentum conservation for the configuration in Fig. 2, and that both $q = P' - P - k$ and k are $\mathcal{O}(\hbar)$, the dominant term is proportional to

$$\int d\Phi(k) \hat{\delta}(2p \cdot k) \hat{\delta}(2P \cdot k). \quad (\text{A5})$$

This integral vanishes [14]. Similar integrals with fewer cut matter lines also occur at the next order in the $\hbar$ expansion and in the calculation of C_0 . We will uniformly regulate them by using their scaling properties to set to unity the timelike component of the external momentum P not entering the cut condition and shifting the one that does, p , as $p \rightarrow p + m\beta P/P^0$ in the quadratic form $(p+k)^2 = m^2$ of the cut condition. The deformed condition is $2p \cdot k + 2m\beta p \cdot P/P^0 = 0$. β is a regulator to be set to zero at the end. Terms proportional to $P \cdot k$ are dropped because they either vanish or yield scaleless integrals. Terms $\mathcal{O}(\beta^2)$ are dropped because $\beta \rightarrow 0$. With this regularization, $\hat{\delta}(2P \cdot k)$ in Eq. (A5) has no support, so the integral vanishes.

The classical contribution due to the difference between X and X' in Eq. (A3) is proportional to

$$I(p, P) = \int d\Phi(k) \frac{\hat{\delta}(2p \cdot k)}{2P \cdot k - i\epsilon}, \quad (\text{A6})$$

while that of Eq. (A5) is proportional to

$$\int d\Phi(k) \frac{q \cdot k \hat{\delta}(2p \cdot k)}{(2P \cdot k - i\epsilon)^2} = -\frac{1}{2} q \cdot \frac{\partial}{\partial P} I(p, P). \quad (\text{A7})$$

It is not difficult to see that $[Z = \sqrt{(p \cdot P)^2 - p^2 P^2}]$,

$$I(p, P) = -\frac{\pi}{2Z} \ln \frac{p \cdot P + Z}{p \cdot P - Z}. \quad (\text{A8})$$

The case $p = P$, corresponding to a zero-energy graviton starting and ending on the same matter line, can either be obtained as the limit $p \rightarrow P$ of $I(p, P)$ or independently. Care is needed in the latter approach to account for a singular propagator regularized by the β deformation.

Thus, the matter contribution C_M to the commutator $[S, Q_s]$ can be written as

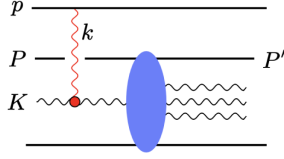


FIG. 3. In gravity and Yang-Mills theory, a soft mediator (red) may attach to external mediators.

$$C_M = -[S, Q_{h,M}], \quad Q_{h,M} = \sum_i Q_{h,i} a_i^\dagger a_i, \quad (\text{A9})$$

where the sum is over all the matter particles. $Q_{h,i}$ is a linear combination of $I(p, P)$ and its derivatives and includes a sum over all the matter particles. This is the result quoted in Eq. (9).

In Yang-Mills theory and in gravity, soft modes may also interact with external mediators (gauge boson or gravitons). Therefore, we must consider the additional contributions to C of the type shown in Fig. 3. Unlike massive particles, there is no requirement for massless mediators to appear both in initial and final states. Moreover, since Q_s is linear in $a_\eta^\dagger$, we can focus on one external mediator at a time, bearing in mind that we must sum over *all* external gravitons *and all* external matter particles. We find, in gravity,

$$C_0 = - \int d\Phi(K) iH(K) \int d\Phi(k) \hat{\delta}(2p \cdot k) \frac{\kappa^2(K \cdot p)^2}{2K \cdot k + i\epsilon}.$$

The soft integral does not vanish after the β deformation [14], $p \rightarrow p + m\beta K/K^0$, and has both finite and divergent terms. Denoting the latter by $Y(\beta, \epsilon)$ where $\epsilon = (4 - d)/2$ is the dimensional regulator, and making use of its momentum (in)dependence, the result for the commutator

of the S matrix and the soft charge is

$$C_0 = 2G \int d\Phi(K) [iH(K)] p \cdot K \left[\log\left(\frac{p \cdot K}{mK^0}\right) + Y \right] = -[S, Q_{h,0}]. \quad (\text{A10})$$

On the second line, we defined the hard charge $Q_{h,0}$,

$$Q_{h,0} = 2G \sum_\eta \int d\Phi(K) p \cdot K \times \left[\log\left(\frac{p \cdot K}{mK^0}\right) + Y(\beta, \epsilon) \right] a_\eta^\dagger(K) a_\eta(K). \quad (\text{A11})$$

A sum over matter particles is implied. This is the result quoted in Eq. (10).

J_{rad} vs BMS frame—a brief literature review—It is well known that the radiated angular momentum suffers from a BMS ambiguity related to the contribution of the static field [2,66–78]. This issue has been sharpened [18,28,33,34,79] in the context of the program of computing radiative observables from scattering amplitudes. Static contributions in the intrinsic BMS frame lead to a perturbative enhancement of the angular momentum flux relative to the power counting of radiative amplitudes. No such contributions occur in the canonical BMS frame [80] since the shear is zero, consistent with the expected power counting from amplitudes. The leading $\mathcal{O}(G^2)$ to the angular momentum flux originally found in Ref. [33] was recovered in Ref. [2] by resumming the contribution of zero-frequency mediators into an eikonal operator [81], which is then used to evaluate expectation value of the angular momentum operator.