

Yu and Tong Reply: In Ref. [1], we addressed the issue of how to separate an evolution operator into holonomic and dynamic parts. We proved that the evolution operator $\hat{U}(t)$ for the subspace spanned by ℓ orthonormal states $\{|\psi_i(t)\rangle\}_{i=1}^{\ell}$ can always be separated into the product,

$$\hat{U}(t) = \hat{\Gamma}(t)\hat{D}(t), \quad (1)$$

where

$$\hat{\Gamma}(t) = \mathcal{P} \exp \left[\int_0^t \hat{P}(\tau) d\tau \right] P(0), \quad (2)$$

$$\hat{D}(t) = P(0)\bar{T} \exp \left[\int_0^t \hat{F}(\tau) d\tau \right], \quad (3)$$

with $P(t) = \sum_{i=1}^{\ell} |\psi_i(t)\rangle \langle \psi_i(t)|$, $\hat{F}(t) = \sum_{i,j=1}^{\ell} F_{ij}(t) \times |\psi_i(0)\rangle \langle \psi_j(0)|$, and $F_{ij}(t) = \langle \psi_i(t) | \dot{\psi}_j(t) \rangle$. Clearly, $\hat{\Gamma}(t)$ is only dependent on the path of the evolution but independent of the evolution details, referred to as the holonomy operator, while $\hat{D}(t)$ depends on the dynamical details of the evolution, referred to as the dynamic operator. Particularly, if the subspace evolves cyclically with period T , then the evolution operator at time T has the matrix representation $U(T) = \Gamma(T)D(T)$ with $\Gamma(T) = \mathcal{P} \exp[\int_0^T A(t)dt]$ and $D(T) = \bar{T} \exp[\int_0^T F(t)dt]$. Note that $\Gamma(T)$ is just the nonadiabatic non-Abelian geometric phase defined by Anandan in Ref. [2].

This is our main result. It unifies the representations of all four types of evolution concerning the adiabatic/non-adiabatic Abelian/non-Abelian geometric phase, and all known decomposition results on geometric and dynamical parts can be derived from our result as special cases.

Here, to confirm that our result is correct and refute the misconceptions appearing in the comment [3], we have the following remarks.

First, in the Letter [1], every step of our proofs is rigorous and all conclusions are derived from these rigorous arguments. Thus, the decomposition of $\hat{U}(t)$ in the form of Eq. (1) works for any $\hat{U}(t)$, and both $\hat{\Gamma}(t)$ and $\hat{D}(t)$ admit closed expressions as shown in Eqs. (2) and (3), which we believe the authors of the comment should also acknowledge.

Second, without identifying any errors in our paper and disregarding the facts conveyed by formulas (1)–(3), the authors of the comment [3] assert that our result was false, claiming that Eq. (1) does not constitute a general separation of $\hat{U}(t)$. Their argument is that $\hat{D}(t)$ depends on $|\psi_i(t)\rangle$ while $|\psi_i(t)\rangle$ depends on $\hat{U}(t)$, and therefore, $\hat{D}(t)$

contains contributions from the holonomy operator via $\hat{U}(t)$. Such an argument is completely untenable. In fact, for any geometric phase theory, the dynamic phase must depend on $\hat{U}(t)$. By definition, the geometric phase only depends on the path information, thus, for any decomposition of $\hat{U}(t)$, the remaining dynamic part must depend on $\hat{U}(t)$, either explicitly or implicitly. This is a common feature of any geometric phase theory, including the adiabatic/nonadiabatic Abelian/non-Abelian geometric phase. The only difference is that the expression of $\hat{D}(t)$ can be further simplified for the adiabatic evolution or one-dimensional evolution while Eq. (3) is a general expression of $\hat{D}(t)$ applying to all evolutions. For example, the well-known Aharonov-Anandan phase [4] reads

$$\gamma = \arg \langle \psi(0) | \psi(T) \rangle - \text{Im} \int_0^T \langle \psi(t) | \dot{\psi}(t) \rangle dt, \quad (4)$$

which corresponds to the one-dimensional cyclic evolution,

$$\langle \psi(0) | \hat{U}(T) | \psi(0) \rangle = e^{i\gamma} e^{\int_0^T \langle \psi(t) | \dot{\psi}(t) \rangle dt}. \quad (5)$$

Clearly, the dynamical phase factor $e^{\int_0^T \langle \psi(t) | \dot{\psi}(t) \rangle dt}$ depends on the evolution of the state $|\psi(t)\rangle$. Note that Eq. (5) is just the matrix representation of formula (1), expressed as Theorem 2 in Ref. [1] upon setting $\ell = 1$.

Third, our result is by no means “in conflict with prior work by Anandan.” As we already mentioned in Ref. [1], by starting from $U(t)$ defined by Anandan in Ref. [2], it is generally impossible to separate the holonomic part from the dynamic part. This is because $U(t)$ is determined by Eq. (4) of Ref. [1]. However, when we use $\hat{U}(t)$ defined by Eq. (5) of Ref. [1] instead of $U(t)$, we find that $\hat{U}(t)$ satisfies Eq. (6) of Ref. [1], which naturally leads to a separation of the holonomic and the dynamic parts. Note that $U(t)$ defined by Anandan is not the evolutionary operator but only the transformation matrix between the basis and the states while $\hat{U}(t)$ used by us is just the evolutionary operator. Anandan’s work indicates that the transformation matrix $U(t)$ cannot be written as the product of the parts generated by $A(t)$ and $K(t)$ while our work shows that the evolution operator $\hat{U}(t)$ can always be separated into the product $\hat{\Gamma}(t)\hat{D}(t)$. Both the results in Ref. [1] and [2] are correct.

Data availability—No data were created or analyzed in this study.

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