

# From the Gravitational Compton Amplitude to Black Hole Perturbation Theory

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We employ a single worldline effective field theory in a Schwarzschild-Tangherlini background to compute the gravitational Compton amplitude up to third post-Minkowskian order. After excising the infrared divergences associated with the Weinberg phase, we establish a direct and exact map to results in black hole perturbation theory with a natural resummation and regularization of the forward divergences of the Compton amplitude. We also outline possible applications of this useful computational bridge for Compton amplitudes that include spin and tidal effects.

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**Introduction**—The ability to detect gravitational waves directly using Earth-based interferometers, along with advances in computational technology for general relativity arising from quantum field theory [1–7], has led to a renewed theoretical emphasis on deriving analytic results in classical gravity. Several new insights have emerged from studies of the inspiral dynamics [8–64] and radiation effects [7,65–82] of two-body mergers. The gravitational Compton amplitude describing graviton scattering off a massive particle is important in this context, serving to probe and characterize stable, curved spacetimes within an effective field theory approach, and is vital for computing results relevant to massive binary scattering via amplitude techniques (see, for instance, [83–102]).

The Compton amplitude at the second post-Minkowskian order has been studied previously in Refs. [103–105]. The full result for a nonspinning mass was first derived by the present authors [104] using worldline effective field theory [2,8,9,106] in flat and curved backgrounds [107–109]. In this Letter, we extend the approach of Ref. [104] to the third post-Minkowskian order. The infrared structure of the amplitude displays the expected form of the Weinberg phase [110], allowing a straightforward extraction of the infrared finite amplitude and the physical differential cross section.

The complete description of gravitational waves scattering against compact objects like black holes, as dictated by general relativity, comes from black hole perturbation theory [111–114] and matching to these results systematically identifies the right point particle effective field theory for modeling them. Playing a central role in analyzing gravitational waves in the ringdown regime of a black hole merger process, black hole perturbation theory studies are most naturally conducted in a partial-wave expansion that encodes strong-field dynamics. Establishing an explicit connection between plane-wave amplitudes from post-Minkowskian perturbation theory and partial-wave amplitudes from black hole perturbation theory is therefore essential for extracting and classifying the effects from the finite size of these objects [52,97,100,102,115–117]. A comparison between these two representations—requiring a projection integral from plane-wave to partial-wave amplitudes—has been achieved at tree level [89,118] but an obstacle when going to higher post-Minkowskian orders is the increasingly complex singular integrals arising from forward-limit divergences. In this Letter, we overcome the challenge by employing a resummation of the leading forward divergences into a compact Newtonian-like contribution [119] that naturally regularizes the singularity. Using it, we establish a direct map between plane-wave and partial-wave amplitudes. We find complete agreement between the scattering amplitudes up to third post-Minkowskian order and therefore obtain the same differential cross section up to next-to-next-to-leading order.

We envision many interesting applications of our results. Perturbative amplitude relations arising from the nonperturbative isospectrality relations in black hole perturbation theory [120] could, for instance, lead to new insight on how to resum perturbative amplitude expressions. We also believe that computations of amplitudes with finite-size

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effects, such as spin and tidal contributions, could receive useful input from the direct link.

The structure of this Letter is as follows. We begin by reviewing the computational formalism and discussing the general structure of the classical Compton amplitude. Then, we outline the structure of the third-order post-Minkowskian amplitude, identifying the contributing diagrams and specifying the integral basis required to determine the integrand. The results for the third-order post-Minkowskian amplitude and next-to-leading order cross section are presented next. This is followed by a section describing the comparison of our post-Minkowskian results with black hole perturbation theory by projecting the gravitational post-Minkowskian Compton amplitude to its counterpart in black hole perturbation theory, given in terms of functions in a spin-weighted spherical harmonic basis. Finally, we present our conclusions.

Throughout, we adopt the mostly minus metric signature and set  $\hbar = c = 1$ .

*Gravitational Compton amplitude*—We start with describing the gravitational interactions of a nonspinning compact object of mass  $M$  in an effective field theory approach using the gauge-fixed Einstein-Hilbert action and a minimally coupled worldline action,

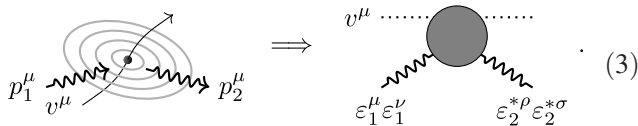
$$S = S_G[g] + S_{\text{wl}}[g, x], \quad (1)$$

where

$$\begin{aligned} S_G[g] &= -\frac{2}{\kappa_d^2} \int d^d x \sqrt{|g|} \left( R - \bar{g}^{\mu\nu} G_\mu G_\nu \right), \\ S_{\text{wl}}[g, x] &= -\frac{M}{2} \int d\tau (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + 1). \end{aligned} \quad (2)$$

The gauge-fixing function is  $G_\mu = \bar{g}^{\nu\lambda} \bar{\nabla}_\nu h_{\lambda\mu} - \frac{1}{2} \bar{g}^{\rho\sigma} \bar{\nabla}_\mu h_{\rho\sigma}$  and the bars denote a fixed background metric and its associated covariant derivatives. To regulate formal infinities that arise in loop level computations, we work in  $d = 4 - 2\epsilon$  dimensions. The gravitational coupling constant is  $\kappa_d^2 \equiv 32\pi G \tilde{\mu}^{2\epsilon}$ , with  $G$  being the four-dimensional Newton's constant, and  $\tilde{\mu}^2 \equiv (e^\gamma/4\pi)\mu^2$ . Here,  $\mu$  is an arbitrary mass scale coming from dimensional regularization, and  $\gamma$  is the Euler-Mascheroni constant.

The object of our study, the scattering of a graviton off a compact object modeled as a point particle, is described by the gravitational Compton amplitude whose relevant kinematics can be summarized as



Here,  $p_i^\mu$  and  $\epsilon_i^\mu \epsilon_i^\nu$  are the momenta and polarization of the gravitons, and  $v^\mu$  denotes the velocity of the compact

object. The frequency of the external gravitons is defined by  $\omega = p_1 \cdot v = p_2 \cdot v$ , and the momentum transferred in the process is  $q^\mu = p_2^\mu - p_1^\mu$ . The magnitude of  $q^\mu$  is related to the spatial angle  $\theta$  between the two graviton momenta and is encoded in  $x = (|q|/2\omega) = \sin(\theta/2)$ ,  $0 \leq x \leq 1$ . The Compton amplitude, which has a nontrivial functional dependence on  $x$ , can be systematically expanded in a post-Minkowskian series,  $n\text{PM}$  signifying the  $n$ th order term, and in a series in the dimensional regulator  $\epsilon$  so that

$$\mathcal{M} = \mathcal{M}_0^{1\text{PM}} + \sum_{n=2}^{\infty} \sum_{j=1-n}^{\infty} \epsilon^j \mathcal{M}_j^{n\text{PM}}, \quad (4)$$

where terms with  $j < 0$  represent infrared divergences. From Weinberg's soft graviton theorem [110], the infrared divergences may be extracted by factoring out the Weinberg phase from the amplitude [121],

$$\begin{aligned} \mathcal{M}_{\text{finite}}(\mu, \epsilon_1, \epsilon_2^*) &= e^{i\bar{\epsilon}/\epsilon} \mathcal{M}|_{\epsilon \rightarrow 0} \\ &= \mathcal{M}_0^{1\text{PM}} + \mathcal{M}_0^{2\text{PM}} + (\mathcal{M}_0^{3\text{PM}} + i\bar{\epsilon} \mathcal{M}_1^{2\text{PM}}) + \dots, \end{aligned} \quad (5)$$

where  $\bar{\epsilon} = 2GM\omega$  is the post-Minkowskian expansion parameter. Note at third post-Minkowskian order that  $\mathcal{M}_{\text{finite}}$  has a contribution from the  $\epsilon$ -linear  $\mathcal{M}_1^{2\text{PM}}$  [122]. More generally, at higher orders in the post-Minkowskian expansion, higher powers of  $\epsilon$  in the amplitude can also contribute to the finite amplitude and hence contribute to the physical differential cross section [123],

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}_{\text{finite}}(\mu, \epsilon_1, \epsilon_2^*)|^2}{(8\pi M)^2}. \quad (6)$$

Therefore, positive powers of  $\epsilon$  in the amplitude cannot be discarded *a priori*. We also note that the  $\mu$  dependence in the amplitude cancels in the cross section.

All Feynman rules required for our computation are extracted from the action in Eq. (1) by inserting the background field expansions,

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \kappa_d h_{\mu\nu}(x), \quad x^\mu(\tau) = v^\mu \tau + z^\mu(\tau), \quad (7)$$

such that the metric is expanded around the Schwarzschild-Tangherlini solution [125] in isotropic coordinates, and the massive particle trajectory around straight-line motion [104,107,108]. Expanding around this solution resums certain classes of Feynman diagrams, reducing the number of diagrams one has to evaluate. To construct the integrand for the third-order Compton amplitude, we add the 13 contributing diagrams [126],

$$\begin{aligned}
 & \text{Diagram with a shaded circle and a wavy line} \equiv -\text{Diagram with three wavy lines and three circles labeled R} + \text{Diagram with three wavy lines and three circles labeled 1} + \left( -\text{Diagram with two wavy lines and two circles labeled 1 and R} + 2 \text{ perms.} \right) \\
 & + \left( -\text{Diagram with two wavy lines and two circles labeled R and 1} + 2 \text{ perms.} \right) + \left( -\text{Diagram with one wavy line and one circle labeled R and 2} + 1 \text{ perm.} \right) + \left( -\text{Diagram with one wavy line and one circle labeled 1 and 2} + 1 \text{ perm.} \right) + \text{Diagram with one wavy line and one circle labeled 3},
 \end{aligned} \quad (8)$$

where we have the deflection and metric insertion subdiagrams,

$$\text{Diagram with a circle labeled R} \equiv \text{Diagram with a wavy line and a dot} \rightarrow \text{Diagram with a wavy line and a dot}, \quad \text{Diagram with a circle labeled n} \equiv \text{Diagram with a wavy line and a dot labeled n}. \quad (9)$$

The deflection subdiagram (also known as the recoil vertex [109]) represents the process of a graviton being absorbed by the compact object, perturbing its motion, and being reemitted. The  $n$ th post-Minkowskian metric insertion represents the graviton interacting with the gravitational potential generated by the compact object at that order in the expansion. Due to the nonlinearity of general relativity, there is an infinite tower of these metric insertion vertices. We refer to our previous paper for more details [104].

The expression for the numerator of the integrand contains scalar products like  $\ell_i \cdot \epsilon_j$ . These can be reduced by expanding the polarization vectors  $\epsilon_i^\mu$  in a basis composed of the external kinematics and an additional spacelike unit vector  $w^\mu$ . After tensor reduction, the integrand is spanned by the integral family,

$$\begin{aligned}
 K_{\vec{\nu}}^{\vec{\lambda}} &= \tilde{\mu}^{4\epsilon} \int_{\mathcal{C}} \frac{\hat{\delta}^{(\nu_1-1)}(\ell_1 \cdot v) \hat{\delta}^{(\nu_2-1)}(\ell_2 \cdot v)}{[\ell_1^2]^{\nu_3} [\ell_2^2]^{\nu_4} [(q - \ell_1 - \ell_2)^2]^{\nu_5}} \\
 &\times \frac{[(\ell_1 + \ell_2)^2]^{\lambda_1} [(q - \ell_1)^2]^{\lambda_2} (\ell_1 \cdot w)^{\lambda_3} (\ell_2 \cdot w)^{\lambda_4}}{[(p_1 + \ell_1)^2 + i0^+]^{\nu_6} [(p_1 + \ell_1 + \ell_2)^2 + i0^+]^{\nu_7}},
 \end{aligned} \quad (10)$$

where  $\vec{\nu} = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7)$ ,  $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ . The superscripts on the  $\delta$  functions signify derivatives and are only needed for the purpose of deriving integration-by-parts relations between the integrals [128–132]. The first three propagators in the denominator originate entirely from the metric insertion vertices in Eq. (9). They are never on shell, since they are forced to be spacelike on the whole integration domain. Therefore, their  $i0^+$  prescription is inconsequential and suppressed in Eq. (10). The next two propagators come from the gravitons traversing between any two-point vertices in Eq. (8), and it is essential to keep track of their  $i0^+$  prescription. It turns out to be trivial to show that the Feynman and retarded prescriptions are identical for the Compton amplitude.

With LiteRed [133,134], we reduce the integral family to eight master integrals: [135]

$$\begin{aligned}
 K_1 &= K_{1,1,0,0,0,1,1}^{0,0,0,0}, & K_2 &= K_{1,1,1,1,0,0,1}^{0,0,0,0}, & K_3 &= K_{1,1,0,1,1,1,0}^{0,0,0,0}, \\
 K_4 &= K_{1,1,1,1,1,0,0}^{0,0,0,0}, & K_5 &= K_{1,1,1,1,1,0,1}^{0,0,0,0}, & K_6 &= K_{1,1,1,1,1,1,0}^{0,0,0,0}, \\
 K_7 &= K_{1,1,1,1,1,1,1}^{0,0,0,0}, & K_8 &= K_{1,1,2,1,1,1,1}^{0,0,0,0}.
 \end{aligned}$$

See Fig. 1 for a diagrammatic representation. To solve the master integrals, we use the method of differential equations [128,129,136–138]. Concretely, we find a canonical basis of master integrals [138] which allows us to easily integrate the required integration-by-parts differential equations that appear in the calculation order by order in  $\epsilon$ . We fix the boundary constants for the three lowest sectors by directly determining the limit of the integrals as  $x \rightarrow 0$ , which is the forward-scattering limit. The boundary constants for the top sector are fixed by imposing the condition that the amplitude be free of all backward scattering singularities as  $x \rightarrow 1$  and of higher power singularities  $x^{-m}$ ,  $m \geq 3$ . More details on our procedure and integral results can be found in Supplemental Material [139], which includes [140–147].

We find that the third post-Minkowskian order amplitude up to  $\mathcal{O}(\epsilon)$ ,

$$\mathcal{M}^{3\text{PM}} = -\frac{\bar{\epsilon}^2}{2\epsilon^2} \mathcal{M}_0^{\text{1PM}} - \frac{i\bar{\epsilon}}{\epsilon} \mathcal{M}_0^{\text{2PM}} + \mathcal{M}_0^{\text{3PM}} + \mathcal{O}(\epsilon), \quad (11)$$

has the expected infrared behavior from Weinberg’s soft theorem [110]. Then, using Eq. (5) and inserting the linear-in- $\epsilon$  term from the second-order post-Minkowskian amplitude, extracted from the  $d$ -dimensional expression presented in [121], we obtain the finite amplitude at third post-Minkowskian order, expressed here in terms of the transcendental functions appearing in the master integrals,

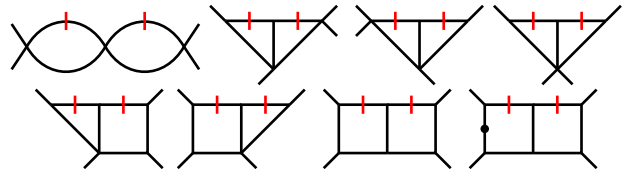


FIG. 1. Diagrams representing the master integrals ordered left to right from  $K_1$  to  $K_4$  and then  $K_5$  to  $K_8$ . Red lines denote velocity-cut propagators. The dot on the second double box signifies a squared propagator.

$$\mathcal{M}_{\text{finite}}^{\text{3PM}} = \mathcal{M}_0^{\text{3PM}} + i\bar{\epsilon}\mathcal{M}_1^{\text{2PM}} = \frac{\pi M(2GM\omega)^3}{\omega} \times \left[ d_1(G_+(x) + i\pi \log(1+x)) + d_2\left(G_-(x) + i\pi \log \frac{4x}{1+x}\right) + d_3 \log^2 \frac{4\omega^2 x^2}{\mu^2} + d_4\left(4\zeta(2) - \log^2 \frac{4\omega^2}{\mu^2}\right) + d_5 \log x^2 + (d_6 + d_7 i\pi) \log \frac{4\omega^2}{\mu^2} + d_{\text{Re}} + d_{\text{Im}} \right], \quad (12)$$

where we have used

$$G_{\pm}(x) = \text{Li}_2(x) \pm \text{Li}_2(-x) + (\log(1-x) \pm \log(1+x)) \log x. \quad (13)$$

The coefficients, which are given in Supplemental Material [139], are all in terms of two gauge-invariant functions,

$$F_1 = \frac{(v \cdot f_1 \cdot f_2^* \cdot v)}{\omega^2}, \quad F_2 = \frac{(v \cdot f_1 \cdot p_2)(v \cdot f_2^* \cdot p_1)}{\omega^4}, \quad (14)$$

where  $f_i^{\mu\nu} = p_i^\mu \epsilon_i^\nu - p_i^\nu \epsilon_i^\mu$ . The poles at  $x \rightarrow 1$  and  $x \rightarrow 1/\sqrt{2}$  are spurious and originate in the tensor and integration-by-parts reduction. We remark that the leading forward divergences up to third post-Minkowskian order in  $\mathcal{M}_{\text{finite}}$  go as

$$3\text{PM}: x^{-2} \log^2 x, \quad 2\text{PM}: x^{-2} \log x, \quad 1\text{PM}: x^{-2}. \quad (15)$$

Using Eqs. (5) and (6), together with the first and second order post-Minkowskian amplitude results in [104], one can directly get the leading order, next-to-leading order, and next-to-next-to-leading order cross sections from the expression,

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{|\mathcal{M}_0^{\text{1PM}}|^2 + 2\text{Re}(\mathcal{M}_0^{\text{1PM}}\mathcal{M}_0^{\text{2PM}})}{(8\pi M)^2} + \frac{|\mathcal{M}_0^{\text{2PM}}|^2 + 2\text{Re}(\mathcal{M}_0^{\text{1PM}}\mathcal{M}_{\text{finite}}^{\text{3PM}})}{(8\pi M)^2} + \dots \quad (16)$$

Working in the lab frame, one can take  $\theta_1 = 0$  and relabel  $\theta_2 = \theta$  in the following general expressions for the external data present in the amplitude,  $v = (1, 0, 0, 0)$ ,  $p_j = \omega(1, \sin\theta_j, 0, \cos\theta_j)$ ,  $\epsilon_j^+ = \frac{1}{\sqrt{2}}(0, \cos\theta_j, i, -\sin\theta_j)$ ,  $\epsilon_j^- = \frac{1}{\sqrt{2}}(0, \cos\theta_j, -i, -\sin\theta_j)$ , so that the cross section can be rephrased as receiving contributions from a helicity-conserving function  $f$  and helicity-reversing function  $g$ ,

$$\frac{d\sigma}{d\Omega} = |f_{\text{amp}}(\theta)|^2 + |g_{\text{amp}}(\theta)|^2, \quad (17)$$

where  $f_{\text{amp}}(\theta) = (1/8\pi M)\mathcal{M}_{\text{finite}}(\mu, \epsilon_1^+, \epsilon_2^{+*})$  and  $g_{\text{amp}}(\theta) = (1/8\pi M)\mathcal{M}_{\text{finite}}(\mu, \epsilon_1^+, \epsilon_2^{*-})$ . Using explicit expressions for the amplitudes, the cross section up to next-to-leading order is

$$\frac{d\sigma}{d\Omega} = (GM)^2 \left[ \frac{(x^2 - 1)^4 + x^8}{x^4} + \bar{\epsilon} \frac{\pi(15(x+1)^2 - 2x)}{4x^3} \times (x-1)^4 + \bar{\epsilon}^2 \left( c_1 G_+(x) + c_2 G_-(x) + c_3 + \pi^2 c_4 + c_5 \log(x) + c_6 \log^2(x) \right) \right], \quad (18)$$

where the coefficients  $c_i$  are provided in Supplemental Material [139]. In the plot shown in Fig. 2, we display the leading order, next-to-leading order, and next-to-next-to-leading order cross sections separately as functions of  $\theta$  at  $\bar{\epsilon} = 0.3$ . The leading order is dominant at all angles and indicates the efficiency of the post-Minkowskian expansion (at  $\bar{\epsilon} \leq 0.3$ ).

*Comparison with black hole perturbation theory*—In black hole perturbation theory, the helicity-conserving and helicity-reversing contributions to the differential cross section are expressed as an infinite sum involving spin-weighted spherical harmonic functions [119,148,149] written here in a modified form,

$$f_{\text{B}}(\theta) = \frac{\pi e^{-i\Phi}}{i\omega} \sum_{l=2}^{\infty} {}_{-2}Y_{l2}(0) {}_{-2}Y_{l2}(\theta) \Delta_{l2}^+, \quad g_{\text{B}}(\theta) = \frac{\pi e^{-i\Phi}}{i\omega} \sum_{l=2}^{\infty} (-1)^l {}_{-2}Y_{l2}(0) {}_{-2}Y_{l2}(\pi - \theta) \Delta_{l2}^-, \quad (19)$$

where  $\Phi \equiv 2\bar{\epsilon} \ln(2\bar{\epsilon}) - \bar{\epsilon}$  and  $\Delta_{l2}^{\pm} = e^{2i\delta_{l2}^+} \pm e^{2i\delta_{l2}^-}$ . We drop the subtraction of 2 in the original  $f_{\text{B}}(\theta)$  to make the definition consistent with the post-Minkowskian amplitude and also multiply by an  $l$ -independent overall phase that does not affect the physical differential cross section. The parity- and  $l$ -dependent phases are obtained from the Mano-Suzuki-Takasugi solution [150] of the Teukolsky equation [113,114]

$$e^{2i\delta_{lm}^{\pm}} = (-1)^{l+1} \frac{C_{\text{TS}}^{(0)} B_{lm}^{(\text{ref})}}{16\omega^4 B_{lm}^{(\text{inc})}}, \quad (20)$$

where  $C_{\text{TS}}^{(0)} = l(l^2 - 1)(l + 2) \pm 6i\bar{\epsilon}$  is the Teukolsky-Starobinsky constant in the spinless limit.  $B^{(\text{ref})}$  and  $B^{(\text{inc})}$  are obtained from the asymptotic behavior of the

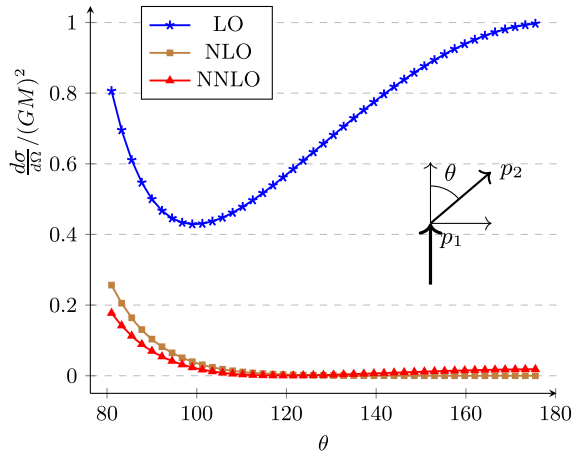


FIG. 2. Differential cross section at  $\bar{\epsilon} = 0.3$  showing leading order (LO), next-to-leading order (NLO), and next-to-next-to-leading order (NNLO) contributions.

Mano-Suzuki-Takasugi solution for the Newman-Penrose scalar  $\psi_4$  and we refer to [150,151] for their explicit values [152]. If we define components of the scattering amplitudes under projection to the spin-weighted spherical harmonic basis as

$$\begin{aligned} (f)^{(l)} &\equiv 2\pi \int_0^\pi f(\theta) {}_{-2}Y_{l2}(\theta) \sin(\theta) d\theta, \\ (g)^{(l)} &\equiv 2\pi \int_0^\pi g(\theta) {}_{-2}Y_{l2}(\pi - \theta) \sin(\theta) d\theta, \end{aligned} \quad (21)$$

it is straightforward to deduce a non-perturbative relation between helicity-conserving and helicity-reversing functions,

$$g_B^{(l)} = \frac{6i\bar{\epsilon}}{(-1)^l l(l^2 - 1)(l + 2)} f_B^{(l)}, \quad (22)$$

as the parity-dependent phase arises only from the Teukolsky-Starobinsky constant. This relation is also proved in [155] based on Zerilli's radial equation [112]; see also Ref. [119].

To demonstrate the equivalence between the Compton amplitude in a post-Minkowskian expansion and expressions from black hole perturbation theory, it is useful to project the Compton amplitude onto the same complete basis of spherical harmonic functions. In order to do so and match known results, we set the scale parameter in the Compton amplitude to  $\mu = 2e^{-\gamma}\omega$ , and first proceed to regularize the forward scattering divergences in Eq. (15) that prevent successful projection. Covariantizing the results of long-range (Newtonian) contributions to gravitational wave scattering in [119,156], we find that a resummation of the leading forward divergent components from each post-Minkowskian order suffices as a natural regularization. While this contribution contains terms at all

orders in the post-Minkowskian expansion, it arises solely from iterations of the tree-level Newtonian potential. Thus, we arrive at

$$\begin{aligned} \mathcal{M}_{\text{finite}}(2e^{-\gamma}\omega, \epsilon_1, \epsilon_2^*) &= 32\pi GM^2 \frac{\Gamma(1 - i\bar{\epsilon})}{\Gamma(1 + i\bar{\epsilon})} \frac{F_1^2}{4x^{2(1-i\bar{\epsilon})}} \\ &+ \sum_{n=2}^{\infty} \mathcal{M}_{\text{finite,rm}}^{\text{nPM}}(2e^{-\gamma}\omega, \epsilon_1, \epsilon_2^*), \end{aligned} \quad (23)$$

where the subscript “rm” denotes remaining terms that are free of the leading forward divergent contribution. We find (to the order we work) that the leading forward divergent contribution can be understood from considering the resummation of graphs with a double cut at each loop,

(24)

The logic behind this argument starts from the one-loop amplitude result, where one arrives at contributions with the form  $(\log(x) + \gamma)[1/x^2]$ . When adding a new rung, these one-loop terms reappear together with new higher-order loop terms. Since the total contribution from summing all loop orders will consist of all lower order contributions plus new ones at each order, it is possible to conjecture the all-loop result from the pattern of explicitly computed terms. We find terms giving rise to  $(e^{2i\gamma\bar{\epsilon}}/x^{2-2i\bar{\epsilon}})$  which is consistent with

$$\frac{\Gamma(1 - i\bar{\epsilon})}{\Gamma(1 + i\bar{\epsilon})} = e^{2i\gamma\bar{\epsilon}} e^{i \sum_{n=1}^{\infty} \frac{2(-1)^n}{2n+1} \bar{\epsilon}^{2n+1} \zeta(2n+1)}$$

in  $32\pi GM^2 [\Gamma(1 - i\bar{\epsilon})/\Gamma(1 + i\bar{\epsilon})] \{F_1^2/4x^{2(1-i\bar{\epsilon})}\}$  in (23). [We note that exponential factors which involve terms such as  $\zeta(3)$  and other transcendental constants begin to appear in the loop diagrams from the fourth post-Minkowskian order, which necessitates the appearance of  $\zeta(j)$  in the phase factor.] A different value of  $\mu$  will result in an extra phase upon resummation which does not affect physics but prevents a complete match when comparing with black hole perturbation theory.

In the lab frame, the scattering functions are

$$\begin{aligned} f_{\text{amp}}^{\text{nt}}(\theta) + f_{\text{amp}}^{\text{rm}}(\theta) &= \frac{\mathcal{M}_{\text{finite}}(2e^{-\gamma}\omega, \epsilon_1^+, \epsilon_2^{+*})}{8\pi M}, \\ g_{\text{amp}}^{\text{nt}}(\theta) + g_{\text{amp}}^{\text{rm}}(\theta) &= \frac{\mathcal{M}_{\text{finite}}(2e^{-\gamma}\omega, \epsilon_1^+, \epsilon_2^{-*})}{8\pi M}, \end{aligned} \quad (25)$$

where  $f_{\text{amp}}^{\text{nt}}(\theta)$  denotes the long-range Newtonian contribution from the first term in (23) and  $f_{\text{amp}}^{\text{rm}}(\theta)$  gets contributions from the remaining terms. The same holds for the helicity-reversing  $g(\theta)$ .

For the resummed Newtonian part, the integration necessary to project to the spherical harmonic basis is well-defined via analytic continuation in  $\bar{\epsilon}$ . An example for  $l = 2$  is

$$f_{\text{amp}}^{\text{nt}(2)} = \frac{\sqrt{5}\pi}{\omega} \left[ -i + \left( 2\gamma - \frac{25}{12} \right) \bar{\epsilon} + i \left( \frac{415}{144} - \frac{25}{6} \gamma + 2\gamma^2 \right) \bar{\epsilon}^2 - \left( \frac{2}{3} \zeta(3) - \frac{25}{6} \gamma^2 + \frac{4}{3} \gamma^3 + \frac{415}{72} \gamma - \frac{5845}{1728} \right) \bar{\epsilon}^3 \right]. \quad (26)$$

The partial wave projection leads to a post-Minkowskian order mixing effect which is clearly illustrated by this example. The  $\zeta(3)$  term, expected in the plane wave amplitude at the fourth post-Minkowskian order, actually contributes at the third post-Minkowskian order in the spherical harmonic basis. This arises from the singular integration of one term in resummed Newtonian part of the amplitude in (23),  $-(8i\pi/3\omega) \int_0^1 dx x (-2Y_{22}) [\bar{\epsilon}^4 \zeta(3)/x^{2-2i\bar{\epsilon}}] = [4\sqrt{5}\pi \bar{\epsilon}^3 \zeta(3)/3(\bar{\epsilon}^2 - 3i\bar{\epsilon} - 2)\omega]$ . The agreement of this term with black hole perturbation theory at  $\bar{\epsilon}^3$ -order thus implies that the corresponding term in the Newtonian part in (23) is correct at the fourth post-Minkowskian order. Similarly, the  $\bar{\epsilon}^0$  term in the spherical harmonic component in (26) actually originates from the first post-Minkowskian amplitude, which explains why we drop the subtraction of 2 in the definition of  $f_B$ .

The remaining terms in Eq. (23) exhibit softer divergences in the projection integral at each post-Minkowskian order and can therefore be integrated directly. As a highly nontrivial check, we compare the projected components explicitly up to  $l = 10$  to  $\bar{\epsilon}^3$ -order and find exact agreement between the amplitude and black hole perturbation theory results,

$$f_{\text{amp}}^{\text{rm}(l)} + f_{\text{amp}}^{\text{nt}(l)} = f_B^{(l)}, \quad g_{\text{amp}}^{\text{rm}(l)} + g_{\text{amp}}^{\text{nt}(l)} = g_B^{(l)}. \quad (27)$$

The match for higher values of  $l$  should be straightforward and we expect this equivalence to continue to hold at fourth post-Minkowskian order directly and at higher post-Minkowskian orders after including all the finite-size effects from either completing the effective action, or equivalently, the four-point tree-level Compton amplitude. In particular, the nonperturbative relations (22) appearing on the black hole perturbation theory side suggest the existence of highly nontrivial nonperturbative relations between the complete helicity-conserving Compton amplitude and the helicity-reversing one.

*Conclusion and outlook*—In this Letter, we explicitly demonstrate that a formalism that systematically incorporates recoil effects in a nonspinning background extends to obtaining the gravitational Compton amplitude to third post-Minkowskian order. We also employ Weinberg’s soft

theorem to factorize infrared singularities in the Compton amplitude and regulate the leading forward divergences of the long-range Newtonian contributions through a resummed expression. This framework allows us to establish an explicit mapping between perturbative scattering amplitudes in the post-Minkowskian expansion and nonperturbative expressions formulated in terms of spin-weighted spherical harmonics within the Mano-Suzuki-Takasugi solution of black hole perturbation theory. It seems to be worthwhile to formally prove the proposed deduced form for the term regulating the all-order resummed Newtonian contribution by explicitly working to higher orders in the post-Minkowskian expansion.

A natural extension of this work is to incorporate finite-size effects—such as spin, tidal responses, and absorption—into gravitational Compton amplitudes, particularly in light of current astrophysical observations. Because the resummed Newtonian contribution systematically handles post-Minkowskian order mixing, the bridge between formalisms established here indicates that black hole perturbation theory might constrain tree-level Compton amplitudes directly at arbitrary spin order in a way that would enable a bypassing of the conventional separation between near- and far-zone dynamics.

The direct observables in the ringdown stage are the quasi-normal modes [157–161], which are naturally obtained from computations within black hole perturbation theory. A central open question is how the discrete quasi-normal-mode frequencies and their associated amplitudes arise from post-Minkowskian scattering amplitudes. Our work suggests that such an extraction can be achieved by partially resumming plane-wave amplitudes and systematically mapping them onto partial-wave observables, thereby providing a new pathway to derive ringdown physics from perturbative scattering, which in turn could be used to characterize the black hole spectrum from an action principle. Moreover, generalizing the framework to higher-multiplicity amplitudes is of considerable interest, as it may provide access to nonlinear features of the quasi-normal-mode spectrum [162–166] and thus further deepen the connection between scattering amplitudes and black hole dynamics. We leave these and other questions to future research.

*Note added*—During the preparation of our manuscript, we became aware of [167] and [168], which have a partial overlap with our work.

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*Data availability*—There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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- [1] Y. Iwasaki, *Lett. Nuovo Cimento* **1S2**, 783 (1971); *Lett. Nuovo Cimento* **1**, 783(E) (1971).
  - [2] W. D. Goldberger and I. Z. Rothstein, *Phys. Rev. D* **73**, 104029 (2006).
  - [3] D. Neill and I. Z. Rothstein, *Nucl. Phys.* **B877**, 177 (2013).
  - [4] N. E. J. Bjerrum-Bohr, J. F. Donoghue, and P. Vanhove, *J. High Energy Phys.* **02** (2014) 111.
  - [5] N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Plante, and P. Vanhove, *Phys. Rev. Lett.* **121**, 171601 (2018).
  - [6] C. Cheung, I. Z. Rothstein, and M. P. Solon, *Phys. Rev. Lett.* **121**, 251101 (2018).
  - [7] D. A. Kosower, B. Maybee, and D. O’Connell, *J. High Energy Phys.* **02** (2019) 137.
  - [8] G. Kälin and R. A. Porto, *J. High Energy Phys.* **11** (2020) 106.
  - [9] G. Mogull, J. Plefka, and J. Steinhoff, *J. High Energy Phys.* **02** (2021) 048.
  - [10] A. Cristofoli, N. E. J. Bjerrum-Bohr, P. H. Damgaard, and P. Vanhove, *Phys. Rev. D* **100**, 084040 (2019).
  - [11] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, *Phys. Rev. Lett.* **122**, 201603 (2019).
  - [12] A. Antonelli, A. Buonanno, J. Steinhoff, M. van de Meent, and J. Vines, *Phys. Rev. D* **99**, 104004 (2019).
  - [13] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, *J. High Energy Phys.* **10** (2019) 206.
  - [14] J. Parra-Martinez, M. S. Ruf, and M. Zeng, *J. High Energy Phys.* **11** (2020) 023.
  - [15] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, *Phys. Lett. B* **811**, 135924 (2020).
  - [16] T. Damour, *Phys. Rev. D* **102**, 124008 (2020).
  - [17] G. Kälin, Z. Liu, and R. A. Porto, *Phys. Rev. Lett.* **125**, 261103 (2020).
  - [18] Z. Bern, A. Luna, R. Roiban, C.-H. Shen, and M. Zeng, *Phys. Rev. D* **104**, 065014 (2021).
  - [19] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, *Phys. Lett. B* **818**, 136379 (2021).
  - [20] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, *J. High Energy Phys.* **07** (2021) 169.
  - [21] E. Herrmann, J. Parra-Martinez, M. S. Ruf, and M. Zeng, *J. High Energy Phys.* **10** (2021) 148.
  - [22] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté, and P. Vanhove, *Phys. Rev. D* **104**, 026009 (2021).
  - [23] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté, and P. Vanhove, *J. High Energy Phys.* **08** (2021) 172.
  - [24] P. H. Damgaard, L. Plante, and P. Vanhove, *J. High Energy Phys.* **11** (2021) 213.
  - [25] A. Brandhuber, G. Chen, G. Travaglini, and C. Wen, *J. High Energy Phys.* **10** (2021) 118.
  - [26] N. E. J. Bjerrum-Bohr, L. Planté, and P. Vanhove, *J. High Energy Phys.* **03** (2022) 071.
  - [27] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, *Phys. Rev. Lett.* **128**, 161103 (2022).
  - [28] C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, *Phys. Lett. B* **831**, 137203 (2022).
  - [29] C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, *Phys. Rev. Lett.* **128**, 161104 (2022).
  - [30] G. U. Jakobsen, G. Mogull, J. Plefka, and J. Steinhoff, *J. High Energy Phys.* **01** (2022) 027.
  - [31] G. U. Jakobsen, G. Mogull, J. Plefka, and J. Steinhoff, *Phys. Rev. Lett.* **128**, 011101 (2022).
  - [32] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, *Phys. Rev. Lett.* **126**, 171601 (2021).
  - [33] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Plante, and P. Vanhove, *J. Phys. A* **55**, 443014 (2022).
  - [34] N. E. J. Bjerrum-Bohr, L. Planté, and P. Vanhove, *arXiv*: 2212.08957.
  - [35] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, *Proc. Sci.* **LL2022** (2022) 051.
  - [36] C. Dlapa, G. Kälin, Z. Liu, J. Neef, and R. A. Porto, *Phys. Rev. Lett.* **130**, 101401 (2023).
  - [37] G. U. Jakobsen and G. Mogull, *Phys. Rev. Lett.* **128**, 141102 (2022).
  - [38] T. Adamo and R. Gonzo, *J. High Energy Phys.* **05** (2023) 088.
  - [39] Z. Bern, D. Kosmopoulos, A. Luna, R. Roiban, and F. Teng, *Phys. Rev. Lett.* **130**, 201402 (2023).
  - [40] A. Luna, N. Moynihan, D. O’Connell, and A. Ross, *J. High Energy Phys.* **08** (2024) 045.
  - [41] P. H. Damgaard, E. R. Hansen, L. Planté, and P. Vanhove, *J. High Energy Phys.* **09** (2023) 183.
  - [42] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, *Phys. Rep.* **1083**, 1 (2024).
  - [43] C. Heissenberg, *Phys. Rev. D* **108**, 106003 (2023).
  - [44] G. U. Jakobsen, G. Mogull, J. Plefka, B. Sauer, and Y. Xu, *Phys. Rev. Lett.* **131**, 151401 (2023).
  - [45] N. E. J. Bjerrum-Bohr, L. Planté, and P. Vanhove, *Springer Nature Singapore*, **10.1007/978-981-19-3079-9-3-1** (2023).

- [46] M. Driesse, G. U. Jakobsen, G. Mogull, J. Plefka, B. Sauer, and J. Usovitsch, *Phys. Rev. Lett.* **132**, 241402 (2024).
- [47] G. Chen and T. Wang, *J. High Energy Phys.* **12** (2025) 213.
- [48] L. Bohnenblust, L. Cangemi, H. Johansson, and P. Pichini, *J. High Energy Phys.* **07** (2025) 261.
- [49] M. Driesse, G. U. Jakobsen, A. Klemm, G. Mogull, C. Nega, J. Plefka, B. Sauer, and J. Usovitsch, *Nature (London)* **641**, 603 (2025).
- [50] G. Chen, J.-W. Kim, and T. Wang, *Phys. Rev. D* **111**, L021701 (2025).
- [51] D. Akpinar, F. Febres Cordero, M. Kraus, A. Smirnov, and M. Zeng, *Phys. Rev. Lett.* **135**, 041602 (2025).
- [52] N. E. J. Bjerrum-Bohr, G. Chen, C. Su, and T. Wang, *Phys. Rev. Lett.* **136**, 071601 (2026).
- [53] S. Foffa, P. Mastrolia, R. Sturani, C. Sturm, and W. J. Torres Bobadilla, *Phys. Rev. Lett.* **122**, 241605 (2019).
- [54] J. Bluemlein, A. Maier, and P. Marquard, *Phys. Lett. B* **800**, 135100 (2020).
- [55] S. Foffa, R. A. Porto, I. Rothstein, and R. Sturani, *Phys. Rev. D* **100**, 024048 (2019).
- [56] J. Bluemlein, A. Maier, P. Marquard, and G. Schaefer, *Nucl. Phys. B* **955**, 115041 (2020).
- [57] J. Bluemlein, A. Maier, P. Marquard, and G. Schaefer, *Phys. Lett. B* **807**, 135496 (2020).
- [58] G. Kälin and R. A. Porto, *J. High Energy Phys.* **02** (2020) 120.
- [59] G. U. Jakobsen, G. Mogull, J. Plefka, and B. Sauer, *Phys. Rev. Lett.* **131**, 241402 (2023).
- [60] G. Kälin and R. A. Porto, *J. High Energy Phys.* **01** (2020) 072.
- [61] M. Driesse, G. U. Jakobsen, G. Mogull, C. Nega, J. Plefka, B. Sauer, and J. Usovitsch, *Phys. Rev. Lett.* **137**, 081402 (2026).
- [62] Z. Bern, E. Herrmann, R. Roiban, M. S. Ruf, A. V. Smirnov, S. Smith, and M. Zeng, *Phys. Rev. Lett.* **137**, 021402 (2026).
- [63] A. Brandhuber, G. R. Brown, P. Pichini, G. Travaglini, and P. Vives Matasan, *J. High Energy Phys.* **07** (2026) 151.
- [64] J.-W. Kim, R. Patil, T. Scheopner, and J. Steinhoff, *J. High Energy Phys.* **03** (2026) 241.
- [65] A. Brandhuber, G. R. Brown, G. Chen, S. De Angelis, J. Gowdy, and G. Travaglini, *J. High Energy Phys.* **06** (2023) 048.
- [66] A. Herderschee, R. Roiban, and F. Teng, *J. High Energy Phys.* **06** (2023) 004.
- [67] A. Elkhidir, D. O'Connell, M. Sergola, and I. A. Vazquez-Holm, *J. High Energy Phys.* **07** (2024) 272.
- [68] A. Georgoudis, C. Heissenberg, and I. Vazquez-Holm, *J. High Energy Phys.* **06** (2023) 126.
- [69] S. Caron-Huot, M. Giroux, H. S. Hannesdottir, and S. Mizera, *J. High Energy Phys.* **01** (2024) 139.
- [70] L. Bohnenblust, H. Ita, M. Kraus, and J. Schlenk, *J. High Energy Phys.* **11** (2024) 109.
- [71] D. Bini, T. Damour, and A. Geralico, *Phys. Rev. D* **108**, 124052 (2023).
- [72] A. Georgoudis, C. Heissenberg, and R. Russo, *J. High Energy Phys.* **03** (2024) 089.
- [73] A. Brandhuber, G. R. Brown, G. Chen, J. Gowdy, and G. Travaglini, *J. High Energy Phys.* **02** (2024) 026.
- [74] S. De Angelis, P. P. Novichkov, and R. Gonzo, *Phys. Rev. D* **110**, L041502 (2024).
- [75] T. Adamo, R. Gonzo, and A. Ilderton, *J. High Energy Phys.* **05** (2024) 034.
- [76] D. Bini, T. Damour, S. De Angelis, A. Geralico, A. Herderschee, R. Roiban, and F. Teng, *Phys. Rev. D* **109**, 125008 (2024).
- [77] F. Alessio and P. Di Vecchia, *J. Phys. A* **57**, 475402 (2024).
- [78] A. Georgoudis, C. Heissenberg, and R. Russo, *Phys. Rev. D* **109**, 106020 (2024).
- [79] G. Brunello and S. De Angelis, *J. High Energy Phys.* **07** (2024) 062.
- [80] A. Brandhuber, G. R. Brown, G. Chen, G. Travaglini, and P. Vives Matasan, *J. High Energy Phys.* **12** (2024) 039.
- [81] L. Bohnenblust, H. Ita, M. Kraus, and J. Schlenk, *J. High Energy Phys.* **12** (2025) 100.
- [82] G. Brunello, S. De Angelis, and D. A. Kosower, *Phys. Rev. D* **114**, 045004 (2026).
- [83] A. Brandhuber, G. Chen, G. Travaglini, and C. Wen, *J. High Energy Phys.* **07** (2021) 047.
- [84] A. Brandhuber, G. Chen, H. Johansson, G. Travaglini, and C. Wen, *Phys. Rev. Lett.* **128**, 121601 (2022).
- [85] W.-M. Chen, M.-Z. Chung, Y.-t. Huang, and J.-W. Kim, *J. High Energy Phys.* **08** (2022) 148.
- [86] R. Aoude, K. Haddad, and A. Helset, *J. High Energy Phys.* **05** (2020) 051.
- [87] M.-Z. Chung, Y.-t. Huang, J.-W. Kim, and S. Lee, *J. High Energy Phys.* **05** (2020) 105.
- [88] M. Chiodaroli, H. Johansson, and P. Pichini, *J. High Energy Phys.* **02** (2022) 156.
- [89] Y. F. Bautista, A. Guevara, C. Kavanagh, and J. Vines, *J. High Energy Phys.* **03** (2023) 136.
- [90] R. Aoude, K. Haddad, and A. Helset, *J. High Energy Phys.* **07** (2022) 072.
- [91] N. E. J. Bjerrum-Bohr, G. Chen, and M. Skowronek, *J. High Energy Phys.* **06** (2023) 170.
- [92] W.-M. Chen, M.-Z. Chung, Y.-t. Huang, and J.-W. Kim, *J. High Energy Phys.* **12** (2022) 058.
- [93] N. E. J. Bjerrum-Bohr, G. Chen, and M. Skowronek, *Phys. Rev. Lett.* **132**, 191603 (2024).
- [94] G. Chen and N. E. J. Bjerrum-Bohr, *Proc. Sci. ICHEP2024 (2025)* 793.
- [95] L. Cangemi, M. Chiodaroli, H. Johansson, A. Ochirov, P. Pichini, and E. Skvortsov, *Phys. Rev. Lett.* **133**, 071601 (2024).
- [96] N. E. J. Bjerrum-Bohr, G. Chen, Y. Miao, and M. Skowronek, *J. High Energy Phys.* **07** (2024) 242.
- [97] Y. F. Bautista, Y.-T. Huang, and J.-W. Kim, *Phys. Rev. D* **111**, 044043 (2025).
- [98] M. Correia and G. Isabella, *J. High Energy Phys.* **03** (2025) 144.
- [99] I. Vazquez-Holm and A. Luna, *J. High Energy Phys.* **07** (2025) 087.
- [100] M. M. Ivanov and Z. Zhou, *Phys. Rev. Lett.* **130**, 091403 (2023).
- [101] M. V. S. Saketh, Z. Zhou, S. Ghosh, J. Steinhoff, and D. Chatterjee, *Phys. Rev. D* **110**, 103001 (2024).
- [102] S. Caron-Huot, M. Correia, G. Isabella, and M. Solon, *Phys. Rev. Lett.* **135**, 191601 (2025).

- [103] D. Akpinar, V. del Duca, and R. Gonzo, *Phys. Rev. D* **112**, 084014 (2025).
- [104] N. E. J. Bjerrum-Bohr, G. Chen, C. J. Eriksen, and N. Shah, *J. High Energy Phys.* **10** (2025) 235.
- [105] D. Akpinar, *Phys. Rev. D* **113**, 045003 (2026).
- [106] R. A. Porto, *Phys. Rev. D* **73**, 104031 (2006).
- [107] D. Kosmopoulos and M. P. Solon, *J. High Energy Phys.* **03** (2024) 125.
- [108] C. Cheung, J. Parra-Martinez, I. Z. Rothstein, N. Shah, and J. Wilson-Gerow, *Phys. Rev. Lett.* **132**, 091402 (2024).
- [109] C. Cheung, J. Parra-Martinez, I. Z. Rothstein, N. Shah, and J. Wilson-Gerow, *J. High Energy Phys.* **10** (2024) 005.
- [110] S. Weinberg, *Phys. Rev.* **140**, B516 (1965).
- [111] T. Regge and J. A. Wheeler, *Phys. Rev.* **108**, 1063 (1957).
- [112] F. J. Zerilli, *Phys. Rev. Lett.* **24**, 737 (1970).
- [113] S. A. Teukolsky, *Phys. Rev. Lett.* **29**, 1114 (1972).
- [114] S. A. Teukolsky, *Astrophys. J.* **185**, 635 (1973).
- [115] M. M. Ivanov, Y.-Z. Li, J. Parra-Martinez, and Z. Zhou, *Phys. Rev. Lett.* **132**, 131401 (2024).
- [116] Y. F. Bautista, M. Khalil, M. Sergola, C. Kavanagh, and J. Vines, *Phys. Rev. D* **110**, 124005 (2024).
- [117] D. Kosmopoulos, D. Perrone, and M. Solon, *Phys. Rev. Lett.* **136**, 211401 (2026).
- [118] Y. F. Bautista, A. Guevara, C. Kavanagh, and J. Vines, *J. High Energy Phys.* **05** (2023) 211.
- [119] S. R. Dolan, *Phys. Rev. D* **77**, 044004 (2008).
- [120] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, Oxford, 1983).
- [121] C. Jordan Eriksen, Gravitational scattering amplitudes from curved space, Other thesis, Niels Bohr Institute, 2026, [arXiv:2603.20261](https://arxiv.org/abs/2603.20261).
- [122] We thank Gustav Uhre Jakobsen for discussions on this point.
- [123] We note that the generalized Compton amplitudes contribute to the cross section only at quantum level (see amplitude in [124] for QED).
- [124] R. N. Lee, M. D. Schwartz, and X. Zhang, *Phys. Rev. Lett.* **126**, 211801 (2021).
- [125] F. R. Tangherlini, *Nuovo Cimento* **27**, 636 (1963).
- [126] We use xAct [127] to carry out the index contractions in the diagrams.
- [127] J. M. Martín-García *et al.*, xAct: Efficient tensor computer algebra for Mathematica, <http://www.xact.es/> (2002–present).
- [128] A. V. Kotikov, *Phys. Lett. B* **254**, 158 (1991).
- [129] T. Gehrmann and E. Remiddi, *Nucl. Phys.* **B580**, 485 (2000).
- [130] A. Smirnov, *J. High Energy Phys.* **10** (2008) 107.
- [131] K. J. Larsen and Y. Zhang, *Phys. Rev. D* **93**, 041701 (2016).
- [132] R. N. Lee, *J. High Energy Phys.* **04** (2015) 108.
- [133] R. N. Lee, [arXiv:1212.2685](https://arxiv.org/abs/1212.2685).
- [134] R. N. Lee, *J. Phys. Conf. Ser.* **523**, 012059 (2014).
- [135] Since  $K_3 = K_2$  and  $K_6 = K_5$ , our master integral basis is, in principle, overdetermined.
- [136] Z. Bern, L. J. Dixon, and D. A. Kosower, *Nucl. Phys.* **B412**, 751 (1994).
- [137] E. Remiddi, *Nuovo Cimento A* **110**, 1435 (1997).
- [138] J. M. Henn, *Phys. Rev. Lett.* **110**, 251601 (2013).
- [139] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/f939-mjhj> for discussions on the details of all master integrals, amplitudes in the lab frame, and a detailed comparison between the post-Minkowskian expanded amplitude and the amplitude derived from black hole perturbation theory.
- [140] J. M. Henn, *J. Phys. A* **48**, 153001 (2015).
- [141] C. Dlapa, X. Li, and Y. Zhang, *J. High Energy Phys.* **07** (2021) 227.
- [142] S. Weinzierl, *Feynman Integrals*, UNITEXT for Physics (Springer, Cham, 2022), 10.1007/978-3-030-99558-4.
- [143] P. A. Baikov, [arXiv:hep-ph/9604254](https://arxiv.org/abs/hep-ph/9604254).
- [144] P. A. Baikov, *Phys. Lett. B* **385**, 404 (1996).
- [145] P. A. Baikov, *Phys. Lett. B* **634**, 325 (2006).
- [146] H. Frellesvig, *J. High Energy Phys.* **04** (2025) 111.
- [147] C. Duhr and F. Dulat, *J. High Energy Phys.* **08** (2019) 135.
- [148] R. A. Matzner and M. P. Ryan, *Phys. Rev. D* **16**, 1636 (1977).
- [149] S. R. Dolan, *Classical Quantum Gravity* **25**, 235002 (2008).
- [150] S. Mano, H. Suzuki, and E. Takasugi, *Prog. Theor. Phys.* **95**, 1079 (1996).
- [151] M. Sasaki and H. Tagoshi, *Living Rev. Relativity* **6**, 6 (2003).
- [152] We recommend the efficient program described in [153] for calculating all the relevant quantities and also point the reader to [154] for a program to calculate phase shifts using [153].
- [153] Black Hole Perturbation Toolkit, <http://bhptoolkit.org/>.
- [154] J. Markovic and M. M. Ivanov, *Phys. Rev. D* **114**, 024002 (2026).
- [155] J. A. H. Futterman, F. A. Handler, and R. A. Matzner, *Scattering from Black Holes*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 1988).
- [156] N. Andersson and B. P. Jensen, [arXiv:gr-qc/0011025](https://arxiv.org/abs/gr-qc/0011025).
- [157] C. V. Vishveshwara, *Nature (London)* **227**, 936 (1970).
- [158] S. Chandrasekhar and S. L. Detweiler, *Proc. R. Soc. A* **344**, 441 (1975).
- [159] E. W. Leaver, *Proc. R. Soc. A* **402**, 285 (1985).
- [160] R. A. Konoplya, *Phys. Rev. D* **68**, 024018 (2003).
- [161] E. Berti, V. Cardoso, and A. O. Starinets, *Classical Quantum Gravity* **26**, 163001 (2009).
- [162] M. Lagos and L. Hui, *Phys. Rev. D* **107**, 044040 (2023).
- [163] K. Mitman *et al.*, *Phys. Rev. Lett.* **130**, 081402 (2023).
- [164] A. Kehagias, D. Perrone, A. Riotto, and F. Riva, *Phys. Rev. D* **108**, L021501 (2023).
- [165] V. Cardoso, J. Redondo-Yuste, U. Sperhake, and F. Tuncer, [arXiv:2603.04501](https://arxiv.org/abs/2603.04501).
- [166] P. Bourg, R. Panosso Macedo, A. Spiers, B. Leather, B. Bonga, and A. Pound, *Phys. Rev. Lett.* **134**, 061401 (2025).
- [167] K. Haddad, M. Driesse, G. U. Jakobsen, and Y. F. Bautista, *J. High Energy Phys.* **05** (2026) 252.
- [168] M. M. Ivanov, Y.-Z. Li, J. Parra-Martinez, and Z. Zhou, [arXiv:2602.06951](https://arxiv.org/abs/2602.06951).