

Universality of Rényi Entropy in Conformal Field Theory

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We use the thermal effective theory to prove that, for the vacuum state in any conformal field theory in d dimensions, the n th Rényi entropy $S_A^{(n)}$ behaves as $S_A^{(n)} = [f/(2\pi n)^{d-1}][\text{Area}(\partial A)/(d-2)\epsilon^{d-2}](1 + O(n))$ in the $n \rightarrow 0$ limit when the boundary of the entanglement domain A is spherical with the UV cutoff ϵ . The theory dependence is encapsulated in the cosmological constant f in the thermal effective action. Using this result, we estimate the density of states for large eigenvalues of the modular Hamiltonian for the domain A . In two dimensions, we can use the hot spot idea, which describes the effective action in the high-temperature limit when the temperature is position-dependent, to derive more powerful formulas valid for arbitrary positive n . We discuss the difference between two and higher dimensions and clarify the applicability of the hot spot idea. We also use the thermal effective theory to derive an analog of the Cardy formula for boundary operators in higher dimensions.

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Introduction and summary—In conformal field theory (CFT) there are universal formulas that describe how physical quantities depend on the theory. Arguably, the most famous example is the Cardy formula for 2D CFT [1], which estimates the density of states $\rho(\Delta)$ as a function of the conformal dimension Δ as

$$\int_{\Delta-\delta}^{\Delta+\delta} d\Delta' \rho(\Delta') = \left(\frac{c}{48\Delta^3}\right)^{1/4} \exp\left(2\pi\sqrt{\frac{c}{3}}\Delta + O(1)\right), \quad (1)$$

where c is the central charge of the theory. The formula is valid for $\Delta \gg c$ if we choose $\delta = O(1)$ [2,3].

The method of thermal effective theory introduced in [4–6] and further developed in [7,8] turned out to have powerful applications to many problems in CFT including the generalization of the Cardy formula to higher dimensions [4,9], the supersymmetric indices [10], the asymptotic properties of (heavy)³ operator product expansion (OPE)

coefficients [7], the internal and spatial symmetry resolutions of partition functions [8,11], their subleading terms [12], and effects of boundaries and defects [13–16]. Compactifying d -dimensional CFT on the thermal circle $\mathbb{S}_\beta^1$ at temperature β^{-1} , the high-temperature limit $\beta^{d-1} \ll \text{vol}(\Sigma_{d-1})$ is described by a quantum field theory on Σ_{d-1} with a mass gap $m_{\text{gap}} = 1/\beta$. It is convenient to use the Kaluza-Klein (KK) ansatz for the d -dimensional metric $G_{\mu\nu}$ on $\mathbb{S}_\beta^1 \times \Sigma_{d-1}$ as

$$G_{\mu\nu} dx^\mu dx^\nu = g_{ij}(\vec{x}) dx^i dx^j + e^{2\phi(\vec{x})} (d\tau + A_i(\vec{x}) dx^i)^2 = e^{2\phi(\vec{x})} [\hat{g}_{ij}(\vec{x}) dx^i dx^j + (d\tau + A_i(\vec{x}) dx^i)^2], \quad (2)$$

where τ is the Euclidean time with the period 1 and $\hat{g}_{ij} = e^{-2\phi} g_{ij}$ is the Weyl-invariant metric on Σ_{d-1} . Because of the conformal invariance in d dimensions, the thermal partition function and related physical observables depend on the KK scalar ϕ only through the Weyl anomaly [7] and do not contribute to the leading terms in the $\beta \rightarrow 0$ limit, which we study in this Letter. Thus, we can safely ignore the ϕ dependence. The rest of the low energy effective action can be expanded in the power of derivatives as

$$S[\hat{g}, A] = \int d^{d-1}x \sqrt{\hat{g}} (-f + c_1 \hat{R} + c_2 F^2 + \cdots), \quad (3)$$

where $\hat{R}$ is the scalar curvature for the metric $\hat{g}_{ij}$, and F_{ij} is the field strength of the KK gauge field, $F_{ij} = \partial_i A_j - \partial_j A_i$.

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The first term measures the volume of Σ_{g-1} . When $A_i = 0$, it gives

$$\int d^{d-1}x \sqrt{\hat{g}} = \beta^{1-d} \text{vol}(\Sigma_{d-1}). \quad (4)$$

The cosmological constant f multiplying it has several important physical interpretations as explained in [7]. When $d = 2$, the Cardy formula (1) shows that f is given by the central charge c as $f = 2\pi c/12$.

The thermal effective theory is also useful when the temperature β^{-1} is spatially dependent. Under certain conditions, which we will clarify in this Letter, leading contributions to the thermal effective action come from the regions where the local temperature is high, and we only need to evaluate the integral of the Lagrangian density in these regions. In [7], this “hot spot idea” was developed and successfully applied to derive the universal formula for OPE coefficients.

In this Letter, we use the thermal effective theory to study properties of the Rényi entropy defined by

$$S_A^{(n)} = \frac{1}{1-n} \log \text{Tr} \rho_A^n, \quad (5)$$

where ρ_A is the reduced density matrix for a subsystem A . In [17], it was conjectured that the Rényi entropy has the universal behavior in the $n \rightarrow 0$ limit as

$$S_A^{(n)} \sim \frac{f}{n^{d-1}} g(A), \quad (6)$$

where the dependence of the CFT is encapsulated in the cosmological constant f , and $g(A)$ depends only on the geometry of the subsystem A . In this Letter, we recast the argument in Sec. 3.1 of [17] in the language of the thermal effective theory, which allows us to justify their use of the position-dependent temperature, to take into account boundary and curvature effects, and to estimate subleading terms in the small n expansion. While [17] conjectured the leading behavior of Rényi entropy based on physical reasoning, and semiclassical phase space picture, our contribution is to provide a systematic and controlled derivation using the thermal effective field theory framework. Specifically, we justify the use of position-dependent temperature rigorously, include boundary and curvature effects explicitly, and estimate subleading corrections in the small n expansion, which are crucial for understanding the domain of validity of the result. In more detail, the subleading correction from the boundary of the entanglement domain is suppressed by one power of n , as shown in Eq. (24), and the bulk curvature correction by two powers [see Eq. (17)].

In the next section, we verify the universal behavior (6) when the boundary of A is spherical and determine $g(A)$ when $\epsilon^{d-2} \ll \text{Area}(\partial A)$ as

$$g(A) = \frac{\text{Area}(\partial A)}{(2\pi)^{d-1}(d-2)\epsilon^{d-2}}. \quad (7)$$

In our third section, we generalize the thermal effective theory to include effects of boundaries and estimate subleading terms to (6) in the small n expansion as

$$S_A^{(n)} = \frac{f}{n^{d-1}} g(A) + O\left(\frac{1}{n^{d-2}}\right), \quad n \ll 1. \quad (8)$$

Note that the boundary thermal effective theory developed here is highly versatile, and one example of its application is presented in Appendix A. The $n \rightarrow 0$ limit is useful since it helps us extract information about the asymptotic behavior of the entanglement spectrum. In the next section, “Entanglement spectrum,” we use the universal formulas (7) and (8) to derive

$$d_\lambda \sim \frac{1}{\sqrt{2\pi d}} \left(\frac{d-1}{d-2} \frac{f}{\lambda^{d+1}} \frac{\text{Area}(\partial A)}{\epsilon^{d-2}} \right)^{\frac{1}{2d}} \times \exp \left[\frac{d}{(d-2)^{\frac{1}{d}}(d-1)^{1-\frac{1}{d}}} \left(\frac{f \lambda^{d-1} \text{Area}(\partial A)}{\epsilon^{d-2}} \right)^{\frac{1}{d}} \right], \quad (9)$$

where d_λ is defined by

$$S_A^{(n)} = \sum_\lambda d_\lambda e^{-2\pi n \lambda}. \quad (10)$$

Furthermore, recent studies in holography [18] have pointed out the existence of new phases at $n < 1$, providing additional motivation to investigate values of n other than 1, including $n = 0$.

In two dimensions, we can use the hot spot idea to derive a result stronger than the combination of Eqs. (7) and (8), which is applicable to arbitrary n as

$$S_A^{(n)} = \frac{c}{6} \left(\frac{1}{n} + 1 \right) \log \frac{|A|}{2\epsilon} + s_a + s_b, \quad (11)$$

where we followed the prescription of [19,20] to impose conformal boundary conditions at the end points a and b of the subsystem A (see Fig. 1) and $s_a = \log \langle 0|a$ is the boundary entropy of a , introduced in [21]. This formula was originally found by using a conformal map to relate $S_A^{(n)}$ to the annulus partition function [20]. The leading term in (11) can also be captured using the twist field approach [22].

One might expect a similar formula for arbitrary n in higher dimensions. However, it turns out that the hot spot idea is not directly applicable in such a case because of the behavior of the effective temperature β^{-1} near the boundary of the entanglement domain, as we will explain in the “Applicability of hot spot idea” section. This does not necessarily mean that (11) does not generalize to higher

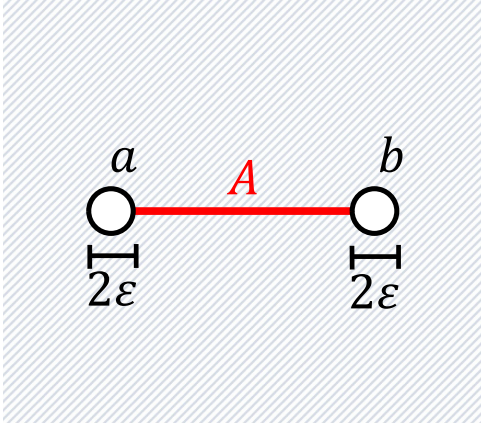


FIG. 1. Regularization prescription in 2D CFT: by inserting a physical boundary of size 2ϵ at the entangling surface, we decompose the CFT Hilbert space.

dimensions, but it suggests that the hot spot idea has a certain limitation. We discuss the necessary conditions for the applicability of the hot spot idea and show that it can be used consistently for the computation in [7] and that the universal formula for OPE coefficients derived there using the hot spot idea remains valid.

Universality of Rényi entropy—Consider the n th Rényi entropy for a spherical entanglement domain A of radius R in a d -dimensional CFT on Minkowski space $\mathbb{R}^{1,d-1}$ with the following metric,

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2, \quad (12)$$

where $d\Omega_{d-2}$ is the metric of the unit transverse sphere $\mathbb{S}^{d-2}$. To calculate the Rényi entropy using the thermal effective theory, it is useful to consider the following coordinate transformation [23],

$$\begin{aligned} t &= R \frac{\sinh(\tau/R)}{\cosh u + \cosh(\tau/R)}, \\ r &= R \frac{\sinh u}{\cosh u + \cosh(\tau/R)}. \end{aligned} \quad (13)$$

After the Euclidean rotation, $\tau \rightarrow -i\tau$, the metric takes the following form,

$$ds^2 = \Xi^2 (d\tau^2 + R^2 (du^2 + \sinh^2 u d\Omega_{d-2}^2)), \quad (14)$$

where $\Xi = (\cosh u + \cosh(\tau/R))^{-1}$ and the periodicity in τ is $2\pi Rn$. By removing the factor Ξ by the Weyl invariance and rescaling τ so that its period is 1, the CFT is defined on $\mathbb{R} \times \mathbb{H}^{d-1}$ with the metric in the KK form (2),

$$ds^2 = d\tau^2 + \frac{1}{(2\pi n)^2} (du^2 + \sinh^2 u d\Omega_{d-2}^2). \quad (15)$$

To define the reduced density matrix, it is necessary to decompose the Hilbert space of the CFT into the one over A

and the one over its complement $\bar{A}$. Here, we perform the decomposition by introducing a physical boundary between the subsystem A and $\bar{A}$, placing the cutoff surface at $r = R - \epsilon$. In the hyperbolic coordinates, this cutoff surface is located at $u = \log(2R/\epsilon)$ in $\mathbb{H}^{d-1}$, to the leading order in (R/ϵ) . In other words, the cutoff corresponds to a regularization of the infinite volume of $\mathbb{H}^{d-1}$.

We now consider the thermal effective theory on $\mathbb{S}_\beta^1 \times \mathbb{H}^{d-1}$. Since the mass gap β^{d-1} is much smaller than the regularized volume of $\mathbb{H}^{d-1}$, the thermal effective theory can be described by a local action. In particular, we can calculate the Rényi entropy by the thermal effective action (3). Here, the inverse temperature β is proportional to the replica number n . Expanding in the power of n , the leading term is

$$-f \int d^{d-1}x \sqrt{\hat{g}} = \frac{f}{(2\pi n)^{d-1}} \frac{\text{Area}(\partial A)}{(d-2)\epsilon^{d-2}}, \quad (16)$$

in the limit of $n \rightarrow 0$ and then $\epsilon^{d-2} \ll \text{Area}(\partial A)$, and the subleading bulk term

$$c_1 \int d^{d-1}x \sqrt{\hat{g}} \hat{R} = -\frac{(d-1)c_1}{(2\pi n)^{d-3}} \frac{\text{Area}(\partial A)}{\epsilon^{d-2}}, \quad (17)$$

is suppressed by n^2 compared to the leading term. In these and later equations, we only exhibit the leading terms in (R/ϵ) since ϵ is the UV cutoff. Note that the powers of (R/ϵ) are the same for both, which will become important when we discuss the applicability of the hot spot idea in the section so titled.

There are also contributions from the boundary at $u = \ln(2R/\epsilon) + O(\epsilon)$. As we will see in “Contribution from Boundary of Entanglement Domain” section, the leading boundary contribution is of the order $1/n^{d-2}$, suppressed by one power of n compared to the leading bulk term (16). Thus, the $n \rightarrow 0$ limit of the Rényi entropy is given by the following universal formula including the estimate of the leading correction:

$$S_A^{(n)} = \frac{f}{(2\pi n)^{d-1}} \frac{\text{Area}(\partial A)}{(d-2)\epsilon^{d-2}} + O\left(\frac{1}{n^{d-2}}\right). \quad (18)$$

So far, we have considered the case when the boundary of A is spherical. It is natural to conjecture that (18) generalizes to arbitrary shapes of A as far as it is connected. A possible approach may be to deform the Weyl-invariant metric $\hat{g}_{ij}$ and use the thermal effective action to understand the responses of the Rényi entropy [24] and other physical quantities. The extension of the map (13) to small deformations of spherical entangling surfaces, as developed in [24–27], appears promising for this purpose. We hope to report this in a future extended publication.

In two dimensions, the polynomial divergence in (16) is replaced by a logarithmic divergence,

$$-f \int d^{d-1}x \sqrt{\hat{g}} = -\frac{f}{\pi n} \log \frac{|A|}{2\epsilon} = -\frac{c}{6n} \log \frac{|A|}{2\epsilon}. \quad (19)$$

Including the estimate of the boundary terms, we obtain

$$S_A^{(n)} = \frac{c}{6n} \log \frac{|A|}{2\epsilon} + O(1), \quad n \ll 1. \quad (20)$$

This reproduces with the $n \rightarrow 0$ limit of the universal formula (11) for the Rényi entropy in two dimensions. Note that the denominator appears as 2ϵ instead of ϵ because, to match the convention used in higher dimensions, we follow the convention shown in Fig. 1, where the boundary size is taken to be 2ϵ rather than ϵ , which is the standard convention in 2D CFT.

The universal formula (18) can be verified for holographic CFT, where the cosmological constant f in the thermal effective action is given [7] as $f = [(4\pi l_{\text{AdS}})^{d-1}/4G_N d^d]$. See Appendix B for details.

We remark that if CFT has a global symmetry G , we can turn on the gauge field and use the method developed in [8,28,29] to obtain the symmetry resolution of the Rényi entropy [30–33] as

$$\lim_{n \rightarrow 0} (S_A^{(n)}(r) - S_A^{(n)}) = \log \frac{d_r^2}{|G|}, \quad (21)$$

where $S_A^{(n)}(r)$ is the n th Rényi entropy within the irreducible representation r , d_r is the quantum dimension of r , and $|G|$ is the order of the group G . Our result (21) is the first step toward the higher dimensional generalization of the results for 2D CFTs [34], where it was shown that the equation holds for arbitrary n .

Contribution from boundary of entanglement domain—To regularize the calculation of the Rényi entropy, we insert a physical boundary between the entanglement domain A and its complement $\bar{A}$. First, we need to specify the boundary condition. In 2D CFT, it has been studied in [19,20]. Assuming that the boundaries a and b at the two end points of A (as shown Fig. 1) are conformal, preserving one half of the bulk conformal symmetry, the boundary contributions to the Rényi entropy (11) are given by

$$S_A^{(n)} = \frac{c}{6} \left(1 + \frac{1}{n}\right) \log \frac{|A|}{2\epsilon} + s_a + s_b, \quad (22)$$

where $s_a = \log \langle 0|a \rangle$ is the boundary entropy of a , introduced in [21].

To generalize this to higher dimensions, we need to take into account boundary terms in the thermal effective action,

$$S[\hat{g}, A] = \int d^{d-1}x \sqrt{\hat{g}} (-f + c_1 \hat{R} + c_2 F^2 + \dots) + \int d^{d-2}x \sqrt{\hat{\sigma}} (-g + b_1 \hat{K} + b_2 \hat{R} + \dots), \quad (23)$$

where σ is the induced metric on the boundary and K is the trace of the extrinsic curvature $K_{ab} = \nabla_a n_b$, where n is a unit vector normal to the boundary. The leading contribution from the boundary effective action is

$$-g \int d^{d-2}x \sqrt{\hat{\sigma}} = -\frac{g}{n^{d-2}} \frac{\text{Area}(\partial A)}{(\pi\epsilon)^{d-2}}. \quad (24)$$

This is suppressed by one power of n relative to the bulk leading term (16) as indicated in (18).

Let us examine the boundary leading contribution (24) explicitly in a 2D CFT. Unlike the computation of the bulk leading contribution (19), the boundary leading contribution is given by a zero-dimensional integral, so no logarithmic divergence appears. Thus, the contribution of the boundary to the Rényi entropy is simply given by a constant $2g$. Here, the factor of 2 accounts for the fact that the entangling surface is disconnected in 2D CFT.

In two dimensions, there are only two local terms one can write using Weyl-invariant metric,

$$S[\hat{g}, A] = -f \int d^{d-1}x \sqrt{\hat{g}} - 2g \int d^{d-2}x \sqrt{\hat{\sigma}}. \quad (25)$$

Thus, we obtain in the $\epsilon \rightarrow 0$ limit for arbitrary n ,

$$S_A^{(n)} = \frac{c}{6} \left(1 + \frac{1}{n}\right) \log \frac{|A|}{2\epsilon} + 2g. \quad (26)$$

In this expression, g plays the role of the boundary entropy. Investigating g in various examples of higher-dimensional CFTs is an interesting future direction. In particular, since the holographic dual of the lattice regularization introduced here is provided in [35], it would be intriguing to use this framework to compute g in holographic CFTs.

Entanglement spectrum—The density matrix ρ can be written in terms of modular Hamiltonian K ,

$$\rho = \frac{e^{-2\pi K}}{\text{Tr} e^{-2\pi K}} \quad (27)$$

The eigenvalues of ρ lie in $[0, 1]$. Consequently, the eigenvalues of K lie in $[0, \infty)$. We would like to estimate the density, i.e., degeneracy of low-lying eigenvalues of ρ . They correspond to the large eigenvalues of K . See [34,36,37] for the computation of the density of large eigenvalues of K in two dimensions and [38] for the holographic computation of the same in higher dimensions.

We recall the derivation of the Cardy formula for the high-energy density of states of a CFT on spatial slice $\mathbb{S}^{d-1}$,

from the leading term in the $\beta \rightarrow 0$ expansion of the partition function of the CFT on the manifold $\mathbb{S}_\beta^1 \times \mathbb{S}^{d-1}$; see, for example, [1,7,8,11,13–15]. In other words, the expansion of $\text{Tr} e^{-\beta H}$ in β provides the density of large eigenvalues of H . Now that we have identified the leading term in the n expansion of the n th Rényi entropy, i.e., we know $\text{Tr} e^{-2\pi n K}$ in the $n \rightarrow 0$ limit, we can determine the entanglement spectrum asymptotically, i.e., the large eigenvalues of K , in the same spirit as the Cardy formula.

In particular, using (10), the inverse Laplace transform of (18) in the saddle point approximation gives us the estimate for d_λ for $\lambda \gg 1$ as (see Appendix C for details),

$$d_\lambda \sim \frac{1}{\sqrt{2\pi d}} \left(\frac{d-1}{d-2} \frac{f}{\lambda^{d+1}} \frac{\text{Area}(\partial A)}{\epsilon^{d-2}} \right)^{\frac{1}{2d}} \times \exp \left[\frac{d}{(d-2)^{\frac{1}{d}} (d-1)^{1-\frac{1}{d}}} \left(\frac{f \lambda^{d-1} \text{Area}(\partial A)}{\epsilon^{d-2}} \right)^{\frac{1}{d}} \right]. \quad (28)$$

Applicability of hot spot idea—In two dimensions, the universal formula (11) for the Rényi entropy originates from the $\epsilon \rightarrow 0$ limit, rather than the $n \rightarrow 0$ limit. That is why the formula is applicable to arbitrary n , not just in the $n \rightarrow 0$ limit. Can we find a similar formula in higher dimensions?

Let us consider the following coordinates:

$$ds^2 = d\tau^2 + \frac{1}{(2\pi n \sin \theta)^2} (d\theta^2 + \cos^2 \theta d\Omega_{d-2}^2), \quad (29)$$

where $0 \leq \theta \leq \pi/2$. They are related to the coordinates (15) by the coordinate transformation $\sinh u = \cot \theta$ with a conformal transformation removing a conformal factor. This metric shows the spatial dependence $\beta(\theta) \sim \sin \theta$ of the temperature $1/\beta$, which diverges linearly at $\theta = 0$. It was suggested in [7] that, when the temperature has spatial dependence and diverges at some points, the thermal effective action is dominated by contributions from the regions near the divergence, which they called the “hot spot.”

In the current setup, the hot spot is expected to be in the vicinity of $\theta = 0$ where $\beta = 0$. In order for the hot spot idea to work, the higher-order terms in the effective action must be suppressed in the region. With the spatial dependence of the temperature given by $\beta^2 = e^{-2\phi(\vec{x})}$, where $\phi(\vec{x})$ is given by (2), the Ricci scalar $\hat{R}$ is evaluated as

$$\hat{R} = \beta^2 R + 2(d-2)\beta^2 \nabla^2 \log \beta + (d-3)(d-2)\beta^2 \partial_a \log \beta \partial^a \log \beta. \quad (30)$$

Therefore, higher-order terms are suppressed only if the logarithmic derivatives of β do not diverge too quickly. Indeed, this is the case for the scalar curvature in two dimensions because of the $(d-2)$ factor in the second and

third terms on the right-hand side in the above equation. However, this is not the case in higher dimensions. For example, in three dimensions, $\nabla^2 \log \beta$ diverges as β^{-2} at the hot spot, and the scalar curvature remains finite $\hat{R} = -6$. Thus, the higher-order terms contribute in the same order as the cosmological constant term, and the high-temperature expansion of the thermal effective action fails near the hot spot.

This is related to the fact that both the cosmological term (16) and the Einstein-Hilbert term (17) in the thermal effective action have the same power in (R/ϵ) , as we observed earlier. On the other hand, they have different powers in n , which is why the small n expansion of the thermal effective action is valid.

The fact that the leading contribution to the Rényi entropy for arbitrary n can be universally expressed in terms of the central charge, seems to be a unique property in two dimensions. Our result suggests that, in higher dimensions, the Rényi entropy for general n depends on the details of the theory (see [39] for related comments in the context of holography).

In contrast, in the setup of the study [7], β vanishes quadratically in the coordinate corresponding to θ , and $\beta^2 \nabla^2 \log \beta$ and $\beta^2 \partial_a \log \beta \partial^a \log \beta$ vanishes at the hot spot. Therefore, the hot spot idea can be consistently applied there, and the universal OPE formula discovered in the work remains unaffected by the subtlety we point out here.

Note added—We recently received [40], where the conjecture (6) was tested for holographic CFTs using the AdS/CFT correspondence. The thermal effective theory approach in this Letter is more general and applicable to any CFT, whether or not it is unitary or has a holographic description, as far as the Rényi entropy is well-defined.

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End Matter

Appendix A: Asymptotic density of boundary states—In two dimensions, the comparison of the open and closed string channels for the annulus zero-point function gives the following asymptotic formula for the density of states $\rho(\Delta)$ in the open string Hilbert space with the boundary condition a ,

$$\begin{aligned} & \int_{\Delta-\delta}^{\Delta+\delta} d\Delta' \rho(\Delta') \\ &= \left(\frac{c}{96\Delta^3} \right)^{1/4} \exp \left(2\pi \sqrt{\frac{c}{6}} \Delta + 2s_a + O_\delta(1) \right). \end{aligned} \quad (\text{A1})$$

The above formula is valid in the regime $\Delta \gg c$ and $\delta = O(1)$. The subscript in $O_\delta(1)$ indicates that the error is in the range bounded by $[\log(2\delta - 1), \log(2\delta + 1)]$, which depends on $\delta > 1/2$ [34].

To generalize this to higher dimensions, consider the CFT on $\mathbb{S}_\beta^1 \times \mathbb{S}_+^{d-1}$, where $\mathbb{S}_+^{d-1}$ is a hemisphere with a $\mathbb{S}^{d-2}$ boundary. The first few terms in the β expansion of the effective action are given by

$$S[\hat{g}, A] = -f \frac{\text{vol}(\mathbb{S}^{d-1})}{2\beta^{d-1}} - g \frac{\text{vol}(\mathbb{S}^{d-2})}{\beta^{d-2}} + c_1 \frac{(d-2)(d-1)\text{vol}(\mathbb{S}^{d-1})}{2\beta^{d-3}} + \dots \quad (\text{A2})$$

By performing the inverse Laplace transform with the saddle point approximation, we obtain the higher-dimensional generalization of (A1),

$$\log \rho(\Delta) = f^{\frac{1}{d}} \frac{d}{d-1} \left(\frac{(d-1)\text{vol}(\mathbb{S}^{d-1})}{2} \right)^{\frac{1}{d}} \Delta^{\frac{d-1}{d}} + f^{\frac{2-d}{d}} g \text{vol}(\mathbb{S}^{d-2}) \left(\frac{(d-1)\text{vol}(\mathbb{S}^{d-1})}{2} \right)^{\frac{2-d}{d}} \Delta^{\frac{d-2}{d}} + O(\Delta^{\frac{d-3}{d}}). \quad (\text{A3})$$

We expect many other applications of the thermal effective theory to boundary conformal field theory. An interesting direction may be to use it to generalize the known universality in two dimensions [41,42] to higher dimensions.

*Appendix B: Test of universal formula in holography—*As a test of the universal formula (18), let us consider a holographic CFT, where the cosmological constant f in the thermal effective action is given [7] as

$$f = \frac{(4\pi l_{\text{AdS}})^{d-1}}{4G_N d^d}, \quad (\text{A4})$$

where l_{AdS} is the anti-de Sitter radius, and G_N is the $d+1$ -dimensional Newton constant. The $n \rightarrow 0$ limit of the Rényi entropy is given by

$$S_A^{(n)} = \left(\frac{2l_{\text{AdS}}}{n} \right)^{d-1} \frac{\text{Area}(\partial A)}{4G_N d^d (d-2) \epsilon^{d-2}} + O\left(\frac{1}{n^{d-2}} \right). \quad (\text{A5})$$

This result agrees with the holographic computation of Rényi entropy based on [40,43]. This can also be regarded as a consistency check of (A4) (We can perform a scheme independent check by comparing coefficients of the logarithmic divergences in even dimensions.)

*Appendix C: Details of entanglement spectrum—*To extract the entanglement spectrum, we consider (18) and rewrite it as

$$S_A^{(n)} = \sum_{\lambda} d_{\lambda} e^{-2\pi n \lambda} = \exp \left[\frac{f}{(2\pi n)^{d-1}} \frac{\text{Area}(\partial A)}{(d-2) \epsilon^{d-2}} + O\left(\frac{1}{n^{d-2}} \right) \right]. \quad (\text{A6})$$

Here, the sum over λ on the right-hand side of the first line runs over the eigenvalues λ of K and d_{λ} is the degeneracy of the eigenvalue λ .

We extract the entanglement spectrum from the asymptotic form of $S_A^{(n)}$ given in (A6) by performing an inverse Laplace transform, which gives

$$d_{\lambda} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dy \left(\sum_{\lambda'} d_{\lambda'} e^{-y\lambda'} \right) e^{y\lambda} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dy S_A^{(n=\frac{y}{2\pi})} e^{y\lambda}. \quad (\text{A7})$$

Because of the factor $e^{y\lambda}$, the large λ asymptotics of d_{λ} is expected to come from the region near $y = 0$ in the saddle point approximation, where we can use (A6) for $S_A^{[n=(y/2\pi)]}$. The saddle is at

$$\lambda_* = f \frac{(d-1)\text{Area}(\partial A)}{(d-2)\epsilon^{d-2}} y^{-d}$$

and gives us the estimate for d_{λ} for $\lambda \gg 1$ as

$$d_{\lambda} \sim \frac{1}{\sqrt{2\pi d}} \left(\frac{d-1}{d-2} \frac{f}{\lambda^{d+1}} \frac{\text{Area}(\partial A)}{\epsilon^{d-2}} \right)^{\frac{1}{2d}} \times \exp \left[\frac{d}{(d-2)^{\frac{1}{d}} (d-1)^{1-\frac{1}{d}}} \left(\frac{f \lambda^{d-1} \text{Area}(\partial A)}{\epsilon^{d-2}} \right)^{\frac{1}{d}} \right]. \quad (\text{A8})$$

We can express this limit more precisely if we assume analyticity in n . In particular, we can use Ingham's theorem [44] (see Theorem IV.21.1 in [45] for a textbook exposition) to obtain the following:

$$\sum_{\lambda=0}^{\Lambda} d_{\lambda} = \frac{1}{\sqrt{2\pi}} \left(\frac{d-2}{(d-1)d^{d-1}} \frac{\epsilon^{d-2}}{f \Lambda^{d-1} \text{Area}(\partial A)} \right)^{\frac{1}{2d}} \times \exp \left[\frac{d}{(d-2)^{\frac{1}{d}} (d-1)^{1-\frac{1}{d}}} \left(\frac{f \Lambda^{d-1} \text{Area}(\partial A)}{\epsilon^{d-2}} \right)^{\frac{1}{d}} + O(\Lambda^{\frac{d-2}{d}}) \right] \times (1 + o(1)). \quad (\text{A9})$$

This is consistent with the naive formula (A8) for the asymptotic spectrum when integrated against a window function supported on $[0, \Lambda]$ in the $\Lambda \rightarrow \infty$ limit. A similar result has been obtained in the context of the Cardy formula in Appendix C of [46] and in [2].