

Impediment to Torsion from Spectral Geometry

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Modifications of standard general relativity that bring torsion into the game have a long-standing history. However, no convincing arguments exist for or against its presence in physically acceptable gravity models. In this Letter, we provide an argument based on spectral geometry (using methods of pseudodifferential calculus) that suggests that the torsion shall be excluded from the consideration. We demonstrate that there is no well-defined functional extending to the torsion-full case of the spectral formulation of the Einstein tensor.

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Introduction—Riemannian and Lorentzian geometries form the fundamental basis of modern theoretical physics, especially in the context of gravity theories. General relativity is a particular example of how geometric concepts influence our understanding of the world around us, ranging from mathematical ideas explaining the expansion of the universe to applications that make our everyday lives easier, such as the global positioning system. Differential operators play an important role in the mathematical description of physical problems. It is well known that their spectra not only encode several characteristics of a system's dynamics but also contain information about certain properties of the underlying geometric structure. This idea lies at the heart of the famous question posed by Mark Kac: Can one hear the shape of a drum? [1], which is often considered the birth of modern spectral geometry—a branch of mathematics that studies the geometric properties of objects by analyzing the spectra of operators associated with them. Spectral methods are now widely used in modern physics; in particular, quantum-mechanical

systems are naturally studied by examining the spectra of the associated Hamiltonians. Later, a groundbreaking idea by Alain Connes led to the formulation of non-commutative geometry—an approach to geometry in which algebraic structures and the spectra of certain operators play an essential role [2,3]. This formulation allows us to go beyond classical spaces, incorporating models that could be referred to as quantum spaces, with potential applications in other branches of physics, such as particle physics [4] and condensed matter physics [5]. Despite the rapid development of new methods and the results achieved in this field, several open questions remain unanswered, even within the purely classical framework.

The main objects of interest in classical spectral geometry are functionals defined for (pseudo-)differential operators on certain vector bundles. More precisely, let M be an even-dimensional $[n \equiv \dim(M) = 2m]$ oriented, closed Riemannian manifold, take $E \rightarrow M$ to be a (complex) vector bundle over M , and consider the algebra $\Psi\text{DO}(E)$ of classical pseudodifferential operators (ΨDO) [6,7] acting on sections of this bundle. As demonstrated by Mariusz Wodzicki [8], this algebra possesses a unique (up to a multiplicative constant) trace, called the Wodzicki residue,

$$\mathcal{W}\text{res}(P) = \int_M \left(\int_{|\xi|=1} \text{Tr}(\sigma_{-n}(P)(x, \xi)) \right) d^n x. \quad (1)$$

Here, Tr denotes the trace over $\text{End}(E)$ (at a given point), and $\sigma_{-n}(P)$ is the symbol of order $-n$ of $P \in \Psi\text{DO}(E)$. We recall that for a differential operator of order d , $P = \sum_{|\alpha|=0}^d (-i)^{|\alpha|} a_\alpha(x) \partial_x^\alpha$, its symbol (in multi-index notation)

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reads $\sum_{|\alpha|=0}^d a_\alpha(x) \xi^\alpha$; this notion can be extended to the full algebra of $\Psi\text{DO}(E)$ —cf. Ref. [9].

One of the crucial observations made in the early days of this field was that the Wodzicki residue of powers of the Laplace operator reproduces certain geometric quantities. More precisely, it has been demonstrated that

$$\begin{aligned} \mathcal{W}\text{res}(f\Delta^{-m}) &\sim \int_M f d\text{vol}_g, \\ \mathcal{W}\text{res}(f\Delta^{-m+1}) &\sim \int_M f R(g) d\text{vol}_g, \end{aligned} \quad (2)$$

i.e., the volume form and the Ricci scalar can be spectrally reconstructed; cf. Refs. [4,10,11]. A natural question that arises is whether it is possible to encode spectrally (in terms of the Wodzicki residue) other geometric quantities, like the metric, the Riemann tensor, the Ricci tensor, or the Einstein tensor. A positive answer for the metric and the Einstein tensor was recently found in Ref. [12]. Spectral methods were also used to examine torsion in Ref. [13], providing an intrinsic method to detect its presence in the Dirac operator. In this Letter, we argue that the presence of torsion in gravity models hinders the spectral reconstruction of geometric quantities within spectral geometry.

Einstein functional with torsion—The natural way of considering the effect of torsion is to study the Einstein functional. To do so, it is convenient to consider M to be a spin manifold and the Dirac operator D to act on the sections of the spinor bundle. The Einstein functional is a bilinear functional on the space of differential forms,

$$\mathcal{G}(u, w) = \mathcal{W}\text{res}(u\{D, w\}D|D|^{-n}), \quad (3)$$

and, for the Dirac operator that comes from the torsion-free Levi-Civita connection, we have

$$\mathcal{G}(u, w) \sim \int_M d\text{vol}_g G^{ab} u_a w_b, \quad (4)$$

where G^{ab} is the Einstein tensor [12]. Such functionals, whose densities are represented as evaluations of tensors on forms, we call tensor-type functionals.

Following this logic, we compute the spectral Einstein functional (3) in the presence of torsion. We have $D = D_0 + B$, with D_0 being the torsionless Dirac operator constructed from the Levi-Civita connection lifted to the spinor bundle, $D_0 = i\gamma^a \nabla_a^{(s)}$, and $B = -(i/8)T_{abc}\gamma^a\gamma^b\gamma^c$, the 3-form perturbation, with γ^i Clifford generators and T the (antisymmetric part of) torsion. With torsion arising from the linear connection, the Dirac operator is uniquely fixed and has a zero-order component, which is linear in torsion.

The square of such a Dirac operator is a Laplace-type operator L , that is, its symbol at any given point x on the manifold is given by $\sigma(L) = \mathbf{a}_2 + \mathbf{a}_1 + \mathbf{a}_0$, where, in

normal coordinates at this point, the respective homogeneous symbols are

$$\begin{aligned} \mathbf{a}_2 &= \left(\delta_{ab} + \frac{1}{3} R_{acbd} x^c x^d \right) \xi_a \xi_b + o(\mathbf{x}^2), \\ \mathbf{a}_1 &= i(P_{ab} x^b + S_a) \xi_a + o(\mathbf{x}), \\ \mathbf{a}_0 &= Q + o(\mathbf{1}), \end{aligned} \quad (5)$$

where R_{acbd} is the Riemann tensor, and P_{ab} , S_a , and Q are tensors valued in endomorphisms of the vector bundle, evaluated at the given point on the manifold. Note that the usual Laplace operator acting on functions corresponds to $P_{ab} = \frac{2}{3} \text{Ric}_{ab}$, where Ric is the Ricci tensor, and $S_a = Q = 0$. For a second-order differential operator O , with a symbol expressed in normal coordinates around x , $\sigma(O) = F^{ab} \xi_a \xi_b + iG^a \xi_a + H + o(\mathbf{1})$, where $F^{ab} = F^{ba}$, G^a , and H are endomorphisms of the fiber at point x , we have

$$\begin{aligned} \mathcal{W}\text{res}(OL^{-m}) &= \frac{\nu_{n-1}}{24} \int_M d\text{vol}_g \text{Tr} [24H + 12G^a S_a \\ &\quad + F^{aa}(-12Q + 6P_{bb} - 2R - 3S_b S_b) \\ &\quad + 2F^{ab}(-6P_{ab} + 2\text{Ric}_{ab} - 3S_a S_b)], \end{aligned} \quad (6)$$

where R is the scalar curvature; see Ref. [14] for a detailed derivation. This fact, together with the knowledge of the symbols of L^{-k} , allows us to compute the density of the Einstein functional [15] using standard techniques of the pseudodifferential calculus (see, e.g., Ref. [9]). The resulting expression is of the form $\mathcal{G}_{D_0}(u, w)(x) + \delta\mathcal{G}(u, w)(x)$, with $\mathcal{G}_{D_0}$ being the standard density of the Einstein functional for the torsionless case and

$$\begin{aligned} \delta\mathcal{G}(u, w)(x) &= \frac{\nu_{n-1}}{2} \text{Tr} \left\{ iu_a w_{bc} [\gamma^a, \gamma^b] \{\gamma^c, B_0\} \right. \\ &\quad + u_a w_b \left[\frac{i}{2} [\gamma^a, \gamma^b] \{\gamma^c, B_c\} \right. \\ &\quad + (\delta^{ab} B_0 - \gamma^a \{\gamma^b, B_0\}) \\ &\quad \left. \left. \times (\{\gamma^c, B_0\} \gamma^c - 2B_0) \right] \right\}, \end{aligned} \quad (7)$$

where the expansions in normal coordinates were used: $B = B_0 + B_a x^a + o(\mathbf{x})$, $w = (w_a + w_{ab} x^b + w_{abc} x^b x^c) \gamma^a + o(\mathbf{x}^2)$, etc. The above expression, however, has an important disadvantage: it cannot represent a well-defined functional of a tensor type, since there is an explicit dependence on the derivatives of w , unless $\text{Tr}([\gamma^a, \gamma^b] \{\gamma^c, B_0\})$ vanishes. This is, however, not possible for the endomorphism B representing a torsion T , unless the torsion itself is zero. One can then argue that perhaps there is a potential modification of the spectral Einstein functional that does not include derivatives and reduces to the classical result in

the torsionless case. In this Letter, we argue that no such coherent modification is possible.

Modified Einstein functionals—By a power-counting argument for pseudodifferential calculus, we argue that potential modifications of the spectral Einstein functional are of the form $\mathcal{G}' = \sum_{k \in \mathbb{Z}} (\alpha_k F_1(k) + \beta_k F_2(k))$, with (only finitely many nonzero) $\alpha_k, \beta_k \in \mathbb{C}$, and

$$\begin{aligned} F_1(k) &= \mathcal{W} \text{res}(u|D|^k w |D|^{-n-k+2}), \\ F_2(k) &= \mathcal{W} \text{res}(u|D|^k w D |D|^{-n-k}). \end{aligned} \quad (8)$$

The previously defined Einstein functional is reconstructed with $F_1(0) + F_2(0)$. To start analyzing the problem, we begin with the expression for F_1 . The main step in this computation is to find the expressions for the symbol of $[|D|^k, w_b \gamma^b]$ —for details, see Ref. [14]. We then analyze the Wodzicki residue of $u|D|^k w |D|^{-n-k+2}$, taking into account only those terms that depend on the derivatives of w or are linear in w but contain contractions of the Riemann tensor. We omit terms that depend only on the forms u and w , but not on their derivatives, and that contain terms quadratic in torsion. As a result, we obtain [14] that, up to an irrelevant global constant,

$$\begin{aligned} F_1(k) &= \frac{2-n}{24} \text{Tr}(\mathbb{1}) u_a w_a R - \frac{k(n+k-2)}{n} \text{Tr}(\mathbb{1}) u_a w_{acc} \\ &\quad - i \frac{k(k+n-2)}{2n} u_a w_{bc} \text{Tr}([\gamma^a, \gamma^b] \{\gamma^c, B_0\}) \\ &\quad + (\dots), \end{aligned} \quad (9)$$

where (...) represents all the remaining terms not pertinent to our analysis. Similarly, by analyzing the symbol of $D|D|^k$, we deduce (see Ref. [14]) that

$$\begin{aligned} F_2(k) &= \left[\frac{n-2}{24} u_a w_a R + \frac{1}{6} u_a w_a G_{ab} \right. \\ &\quad \left. + \frac{k(n+k)}{n+2} u_a w_{acc} - \frac{4k(n+k)}{n(n+2)} u_a w_{bab} \right] \text{Tr}(\mathbb{1}) \\ &\quad + i \left(\frac{k(n+k)}{2n} + \frac{1}{2} \right) u_a w_{bc} \text{Tr}([\gamma^a, \gamma^b] \{\gamma^c, B_0\}) \\ &\quad + (\dots). \end{aligned} \quad (10)$$

Let us now consider the modified spectral Einstein functionals $\mathcal{G}'$ and the necessary and sufficient conditions for the result not to depend on the derivatives of w . First, observe that to eliminate mixed second-order derivatives of w in $\sum_k \beta_k F_2(k)$, we must have $\sum_k \beta_k k(n+k) = 0$. Next, we must independently eliminate the second-order derivatives in $\sum_k \alpha_k F_1(k)$, leading to $\sum_k \alpha_k k(n+k-2) = 0$.

For terms with first-order derivatives, we easily deduce that $\sum_k \beta_k = 0$ or that $\text{Tr}([\gamma^a, \gamma^b] \{\gamma^c, B_0\})$ must vanish identically. However, the former is not acceptable because

first-order derivatives would be eliminated, and the standard Einstein tensor (even in the torsionless case) would have to vanish. Therefore, the only physically acceptable solution is that $\text{Tr}([\gamma^a, \gamma^b] \{\gamma^c, B_0\})$ vanishes. This is impossible in the case where the perturbation is a 3-form, which is the case with antisymmetric torsion [17].

Conclusions—We have demonstrated that spectral geometry imposes a strong obstruction to the presence of torsion in geometric models: for the spectral Einstein functional to be well defined, the torsion T must vanish [18]. Note that, of course, if torsion exists as a fundamental field in an Einstein-Cartan type of gravity theory, then the result of this Letter would point out that it is not spectral in nature, that is, it does not arise from the spectral properties of relevant operators. On the other hand, our results are restricted to the case of torsion arising from a linear connection and cannot be applied to any effective torsion-like fields (e.g., an axial vector field in four dimensions). Therefore, the results presented in this Letter pertain to gravity as a fundamental theory and do not apply directly to models in which classical gravity emerges as an effective theory, such as in string theory or loop quantum gravity (LQG). Specifically, in models arising from the former, the geometric torsion is commonly identified with the strength of the Kalb-Ramond (KR) field [19,20]. However, the absence of torsion does not necessarily imply the vanishing of the KR field; rather, it imposes the condition of its closedness. This could have implications for predictions in magnetic monopole models, as illustrated in Ref. [21]. Conversely, in models originating from LQG, the presence of fermionic matter fields allows for potential measurements of the Barbero-Immirzi (BI) parameter, which is determined by the torsion and effectively governs the strength of the resulting four-fermion interaction, as discussed in Ref. [22]. Taken literally, our results would predict the absence of such interactions. Moreover, a specific class of teleparallel models may be characterized by a self-adjoint spin-Dirac operator, incorporating the Weitzenböck connection, as addressed in the present work. The results presented herein demonstrate that, for such an operator, the antisymmetric component of the torsion must necessarily vanish, thereby rendering these teleparallel gravity models incompatible with spectral geometry. The remaining components of the torsion—namely, the vectorial and pure tensor parts—either give rise to a non-Hermitian Dirac operator or have no effect on the spinorial degrees of freedom.

Despite the potentially compelling possibilities of analyzing gravity models with torsion, describing the twisting of the reference frame along geodesics, the structural arguments based on spectral geometry provide a strong indication that physically acceptable models should be based on torsionless connections. Thus, the widely used assumption of describing physical systems based on the Levi-Civita connection seems to be, from the spectral

perspective, more fundamental than previously supposed. We speculate that spectral geometry methods could lead to further structural obstructions for the building blocks of modified gravity models. We postpone this intriguing possibility for future research.

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