

Thrust Distribution in Electron-Positron Annihilation at Full Next-to-Next-to-Next-to-Leading-Logarithmic Accuracy Including Next-to-Next-to-Leading-Order Terms in QCD

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We consider the thrust (T) distribution in electron-positron (e^+e^-) annihilation into hadrons and we perform the all-order resummation of the large logarithms of $1 - T$ up to full next-to-next-to-next-to-leading logarithmic (N^3LL) accuracy in QCD, also including the impact of next-order logarithmic corrections (N^4LL). We consistently combine resummation with the known fixed-order results up to next-to-next-to-leading order (NNLO). All perturbative terms up to $\mathcal{O}(\alpha_s^3)$ are included in our calculation, which, thanks to a unitarity constraint, exactly reproduces, after integration over T , the next-to-next-to-next-to-leading order (N^3LO) result for the total cross section of e^+e^- into hadrons. We perform resummation in the Laplace-conjugated space, which ensures the factorization of the kinematic momentum conservation constraint, and compare our results with those obtained using the resummation formalism in the physical (T) space. We find that the differences in the spectra obtained with the two different formalisms are sizable. Nonperturbative corrections are included using an analytic hadronization model, depending on two free parameters. Finally, we present a comparison of our spectra with experimental data at the Z -boson mass (m_Z) energy, which enables us to extract the value of the QCD coupling $\alpha_s(m_Z^2) = 0.1181 \pm 0.0018$, fully consistent with the world average. We explicitly show that resumming Sudakov logarithms in Laplace-conjugated space and evaluating the inverse Laplace transform exactly is crucial in order to obtain an accurate determination of the QCD coupling.

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One of the key parameters of the standard model is the QCD coupling constant α_s , which governs the strength of the strong interactions. While QCD predicts the (logarithmic) energy scaling of the coupling via the renormalization group equation, the value of the coupling at an initial (or reference) scale has to be determined by comparing theory predictions against experimental data. Among the fundamental couplings, α_s is by far the least well-known. Increasing its accuracy, being *per se* interesting to test the overall consistency of QCD, is crucial for improving the

theoretical predictions for hard scattering processes at present and future high-energy colliders, such as the Large Hadron Collider (LHC).

A classic method to obtain a precise determination of α_s involves the analysis of shape variables in high-energy electron-positron annihilation into hadrons. Shape variables are functions of the space momenta of all the particles in the final state, which characterize the topology of an event. Shape variables are strongly sensitive to QCD radiation and thus to the precise value of the strong coupling, offering then a method for an accurate extraction of α_s . Among the first and most widely used shape variables is the thrust T [1] defined as

$$T \equiv 1 - \tau = \max_{\mathbf{n}} \frac{\sum_i |\mathbf{p}_i \cdot \mathbf{n}|}{\sum_i |\mathbf{p}_i|}, \quad (1)$$

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where the sum is over all final-state particles i with three-momentum $\mathbf{p}_i$ and the maximum is taken with respect to the direction of the unit three-vector $\mathbf{n}$. The thrust variable thus maximizes the sum of the moduli of the projected three-momenta along the vector $\mathbf{n}$; the value of $\mathbf{n}$ which reaches this maximum is called the thrust axis $\mathbf{n}_T$. As is clear from its definition in Eq. (1), the thrust is an infrared (soft and collinear) safe quantity, that is, it is insensitive to the emission of zero-momentum (soft) particles and to the splitting of one particle into two collinear ones. Therefore the thrust distribution can be safely calculated in a perturbative QCD expansion in powers of α_s . It can easily be shown that the allowed values of the thrust lie in the range $1/2 \leq T \leq 1$: it approaches $T = 1$ ($\tau = 0$) in the two-jet limit (pencil-like event) and $T = 1/2$ ($\tau = 1/2$) in the opposite limit of a completely isotropic event.

Away from the two-jet region, the perturbative QCD series for the thrust distribution is well behaved, and calculations based on the truncation at a fixed order in α_s are reliable. Since hadron production away from the back-to-back region has to be accompanied by the radiation of at least one hard recoiling parton, the leading-order (LO) term for this observable is $\mathcal{O}(\alpha_s)$. The fixed-order expansion of the thrust distribution for $T \neq 1$ is known up to $\mathcal{O}(\alpha_s^3)$. It has been obtained by numerical Monte Carlo integration of the fully differential cross section for three-jet production in electron-positron annihilation at next-to-next-to-leading (NNLO) order in QCD [2–8].

However, at large values of T ($\tau = 1 - T \ll 1$), the fixed-order QCD perturbative expansion does not provide a good approximation, because its coefficients contain large logarithmic corrections of infrared (soft and collinear) origin, of the type $\ln \tau$. The resummation of such logarithms to all orders in α_s has been formulated in the seminal papers [9,10]. More recent studies can be found in Refs. [11–23].

Resummed and fixed-order calculations must be consistently combined with each other at intermediate values of T , in order to obtain accurate QCD predictions over a wide kinematical region.

In this Letter, we present a resummed calculation of the infrared (or Sudakov) logarithms occurring in the thrust distribution in the back-to-back region, up to next-to-next-to-next-to-next-to-leading logarithmic ($N^4\text{LL}$) accuracy in full QCD. We also match the resummed calculation with the corresponding NNLO results in Ref. [5]. Therefore all perturbative terms up to $\mathcal{O}(\alpha_s^3)$ are consistently included in our calculation.

We implement a unitarity constraint in the resummation formalism in such a way that our calculation exactly reproduces, after integration over T , the corresponding fixed-order result for the total cross section of electron-positron annihilation into hadrons up to the next-to-next-to-next-to-leading order ($N^3\text{LO}$) [24,25].

We perform resummation in the Laplace-conjugated space, which ensures the factorization of the kinematic

momentum conservation constraint, and compare our results with those obtained with the resummation formalism in the physical (T) space. We find that the differences in the spectra obtained with the two different formalisms are sizable. This is because the kinematical constraints of momentum conservation do not factorize in thrust space and thus exponentiation of large Sudakov logarithms is strictly valid only in Laplace space.

We include nonperturbative (NP) effects in the thrust distribution [26–33], by using an analytic hadronization model depending on two free parameters, which enables us to perform an extraction of the value of the QCD coupling, by comparing our results with LEP and SLD data at the Z-boson mass (m_Z) energy. We explicitly show that resumming Sudakov logarithms in Laplace-conjugated space and exactly evaluating the inverse Laplace transform is crucial to obtain an accurate determination of the QCD coupling. The result of our fit, $\alpha_s(m_Z^2) = 0.1181 \pm 0.0018$, is fully consistent with the world average [34] and substantially higher than other determinations of α_s based on thrust-space resummation [15,23,35]. (See also Refs. [36,37] and the discussion in Sec. 9.4.4 of Ref. [34].)

We start by considering the differential thrust distribution $d\sigma/d\tau$ and we define the cumulative distribution

$$R_T(\tau) \equiv \frac{1}{\sigma_{\text{tot}}} \int_0^\tau d\tau' \frac{d\sigma}{d\tau'}, \quad (2)$$

where σ_{tot} is the total cross section of e^+e^- annihilation into hadrons.

The upper limit of τ , τ_{max} , (or, equivalently, the lower limit on T , $T_{\text{min}} = 1 - \tau_{\text{max}}$) depends on the number (n) of final-state partons and approaches from below the value $\tau_{\text{max}} = 1/2$, in the (formal) limit of infinite parton emissions ($n \rightarrow \infty$). The knowledge of τ_{max} is important in order to correctly normalize the cross section. In the massless approximation, for three particles ($n = 3$), $\tau_{\text{max}} = 1/3$, corresponding to a symmetric trigonal-planar configuration. For four particles ($n = 4$), $\tau_{\text{max}} = 1 - 1/\sqrt{3} = 0.4226497\dots$, which corresponds to final-state three-momenta forming a regular tetrahedron. For more than four particles, as far as we know, there are no known values of τ_{max} in the literature. For five particles ($n = 5$), τ_{max} was previously estimated to be $\tau_{\text{max}} = 0.4275$ [16]. The estimate was obtained by considering at which bin the cross section of the NNLO Monte Carlo calculation [6] vanishes. In general, finding the upper limit of τ for a generic number n of final-state particles is a nontrivial kinematical problem, as it involves a double optimization. In this Letter, to find the maximum value of τ for a given n , we have used stochastic optimization algorithms, such as the genetic algorithm and the particle swarm optimization. The results for $4 < n \leq 14$ are shown in Table I. The τ_{max} value for $n = 14$ is 0.28% smaller than the asymptotic value $\tau_{\text{max}} = 1/2$ for $n \rightarrow \infty$. The numbers in the table should be interpreted as lower

TABLE I. Maximum kinematically allowed value of τ as a function of the number n of particles in the final state.

n	3	4	5	6	7	8	9	10	11	12	13	14
$\tau_{\max}$	0.333333	0.422649	0.454	0.463	0.472	0.475	0.479	0.481	0.483	0.484	0.485	0.486

bounds to the maximum value of τ , since a stochastic algorithm is not guaranteed to find the absolute maximum. However, we observe that the results obtained with these algorithms for $n = 3$ and $n = 4$ agree (within the numerical precision of $\mathcal{O}(10^{-6})$ or better) with the exact analytical values. Conservatively, we report the values for $n > 4$ particles with an accuracy of $\mathcal{O}(10^{-3})$.

The cumulant cross section $R_T(\tau)$ in Eq. (2) can be factorized as follows [9,10]:

$$R_T(\tau) = C(\alpha_S(Q^2))\Sigma(\tau, \alpha_S(Q^2)) + D(\tau, \alpha_S(Q^2)). \quad (3)$$

$C(\alpha_S)$ is a hard-virtual factor and $D(\tau, \alpha_S)$ a short-distance remainder function vanishing at small τ , with standard (fixed-order) perturbative expansions:

$$C(\alpha_S) = 1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n C_n, \quad D(\tau, \alpha_S) = \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n D_n(\tau). \quad (4)$$

$\Sigma(\tau, \alpha_S)$ is a long-distance dominated form factor, which contains all the logarithms of soft and collinear origin enhanced at small τ , of the type $\alpha_S^m \ln^m \tau$ ($1 \leq m \leq 2n$); it can be written in an exponential form in the Laplace-conjugated space:

$$\Sigma(\tau, \alpha_S) = \frac{1}{2\pi i} \int_C \frac{dN}{N} e^{N\tau} e^{\mathcal{F}(\alpha_S, L)}, \quad (5)$$

where N is a complex variable conjugated to τ and the contour of integration C runs parallel to the imaginary axis, i.e., from $N = \gamma - i\infty$ to $N = \gamma + i\infty$ with γ a real number that lies to the right of all singularities of the integrand. The exponent $\mathcal{F}(\alpha_S, L)$ in Eq. (5) resums to all orders in α_S , towers (infinite series) of logarithms $L = \ln N$, which are large for $N \rightarrow \infty$ (and correspond, in physical space, to $\ln \tau$ terms, enhanced in the two-jet region $\tau \rightarrow 0$),

$$\mathcal{F}(\alpha_S, L) = L f_1(\lambda) + \sum_{n=2}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^{n-2} f_n(\lambda), \quad (6)$$

with $\lambda \equiv \beta_0 \alpha_S L / \pi$ and β_0 is the first-order coefficient of the QCD β function. At large N , $L \gg 1$ and $\alpha_S L$ is assumed to be $\mathcal{O}(1)$, implying that the rhs of Eq. (6) has a customary perturbative expansion in powers of α_S . The truncation of such a (function) series at a given order yields the resummation of towers of logarithmic corrections. Customarily, the approximation at N^n LL accuracy

necessitates the contributions up to $f_{n+1}(\lambda)$. We have explicitly evaluated the resummation functions $f_n(\lambda)$, with $1 \leq n \leq 5$ (explicit expressions are given in Appendix A), thus reaching the N^4 LL accuracy. We have also evaluated up to $\mathcal{O}(\alpha_S^3)$ both the coefficients of the hard factor $C(\alpha_S)$ and, by using the numerical NNLO results of Ref. [5], the remainder function $D(\alpha_S)$. By evaluating numerically the inverse Laplace transform ($N \mapsto \tau$) in Eq. (5), we have been able to perform the resummation of the large logarithms $\ln N$ in Laplace space (or N space) up to N^4 LL accuracy, matching the results up to $\mathcal{O}(\alpha_S^3)$, i.e., with the N^3 LO hard virtual corrections at small τ and the NNLO hard real (and real-virtual) corrections at large τ . The expression of the factor given in Eq. (5) involves an integration over the (formally nonintegrable) Landau singularity of the QCD running, which manifests itself in singularities of the $f_n(\lambda)$ functions at the points $\lambda = 1/2$ and $\lambda = 1$ (i.e., $N \sim N_L = \exp\{\pi/[2\beta_0 \alpha_S(Q^2)]\}$ and $N \sim N'_L = \exp\{\pi/[\beta_0 \alpha_S(Q^2)]\}$, respectively). These singularities, which signal the onset of nonperturbative phenomena at very large values of N (i.e., in the region of very small τ), have been regularized through the so-called minimal prescription of Ref. [38], in which the contour of integration C lies to the right of all physical singularities but to the left of the (unphysical) Landau pole. The results obtained by using this prescription converge asymptotically to the perturbative series and do not include any power correction.

A commonly used alternative to the numerical computation of the inverse Laplace transform is an analytic approximate calculation of the latter. In this case, the Taylor expansion of the integrand in Eq. (5) around the point [9,10,39]

$$\ln N = \ln(1/\tau) \equiv \ell \quad (7)$$

gives

$$\Sigma(\tau, \alpha_S) = \frac{1}{2\pi i} \int_C \frac{dN}{N} e^{N\tau} \exp \left[\sum_{k=0}^{\infty} \frac{\mathcal{F}^{(k)}(\alpha_S, \ell)}{k!} \ln^k(\tau N) \right], \quad (8)$$

where $\mathcal{F}^{(k)}(\alpha_S, \ell) \equiv (\partial^k / \partial \ell^k) \mathcal{F}(\alpha_S, \ell)$. Since it is not possible to evaluate the series on the rhs of Eq. (8) exactly, a hierarchy is defined in τ space. The N^n LL accuracy in τ space is defined by keeping in Eq. (8) the dominant logarithmic terms $\alpha_S^{n-1} (\alpha_S \ell)^k$, up to a given n and for all k . However, we stress that the resulting resummation

formula in τ space is only an approximation of the resummation formula in N space. In particular, the resummation in τ space at a given logarithmic accuracy does not resum all the $\ln N$ terms of the corresponding logarithmic accuracy in N space. The crucial point is that there is not an exact correspondence between $\ln N$ and $\ln(1/\tau)$ terms. The kinematic constraints of momentum conservation factorize only in N space, not in τ space, and thus exponentiation is strictly valid only in N space.

Let us also remark that the above analytic approximation is not a saddle-point expansion, which implies that the resulting expansion is not a truly asymptotic one for large N . The true saddle point is obtained by finding the value of N for which the logarithm of the integrand in Eq. (5) is stationary, that is, by solving the following equation in N [9]:

$$N\tau = 1 - f_1\left(\beta_0 \frac{\alpha_S}{\pi} \ln N\right) - \beta_0 \frac{\alpha_S}{\pi} \ln(N) f'_1\left(\beta_0 \frac{\alpha_S}{\pi} \ln N\right) - \beta_0 \sum_{n=2}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^{n-1} f'_n\left(\beta_0 \frac{\alpha_S}{\pi} \ln N\right), \quad (9)$$

where $f'_n(\lambda) \equiv df_n(\lambda)/d\lambda$. The value of N in Eq. (7) is instead the solution of the saddle point equation of the *free* theory ($\alpha_S \rightarrow 0$): $N = 1/\tau$. While the last term in the rhs of Eq. (9) is a $\mathcal{O}(\alpha_S)$ correction to the saddle point of the free theory, the same is not true for the second and the third terms, which are both $\mathcal{O}(1)$, since $\alpha_S \ln N$ is $\mathcal{O}(1)$. We have thus proved that the analytic formula based on the expansion around the point in Eq. (7) is not a fully consistent saddle point expansion and therefore not a correct asymptotic expansion for large N .

As observed in Ref. [10], while the resummation procedure uniquely determines the structure of the large logarithms in the two-jet region, it leaves an ambiguity on the non-logarithmic contributions, which is partially solved with the matching against the fixed-order result. In particular, contrary to the fixed-order case, the resummed distribution does not fulfill the exact kinematic boundary. This ambiguity can be exploited to impose the following physical constraint:

$$R_T(\tau_{\max}) = 1, \quad (10)$$

which is violated by higher-order terms beyond the nominal fixed-order accuracy of the calculation [e.g., by $\mathcal{O}(\alpha_S^4)$ terms at $N^3\text{LL} + \text{NNLO}$]. The requirement in Eq. (10), which follows from perturbative unitarity, can be fulfilled in τ space using the following replacement [10,40]:

$$\ell \equiv \ln(1/\tau) \mapsto \tilde{\ell} \equiv \ln(1/\tau - 1/\tau_{\max} + 1) \stackrel{\tau \ll 1}{=} \ell + \mathcal{O}(\tau), \quad (11)$$

which indeed acts as a perturbative unitarity constraint, since $\Sigma(\tau = \tau_{\max}, \alpha_S)|_{\ell \rightarrow \tilde{\ell}} = 1$. An analogous constraint

can be imposed in N space by means of the following replacement:

$$L \equiv \ln N \mapsto \tilde{L} \equiv \ln(N - N_c) \stackrel{N \gg 1}{=} L + \mathcal{O}(1/N), \quad (12)$$

with N_c a constant such that $\Sigma(\tau = \tau_{\max}, \alpha_S)|_{L \rightarrow \tilde{L}} = 1$. Since resummed perturbation theory controls the small- τ region, the unitarity constraint mainly acts on suppressing resummed contributions in the large- τ region. Various strategies to suppress resummed contributions in the hard region have been proposed in the literature, typically relying on switching or profile functions depending on ad-hoc boundaries and/or parameters. Our method of applying the unitarity physical constraint allows us to suppress resummed contributions in the hard region without introducing ad-hoc parameters. We note that the replacements in Eqs. (11) and (12), besides affecting $\Sigma(\tau, \alpha_S)$, have also an impact on the remainder function $D(\tau, \alpha_S)$, which has to be properly taken into account. We also observe that in the very large- τ region ($\tau \lesssim \tau_{\max}$), thrust distribution is affected by instabilities from higher-order enhanced corrections of soft and collinear origin near the kinematic fixed-order boundaries [41]. A proper treatment of these regions requires a resummation of such corrections to all orders, which gives rise to the so called Sudakov shoulder [41–44].

In the following part of this Letter, we present Laplace-space (N -space) resummed results and we compare them with corresponding results in τ space. We will show that the formally subleading terms neglected in the τ -space resummation actually give a nonnegligible contribution and turn out to be crucial for a precise extraction of the value of α_S .

In Fig. 1 we show our results for the thrust distribution at $Q = m_Z = 91.1876$ GeV with $\alpha_S(m_Z^2) = 0.118$. We plot results obtained via Laplace-space resummation at $\text{NLL} + \text{LO}$, $\text{NNLL} + \text{NLO}$ and $\text{N}^3\text{LL} + \text{NNLO}$ accuracies (solid bands), where uncertainty bands correspond to renormalization scale variation by a factor of 2 around the central value $\mu_R = Q$. We also show the impact of N^4LL corrections (solid line), which is very small (differences with respect to $\text{N}^3\text{LL} + \text{NNLO}$ result at few permille level). This is not unexpected because a full next-order accuracy would also require the inclusion of the presently unknown $\mathcal{O}(\alpha_S^4)$ corrections both in the hard factor $C(\alpha_S)$ and in the remainder function $D(\tau, \alpha_S)$.

In Fig. 1 we also show (dashed lines) perturbative predictions obtained by using momentum-space resummation at $\text{NLL} + \text{LO}$, $\text{NNLL} + \text{NLO}$, $\text{N}^3\text{LL} + \text{NNLO}$, and the impact of N^4LL terms. We observe that the differences among the predictions obtained with resummation in Laplace space and τ space are substantial; these differences are larger than the perturbative uncertainties estimated through both scale variation and the difference between two consecutive perturbative orders (momentum violation effects of similar size have been observed in Ref. [45]).

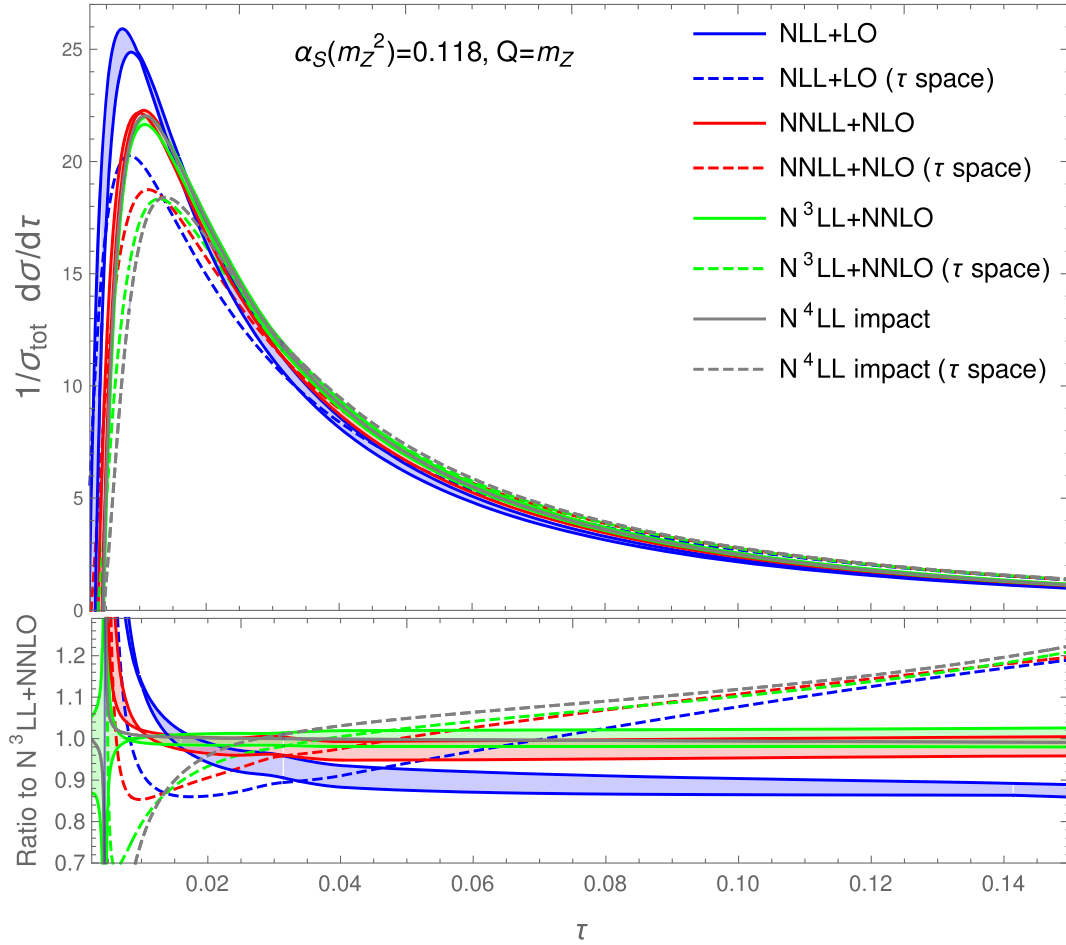


FIG. 1. The thrust distribution at $Q = 91.1876$ GeV at various perturbative orders in QCD. Results obtained through resummation in Laplace-conjugated space (solid bands) are compared with physical τ -space approximated results (dashed lines). The lower panel shows the ratios relative to the $N^3\text{LL} + \text{NNLO}$ prediction at the central value of the scale $\mu_R = Q$.

Up to now, our results entirely relied on perturbation theory. However, in order to compare our predictions with experimental data, we need to include nonperturbative (NP) hadronization effects. In this Letter, we use an analytic model based on a correlation [9] or shape function [46] $f_{NP}(\tau_h, \tau)$ depending on very few NP parameters, such that

$$\frac{d\sigma_h}{d\tau_h} = \int d\tau \frac{d\sigma}{d\tau} f_{NP}(\tau, \tau_h), \quad (13)$$

where $d\sigma_h/d\tau_h$ is the hadronic thrust distribution and τ_h is the hadronic thrust variable. We have tried various ansatz for $f_{NP}(\tau, \tau_h)$ and we have found that we can have a very good description of the LEP and SLD data at the Z boson peak ($Q = m_Z$) [47–52], by means of a Gaussian function depending on two free parameters only:

$$f_{NP}(\tau_h, \tau) = \frac{1}{\sqrt{2\pi}\sigma_{NP}} \exp \left[-\frac{(\tau_h - \tau - \delta_{NP})^2}{2\sigma_{NP}^2} \right], \quad (14)$$

where δ_{NP} represents a shift and σ_{NP} a smearing of the perturbative prediction. We have also tried to use a more general parameterization of NP effects including an extra asymmetry (or skewness) free parameter to our model (i.e., using a skew normal distribution). However, the inclusion of such an additional free parameter only marginally improves the description of the data with respect to the simpler two parameter model.

We finally performed a three-parameter (δ_{NP} , σ_{NP} and $\alpha_S(m_Z^2)$) fit in the small/intermediate τ region ($0 < \tau < 0.15$), obtaining, at $N^3\text{LL} + \text{NNLO}$ accuracy, the following values:

$$\alpha_S(m_Z^2) = 0.1181 \pm 0.0018, \quad \delta_{NP} = 0.0071 \pm 0.0007, \\ \sigma_{NP} = 0.0060 \pm 0.0013, \quad (15)$$

with $\chi^2/\text{d.o.f} \simeq 1.4$ (d.o.f is the number of degrees of freedom, that is 45). The uncertainties on the fitted parameters are calculated from $\Delta\chi^2 = 1$ contours and include experimental and theoretical (perturbative) errors,

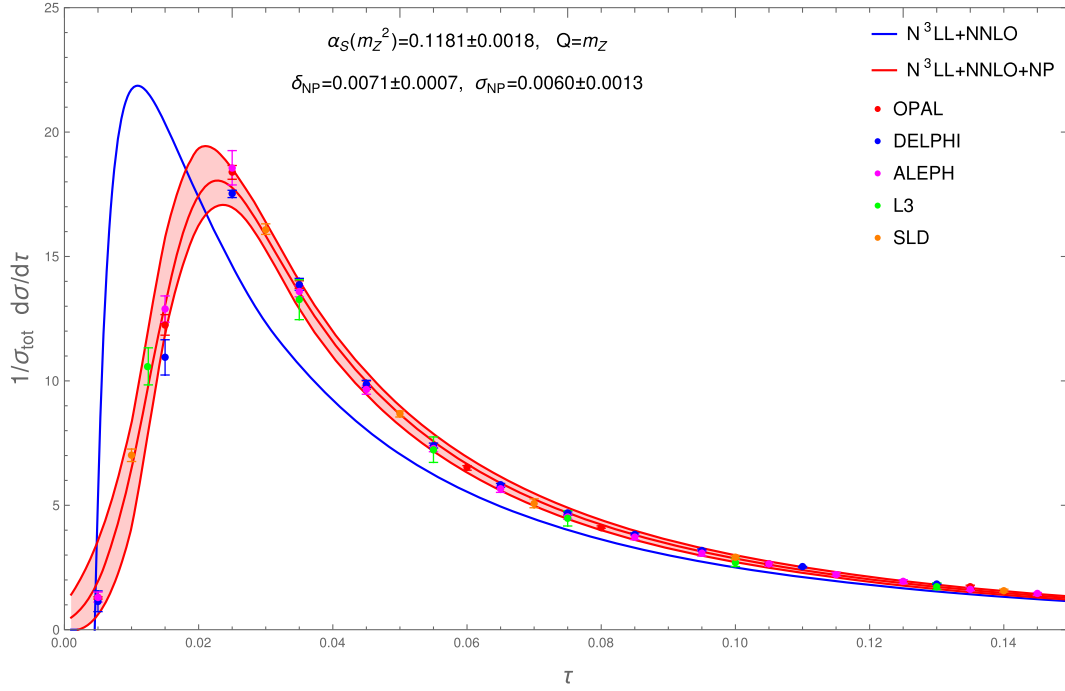


FIG. 2. The thrust distribution at $Q = 91.1876$ GeV at $N^3\text{LL} + \text{NNLO}$ in QCD without (blue solid line) and with (red band) the inclusion of NP effects.

the latter estimated by means of a renormalization scale variation of a factor of two around the central value $\mu_R = Q$.

A comparison of our best prediction against experimental LEP and SLD data at the Z boson peak ($Q = m_Z$) is shown in Fig. 2, where the uncertainty band is obtained by a variation of the extracted parameters within the uncertainties. As a check of our results, we have performed the fit at NNLL + NLO accuracy, which gives a result (fully consistent with the $N^3\text{LL} + \text{NNLO}$): $\alpha_S(m_Z^2) = 0.1194 \pm 0.0020$, $\delta_{\text{NP}} = 0.0071 \pm 0.0007$, $\sigma_{\text{NP}} = 0.0062 \pm 0.0014$ with $\chi^2/\text{d.o.f} \simeq 1.2$ (the slight reduction of the $\chi^2/\text{d.o.f}$ is due to the larger theoretical uncertainty from scale variations). We made also a fit including the (known) $N^4\text{LL}$ corrections, which, as expected by the smallness of the latter, gives basically the same results of the $N^3\text{LL} + \text{NNLO}$ analysis ($\alpha_S(m_Z^2) = 0.1177 \pm 0.0018$ with δ_{NP} , σ_{NP} and χ^2 unchanged). The stability of the fit against changes in perturbative orders is a strong indication that the extracted values of the NP parameters are largely independent of missing higher-order perturbative corrections.

Conversely, an analogous fit performed using the τ -space resummation formalism, gives a sensibly lower value of the strong coupling $\alpha_S(m_Z^2) = 0.1120 \pm 0.0019$ (with $\delta_{\text{NP}} = 0.0083 \pm 0.0010$ and $\sigma_{\text{NP}} = 0.0055 \pm 0.0020$ and $\chi^2/\text{d.o.f} \simeq 1.3$). That is a clear indication that the approximations of usual τ -space resummation formalism have to be properly taken into account in order to obtain a reliable determination of $\alpha_S(m_Z^2)$.

Finally, let us note that hadronization effects are substantial at the peak, shifting its position from $\tau \sim 0.01$ to $\tau \sim 0.02$.

In this Letter, we presented a resummed calculation of the thrust distribution in electron-positron annihilation into hadrons up to full $N^3\text{LL} + \text{NNLO}$ accuracy (including also the $N^4\text{LL}$ terms available) in QCD perturbation theory. We have performed the resummation directly in the Laplace-conjugated space and we have evaluated exactly (in numerical way) the inverse Laplace transform. We compared our theoretical distributions with LEP and SLD data at the Z-boson mass (m_Z) energy, which enabled us to extract the value of the QCD coupling $\alpha_S(m_Z^2) = 0.1181 \pm 0.0018$, fully consistent with the world average. We have also shown that the commonly used (approximate) analytic formalism in the physical space gives quite different results with respect to the Laplace-space formalism, with a corresponding lower determination of $\alpha_S(m_Z^2)$.

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End Matter

Appendix A: Resummation coefficients—In this Appendix we give the explicit expressions of the resummation coefficients up to N⁴LL and N³LO accuracy. The functions $f_i(\lambda)$ in Eq. (6) are

$$f_1(\lambda) = -\frac{A_1}{\beta_0 \lambda} [\tilde{\lambda} \ln \tilde{\lambda} - 2\tilde{\lambda} \ln \bar{\lambda}], \quad (\text{A1})$$

$$f_2(\lambda) = \frac{A_2}{\beta_0^2} \ln \frac{\tilde{\lambda}}{\lambda^2} + \frac{2\gamma_E A_1}{\beta_0} \ln \frac{\tilde{\lambda}}{\lambda} + \frac{A_1 \beta_1}{\beta_0^3} \left[\ln^2 \bar{\lambda} - \frac{\ln^2 \tilde{\lambda}}{2} - \ln \frac{\tilde{\lambda}}{\lambda^2} \right] + \frac{B_1}{\beta_0} \ln \bar{\lambda}, \quad (\text{A2})$$

$$\begin{aligned} f_3(\lambda) = & -\frac{A_3 \lambda^2}{\beta_0^2 \tilde{\lambda} \bar{\lambda}} + \frac{\beta_1 A_2}{\beta_0^3 \tilde{\lambda} \bar{\lambda}} [3\lambda^2 + \bar{\lambda} \ln \tilde{\lambda} - 2\tilde{\lambda} \ln \bar{\lambda}] - \frac{2\gamma_E A_2 \lambda}{\beta_0 \tilde{\lambda} \bar{\lambda}} + \frac{\gamma_E^2 A_1 \lambda (2\lambda - 3)}{\tilde{\lambda} \bar{\lambda}} \\ & - \frac{2\beta_1 \gamma_E A_1}{\beta_0^2 \tilde{\lambda} \bar{\lambda}} [\tilde{\lambda} \ln \bar{\lambda} - \bar{\lambda} \ln \tilde{\lambda} - \lambda] - \frac{\beta_1^2 A_1}{\beta_0^4} \left[\frac{\lambda^2}{\tilde{\lambda} \bar{\lambda}} - \frac{\ln \tilde{\lambda}}{\bar{\lambda}} (2\lambda + \ln \bar{\lambda}) + \frac{\ln^2 \tilde{\lambda}}{2\tilde{\lambda}} + \frac{2\lambda}{\tilde{\lambda}} \ln \tilde{\lambda} \right] \\ & - \frac{\beta_2 A_1}{\beta_0^3 \tilde{\lambda} \bar{\lambda}} [\lambda^2 + \tilde{\lambda} \bar{\lambda} (\ln \tilde{\lambda} - 2 \ln \bar{\lambda})] - \frac{B_2 \lambda}{\beta_0 \tilde{\lambda}} + \frac{B_1 \beta_1}{\beta_0^2 \tilde{\lambda}} [\lambda + \ln \bar{\lambda}] - \frac{\gamma_E B_1 \lambda}{\tilde{\lambda}}, \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} f_4(\lambda) = & -\frac{A_4 \lambda^2 (2\lambda^2 - 6\lambda + 3)}{3\beta_0^2 \tilde{\lambda}^2 \bar{\lambda}^2} + \frac{\beta_1 A_3}{6\beta_0^3 \tilde{\lambda}^2 \bar{\lambda}^2} [15\lambda^2 + 10(\lambda - 3)\lambda^3 + 3\bar{\lambda}^2 \ln \tilde{\lambda} - 6\tilde{\lambda}^2 \ln \bar{\lambda}] + \frac{\gamma_E A_3 \lambda (3\lambda - 2)}{\beta_0 \tilde{\lambda}^2 \bar{\lambda}^2} \\ & + \frac{\gamma_E^2 A_2 \lambda (4\lambda^3 - 12\lambda^2 + 15\lambda - 6)}{\tilde{\lambda}^2 \bar{\lambda}^2} + \frac{\beta_1 \gamma_E A_2}{\beta_0^2 \tilde{\lambda}^2 \bar{\lambda}^2} [\lambda(2 - 3\lambda) + 2\bar{\lambda}^2 \ln \tilde{\lambda} - 2\tilde{\lambda}^2 \ln \bar{\lambda}] + \frac{2\beta_2 A_2 \lambda^3 (4\lambda - 3)}{3\beta_0^3 \tilde{\lambda}^2 \bar{\lambda}^2} \\ & - \frac{\beta_1^2 A_2}{6\beta_0^4 \tilde{\lambda}^2 \bar{\lambda}^2} [\lambda^2 (22\lambda^2 - 30\lambda + 9) + 3\bar{\lambda}^2 \ln^2 \tilde{\lambda} + 3\tilde{\lambda}^2 \ln \tilde{\lambda} - 6\tilde{\lambda}^2 \ln \bar{\lambda} (\ln \bar{\lambda} + 1)] + \frac{\beta_1 \gamma_E^2 A_1}{\beta_0 \tilde{\lambda}^2 \bar{\lambda}^2} (2\bar{\lambda}^2 \ln \tilde{\lambda} - \tilde{\lambda}^2 \ln \bar{\lambda}) \\ & + \frac{\beta_0 \gamma_E^3 A_1 \lambda (12\lambda^3 - 36\lambda^2 + 39\lambda - 14)}{3\bar{\lambda}^2 \tilde{\lambda}^2} - \frac{\beta_1^2 \gamma_E A_1}{\beta_0^3 \tilde{\lambda}^2 \bar{\lambda}^2} [(4\lambda - 3)\lambda^2 - \tilde{\lambda}^2 \ln^2 \bar{\lambda} + \bar{\lambda}^2 \ln^2 \tilde{\lambda}] \\ & - \frac{\beta_1^3 A_1}{6\beta_0^5 \tilde{\lambda}^2 \bar{\lambda}^2} [4(3 - 4\lambda)\lambda^3 + 2\bar{\lambda}^2 \ln \bar{\lambda} (\ln^2 \bar{\lambda} - 3\lambda^2) + 12\bar{\lambda}^2 \lambda^2 \ln \tilde{\lambda} - \bar{\lambda}^2 \ln^3 \tilde{\lambda}] + \frac{\beta_2 \gamma_E A_1 \lambda^2 (4\lambda - 3)}{\beta_0^2 \tilde{\lambda}^2 \bar{\lambda}^2} \\ & + \frac{\beta_1 \beta_2 A_1}{6\beta_0^4 \tilde{\lambda}^2 \bar{\lambda}^2} [\lambda^2 (2\lambda(3 - 7\lambda) + 3) + 3(8\lambda^2 - 4\lambda + 1)\bar{\lambda}^2 \ln \tilde{\lambda} - 6\tilde{\lambda}^2 (-2\bar{\lambda} \lambda + 1) \ln \bar{\lambda}] \\ & - \frac{\beta_3 A_1}{6\beta_0^3 \tilde{\lambda}^2 \bar{\lambda}^2} [(2(\lambda - 3)\lambda + 3)\lambda^2 - 6(2\lambda^2 - 3\lambda + 1)^2 \ln \bar{\lambda} + 3(2\lambda^2 - 3\lambda + 1)^2 \ln \tilde{\lambda}] - \frac{B_3 (2 - \lambda) \lambda}{2\beta_0 \tilde{\lambda}^2} - \frac{\gamma_E B_2 (2 - \lambda) \lambda}{\tilde{\lambda}^2} \\ & - \frac{\beta_1 B_2 (\lambda^2 - 2\lambda - 2 \ln \bar{\lambda})}{2\beta_0^2 \tilde{\lambda}^2} - \frac{\beta_0 \gamma_E^2 B_1 (2 - \lambda) \lambda}{2\tilde{\lambda}^2} + \frac{\beta_1 \gamma_E B_1 \ln \bar{\lambda}}{\beta_0 \tilde{\lambda}^2} - \frac{\beta_2 B_1 \lambda^2}{2\beta_0^2 \tilde{\lambda}^2} + \frac{\beta_1^2 B_1 (\lambda - \ln \bar{\lambda}) (\lambda + \ln \bar{\lambda})}{2\beta_0^3 \tilde{\lambda}^2}, \end{aligned} \quad (\text{A4})$$

$$\begin{aligned}
 f_5(\lambda) = & -\frac{A_5\lambda^2}{6\beta_0^2\tilde{\lambda}^3\bar{\lambda}^3}P_1(\lambda) - \frac{A_4}{18\beta_0^3\tilde{\lambda}^3\bar{\lambda}^3}[12\lambda P_5(\lambda)\beta_0^2\gamma_E - \beta_1(6\tilde{\lambda}^3\ln\tilde{\lambda} - 12\bar{\lambda}^3\ln\bar{\lambda} + 7\lambda^2P_1(\lambda))] \\
 & + \frac{A_3}{18\beta_0^4\tilde{\lambda}^3\bar{\lambda}^3}[-9\beta_1^2\tilde{\lambda}^3\ln^2\tilde{\lambda} + 18\beta_1^2\tilde{\lambda}^3\ln^2\bar{\lambda} + 6(6\gamma_E\beta_0^2 - \beta_1)\beta_1\tilde{\lambda}^3\ln\tilde{\lambda} - 12(3\gamma_E\beta_0^2 - \beta_1)\beta_1\tilde{\lambda}^3\ln\bar{\lambda} \\
 & + \lambda(9\beta_0\beta_2\lambda(4\lambda^3 - 18\lambda^2 + 19\lambda - 6) + \beta_1^2(-52\lambda^4 + 234\lambda^3 - 303\lambda^2 + 150\lambda - 24)) + 12\gamma_E\beta_1\beta_0^2P_5(\lambda) \\
 & - 18\gamma_E^2\beta_0^4(3P_5(\lambda) - 2\lambda P_1(\lambda))] + \frac{A_2}{18\beta_0^5\tilde{\lambda}^3\bar{\lambda}^3}[\lambda(-12\gamma_E(\beta_0\beta_2 - \beta_1^2)\beta_0^2\lambda(18\lambda^2 - 25\lambda + 9) \\
 & + \lambda(3\beta_0^2\beta_3\lambda(20\lambda^3 - 54\lambda^2 + 45\lambda - 12) + \beta_1^3(100\lambda^4 - 342\lambda^3 + 339\lambda^2 - 114\lambda + 6) \\
 & - 2\beta_0\beta_2\beta_1(80\lambda^4 - 252\lambda^3 + 237\lambda^2 - 75\lambda + 3)) - 12\gamma_E^3\beta_0^6\lambda P_3(\lambda)) \\
 & + 6\beta_1^3\tilde{\lambda}^3\ln^3\tilde{\lambda} - 12\beta_1^3\tilde{\lambda}^3\ln^3\bar{\lambda} - 36\gamma_E\beta_0^2\beta_1^2(\tilde{\lambda}^3\ln^2\tilde{\lambda} - \tilde{\lambda}^3\ln^2\bar{\lambda}) \\
 & + 6\beta_1(12\gamma_E^2\beta_0^4 - (\beta_1^2 - \beta_0\beta_2)(6\lambda - 1))\tilde{\lambda}^3\ln\tilde{\lambda} - 12\beta_1(3\gamma_E^2\beta_0^4 - (\beta_1^2 - \beta_0\beta_2)(3\lambda - 1))\tilde{\lambda}^3\ln\bar{\lambda}] \\
 & + \frac{2A_1}{\tilde{\lambda}^3\bar{\lambda}^3}\left[\frac{\beta_1^4}{24\beta_0^6}(-56\lambda^6 + 108\lambda^5 - 66\lambda^4 + 12\lambda^3 - 8(4\lambda - 3)\tilde{\lambda}^3\lambda^2\ln\tilde{\lambda} + 4(2\lambda - 3)\tilde{\lambda}^3\lambda^2\ln\bar{\lambda} \right. \\
 & + 12\tilde{\lambda}^3\lambda\ln^2\tilde{\lambda} - 12\tilde{\lambda}^3\lambda\ln^2\bar{\lambda} - \tilde{\lambda}^3\ln^4\tilde{\lambda} + 2\tilde{\lambda}^3\ln^3\tilde{\lambda} + 2\tilde{\lambda}^3\ln^4\bar{\lambda} - 4\tilde{\lambda}^3\ln^3\bar{\lambda}) \\
 & + \frac{\gamma_E\beta_1^3}{6\beta_0^4}(2\tilde{\lambda}^3\ln^3\tilde{\lambda} - 3\tilde{\lambda}^3\ln^2\tilde{\lambda} - 12\lambda\tilde{\lambda}^3\ln\tilde{\lambda} - 2\tilde{\lambda}^3\ln^3\bar{\lambda} + 3\tilde{\lambda}^3\ln^2\bar{\lambda} + 6\lambda\tilde{\lambda}^3\ln\bar{\lambda} + \lambda^2P_2(\lambda)) \\
 & + \frac{\beta_1^2\gamma_E^2}{2\beta_0^2}(4\lambda^4 - 6\lambda^2 + 3\lambda - 2\tilde{\lambda}^3\ln^2\tilde{\lambda} + 2\tilde{\lambda}^3\ln\tilde{\lambda} + \tilde{\lambda}^3\ln^2\bar{\lambda} - \tilde{\lambda}^3\ln\bar{\lambda}) \\
 & - \frac{\beta_1^2\beta_2}{36\beta_0^5}(-164\lambda^6 + 306\lambda^5 - 165\lambda^4 + 12\lambda^3 + 6\lambda^2 + 18\tilde{\lambda}^3\lambda\ln^2\tilde{\lambda} - 18\tilde{\lambda}^3\lambda\ln^2\bar{\lambda} \\
 & + 12\ln\bar{\lambda}(3\lambda\tilde{\lambda}^2 - 1)\tilde{\lambda}^3 + (144\lambda^6 - 576\lambda^5 + 900\lambda^4 - 690\lambda^3 + 270\lambda^2 - 54\lambda + 6)\ln\tilde{\lambda}) \\
 & + \frac{\beta_1\gamma_E^3}{6}(-\lambda P_3(\lambda) + 8\tilde{\lambda}^3\ln\tilde{\lambda} - 2\tilde{\lambda}^3\ln\bar{\lambda}) \\
 & - \frac{\gamma_E\beta_1\beta_2\lambda}{3\beta_0^3}(-6\tilde{\lambda}^3\ln\tilde{\lambda} + 3\tilde{\lambda}^3\ln\bar{\lambda} + \lambda P_2(\lambda)) + \frac{\beta_1\beta_3}{18\beta_0^4}(-28\lambda^6 + 72\lambda^5 - 51\lambda^4 + 6\lambda^3 + 3\lambda^2 \\
 & - 3(2\lambda(\lambda(8\lambda - 9) + 3) - 1)\tilde{\lambda}^3\ln\tilde{\lambda} + 3(\lambda(\lambda(4\lambda - 9) + 6) - 2)\tilde{\lambda}^3\ln\bar{\lambda}) + \frac{\beta_0^2\gamma_E^4}{12}\lambda(2\lambda - 3)P_4(\lambda) \\
 & + \frac{\gamma_E\lambda^2\beta_3}{6\beta_0^2}P_2(\lambda) + \frac{\beta_2^2}{36\beta_0^4}(-20\lambda^6 - 18\lambda^5 + 69\lambda^4 - 42\lambda^3 + 6\lambda^2 - 12\tilde{\lambda}^3\tilde{\lambda}^3\ln\bar{\lambda} + 6\tilde{\lambda}^3\tilde{\lambda}^3\ln\tilde{\lambda}) \\
 & - \frac{\beta_4}{36\beta_0^3}(\lambda^2P_1(\lambda) - 12\tilde{\lambda}^3\tilde{\lambda}^3\ln\bar{\lambda} + 6\tilde{\lambda}^3\tilde{\lambda}^3\ln\tilde{\lambda}) - \frac{\gamma_E^2\beta_2}{2\beta_0}\lambda(4\lambda^3 - 6\lambda + 3)\Big] \\
 & - \frac{B_4\lambda}{3\beta_0\tilde{\lambda}^3}P_6(\lambda) + \frac{B_3}{3\beta_0^2\tilde{\lambda}^3}[(\lambda P_6(\lambda) + 3\ln\bar{\lambda})\beta_1 - 3\lambda P_6(\lambda)\beta_0^2\gamma_E] \\
 & - \frac{B_2}{3\beta_0^3\tilde{\lambda}^3}[3\lambda P_6(\lambda)\gamma_E^2\beta_0^4 - 6\ln\bar{\lambda}\beta_1\gamma_E\beta_0^2 - (\lambda - 3)\lambda^2\beta_2\beta_0 + ((\lambda - 3)\lambda^2 + 3\ln^2\bar{\lambda})\beta_1^2] \\
 & - \frac{B_1}{6\beta_0^4\tilde{\lambda}^3}[2\lambda P_6(\lambda)\gamma_E^3\beta_0^6 + 3(\lambda P_6(\lambda) - 2\ln\bar{\lambda})\beta_1\gamma_E^2\beta_0^4 + 6\lambda\beta_2\gamma_E\beta_0^3 \\
 & + ((3 - 2\lambda)\lambda^2\beta_3 - 6(-\ln^2\bar{\lambda} + \ln\bar{\lambda} + \lambda)\beta_1^2\gamma_E)\beta_0^2 + 2\lambda(\lambda(2\lambda - 3) - 3\ln\bar{\lambda})\beta_1\beta_2\beta_0 \\
 & + (-2\ln^3\bar{\lambda} + 3\ln^2\bar{\lambda} + 6\lambda\ln\bar{\lambda} + (3 - 2\lambda)\lambda^2)\beta_1^3],
 \end{aligned} \tag{A5}$$

$$\begin{aligned}
 P_1(\lambda) &= 4\lambda^4 - 18\lambda^3 + 33\lambda^2 - 24\lambda + 6, & P_2(\lambda) &= 12\lambda^3 - 36\lambda^2 + 32\lambda - 9, \\
 P_3(\lambda) &= -24\lambda^5 + 108\lambda^4 - 198\lambda^3 + 193\lambda^2 - 99\lambda + 21, & P_4(\lambda) &= 28\lambda^4 - 84\lambda^3 + 105\lambda^2 - 63\lambda + 15, \\
 P_5(\lambda) &= 7\lambda^2 - 9\lambda + 3, & P_6(\lambda) &= \lambda^2 - 3\lambda + 3,
 \end{aligned} \tag{A6}$$

where $\lambda = (\beta_0/\pi)\alpha_s \ln N$, $\bar{\lambda} = 1 - \lambda$, $\tilde{\lambda} = 1 - 2\lambda$, and $\gamma_E = 0.5772 \dots$ is the Euler-Mascheroni constant. The QCD β function coefficients read, up to five loops [54–58],

$$\beta_0 = \frac{11C_A - 2n_f}{12}, \quad \beta_1 = \frac{1}{24}(17C_A^2 - 5C_An_f - 3C_Fn_f), \tag{A7}$$

$$\beta_2 = \frac{1}{64} \left[\frac{2857}{54}C_A^3 - \left(\frac{1415}{54}C_A^2 + \frac{205}{18}C_AC_F - C_F^2 \right)n_f + \left(\frac{79}{54}C_A + \frac{11}{9}C_F \right)n_f^2 \right], \tag{A8}$$

$$\beta_3 = \frac{1093}{186624}n_f^3 + n_f^2 \left(\frac{809\zeta_3}{2592} + \frac{50065}{41472} \right) + n_f \left(-\frac{1627\zeta_3}{1728} - \frac{1078361}{41472} \right) + \frac{891}{64}\zeta_3 + \frac{149753}{1536}, \tag{A9}$$

$$\beta_4 = -0.00179929n_f^4 - 0.225857n_f^3 + 17.156n_f^2 - 181.799n_f + 524.558, \tag{A10}$$

where n_f is the number of QCD active (effective massless) flavors at the hard scale Q ($n_f = 5$ for $Q = m_Z$) and $\zeta_3 = 1.20206 \dots$. The coefficients A_i and B_i are extracted from the fixed-order calculations for the thrust distribution [12,59–68] and the cusp anomalous dimensions [69–72],

$$A_1 = C_F, \quad A_2 = \frac{C_F}{2} \left[\left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9}n_f \right], \tag{A11}$$

$$A_3 = \frac{25 + 3\pi^2}{243}n_f^2 + \frac{288\zeta_3 - 72\pi^2 - 2381}{324}n_f + \frac{77515 - 390\pi^2 - 35640\zeta_3 + 198\pi^4}{1080}, \tag{A12}$$

$$\begin{aligned}
 A_4 &= \left(\frac{4\zeta_3}{81} - \frac{\pi^2}{27} + \frac{91}{4374} \right) n_f^3 + \left(-\frac{407\zeta_3}{108} - \frac{31\pi^4}{1080} + \frac{23\pi^2}{12} + \frac{147199}{46656} \right) n_f^2 \\
 &+ \left(-\frac{125\zeta_5}{18} - \frac{7\pi^2\zeta_3}{36} + \frac{71057\zeta_3}{648} + \frac{143\pi^4}{180} - \frac{266\pi^2}{9} - \frac{1820789}{15552} \right) n_f \\
 &+ \frac{3025\zeta_5}{12} - 6\zeta_3^2 + \frac{99\pi^2\zeta_3}{8} - \frac{24451\zeta_3}{24} - \frac{2033\pi^6}{15120} - \frac{759\pi^4}{160} + \frac{20483\pi^2}{144} + \frac{4311229}{5184},
 \end{aligned} \tag{A13}$$

$$A_5 = 0.001302\Gamma_{\text{cusp}}^{(4)} + 16445.4 - 5225.93n_f + 567.497n_f^2 - 24.5952n_f^3 + 0.356813n_f^4, \tag{A14}$$

$$B_1 = -\frac{3C_F}{2}, \quad B_2 = C_AC_F \left(\frac{3\zeta_3}{2} - \frac{57}{16} + \frac{11\pi^2}{24} \right) + C_F^2 \left(-3\zeta_3 + \frac{\pi^2}{4} - \frac{3}{16} \right) + C_Fn_f \left(\frac{5}{8} - \frac{\pi^2}{12} \right), \tag{A15}$$

$$\begin{aligned}
 B_3 &= \left(-\frac{20\zeta_3}{81} + \frac{173\pi^2}{972} - \frac{191}{324} \right) n_f^2 + \left(\frac{62\zeta_3}{9} + \frac{52\pi^4}{405} - \frac{1109\pi^2}{162} + \frac{2467}{108} \right) n_f \\
 &- \frac{155\zeta_5}{9} - \frac{4\pi^2\zeta_3}{81} - \frac{481\zeta_3}{27} - \frac{2539\pi^4}{1080} + \frac{26189\pi^2}{432} - \frac{18797}{108},
 \end{aligned} \tag{A16}$$

$$B_4 = 2592.79 - 620.792n_f + 44.0707n_f^2 - 0.949394n_f^3, \tag{A17}$$

where $\zeta_5 = 1.03693 \dots$ and $\Gamma_{\text{cusp}}^{(4)} \simeq 49873.5$ for $n_f = 5$ [72].

The hard-virtual coefficients C_i are derived from the quark form factor [73–76] and the constant terms in the collinear/soft functions [12,59–68,77]

$$C_1 = C_F \left(-\frac{5}{4} + \frac{3\gamma_E}{2} - \gamma_E^2 \right), \quad (\text{A18})$$

$$\begin{aligned} C_2 = & C_A C_F \left(\frac{41\zeta_3}{6} - \frac{3\gamma_E \zeta_3}{2} - \frac{491}{96} + \frac{57\gamma_E}{16} - \frac{169\gamma_E^2}{144} - \frac{11\gamma_E^3}{12} - \frac{275\pi^2}{864} - \frac{11\gamma_E \pi^2}{24} + \frac{\gamma_E^2 \pi^2}{12} + \frac{31\pi^4}{1440} \right) \\ & + C_F^2 \left(-\frac{9\zeta_3}{2} + 3\gamma_E \zeta_3 + \frac{41}{32} - \frac{27\gamma_E}{16} + \frac{19\gamma_E^2}{8} - \frac{3\gamma_E^3}{2} + \frac{\gamma_E^4}{2} + \frac{17\pi^2}{96} - \frac{\gamma_E \pi^2}{4} + \frac{19\pi^4}{720} \right) \\ & + C_F n_f \left(-\frac{5\zeta_3}{6} + \frac{35}{48} - \frac{5\gamma_E}{8} + \frac{11\gamma_E^2}{72} + \frac{\gamma_E^3}{6} + \frac{25\pi^2}{432} + \frac{\gamma_E \pi^2}{12} \right), \end{aligned} \quad (\text{A19})$$

$$C_3 = -0.573462n_f^2 + 4.90384n_f + 13.4156. \quad (\text{A20})$$

To avoid cumbersome formulas, in the higher-order terms we have replaced the explicit value of the color factors of $SU(N_c = 3)$, namely $C_F = 4/3$, $C_A = 3$.