

Topological Phase Transitions in a Constrained Two-Qubit Quantum Control Landscape

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In optimal quantum control, control landscape phase transitions (CLPTs) indicate sharp changes occurring in the set of optimal protocols, as a physical model parameter is varied. Here, we demonstrate the existence of a new class of CLPTs, associated with changes in the topological properties of the optimal level set in a two-qubit state-preparation problem. In particular, the distance distribution of control protocols sampled through stochastic homotopic dynamics reveals discontinuous changes in the number of connected components in the optimal level set, as a function of the protocol duration. We demonstrate how topological CLPTs can be detected in modern-day experiments.

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Optimal control theory is central for modern-day quantum technologies where quantum properties are preserved and exploited through precise and efficient hardware manipulations [1–4]. Despite the numerous advances in recent years, complex quantum systems remain difficult to control. The procedure aiming at finding the optimal control protocol [Fig. 1(a)] often terminates in a suboptimal solution, while the true-optimum remains inaccessible or extremely difficult to find [5–7]. This situation is widespread in optimization problems [8–16] where one is typically interested in characterizing the problem’s complexity. Common approaches analyze the critical sets of the optimization landscape—stationary points of the cost function that assigns a figure of merit to each control protocol [Fig. 1(b)] [17–22]. Among the critical sets, one distinguishes the optimal level set, consisting of protocols with the best performance in the control task.

The quantum control landscape depends on external model parameters, e.g., the total protocol duration. Varying these can lead to sharp deformations of the optimal level set; in this case, specific order parameters designed to capture statistical properties of optimal protocols [23–26], exhibit nonanalytic points (cusps, jump discontinuities, etc.) called “Control Landscape Phase Transitions”

(CLPTs) [27], in analogy to transitions in statistical mechanics. As the optimization complexity depends on the structure of the landscape, a thorough understanding of

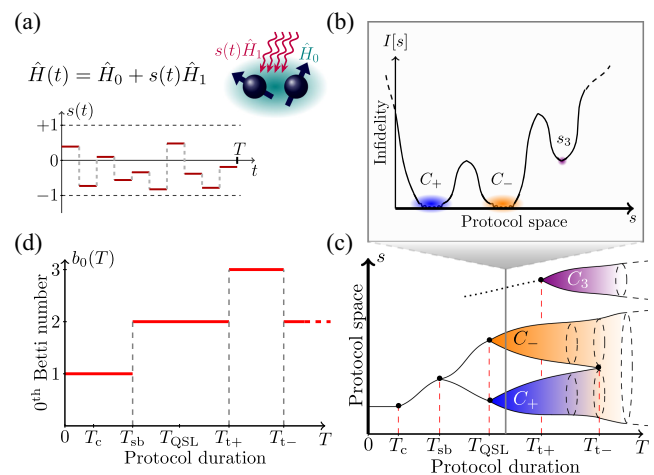


FIG. 1. Topological transitions in the two-qubit quantum control landscape in Eq. (1). (a) Piecewise-constant protocols $s(t)$ control a system H_0 by modulating the external drive H_1 for a total duration T . (b) Illustration of the optimization landscape $I_T[s]$ for a fixed protocol duration T . Depending on T , the landscape possesses a different number of basins of attraction; in this schematic, two basins are associated with the connected components C_+ , C_- of the infinite-dimensional optimal level set and one with the local trap s_3 . (c) Sketch of the optimal level set $\{\arg \min_s I_T[s]\}$ as a function of T . Topological transitions at T_{sb} , T_{t+} , and T_{t-} mark sudden change in the number of connected components $b_0(T)$. (d) Topological CLPTs correspond to jump discontinuities of a topological invariant known as the 0th Betti number $b_0(T)$ as a function of T ; the T_{QSL} transition does not affect $b_0(T)$.

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CLPTs is of central importance for the characterization of quantum control problems. So far, CLPTs have been associated with local changes in the optimal level set, i.e., changes involving the neighborhood of isolated optimal protocols [28]. In this Letter, we raise the question of whether CLPTs can exist, associated with changes in the topology (i.e., global changes) of the optimal level set.

Investigating topological properties requires probing the global structure of the systems of interest [29–33]. In the case of quantum control landscapes, an analytical parametrization of the optimal level set is generally missing, so methods of differential topology are not applicable [3,34]. Conversely, numerical methods do not necessarily give information about topological properties; e.g., “bang-bang” protocols (alternating sequences of pulses of fixed amplitude) do not allow for generic continuous transformations (homotopies), used in a topological analysis.

In this Letter, we study a prototypical quantum control problem, and report a new class of control landscape phase transitions, characterized by changes in the topological structure of the optimal level set [Fig. 1(c)]. We develop methods to explore different connected components via stochastic homotopic transformations, and find discontinuous changes in their number—a topological invariant known as the 0th Betti number [Fig. 1(d)] [35]. Finally, in the End Matter we discuss how modern-day quantum experiments can track topological CLPTs via standard measurements of observables (e.g., qubit polarization).

Model—Consider a state preparation problem associated with the two-qubit Hamiltonian

$$\hat{H}(t) = -J\hat{S}_1^z\hat{S}_2^z - h_z(\hat{S}_1^z + \hat{S}_2^z) - s(t)h_x(\hat{S}_1^x + \hat{S}_2^x), \quad (1)$$

where $\hat{S}^\alpha$ ($\alpha = x, y, z$) are the $\mathfrak{su}(2)$ generators describing the qubit, and $s(t)$ is the control protocol. Let us fix $J = 2$, $h_z = 1$, $h_x = \sqrt{5}$ and set the initial $|\psi_0\rangle$ and target $|\psi_*\rangle$ states as ground states of $\hat{H}[h_x = -2]$ and $\hat{H}[h_x = +2]$, respectively. We restrict the protocol space to bounded functions $|s(t)| \leq 1$, reflecting physical limitations in the output power of the applied external field [36].

The infidelity $I_T[s] = 1 - |\langle\psi_*|\hat{U}[s](T)|\psi_0\rangle|^2$ defines the cost function of the state-preparation problem; it depends on the duration of the quantum evolution $T(t \in [0, T])$ and on the time-dependent protocol $s(t)$. The mapping $s \mapsto I_T[s]$ from protocol space to the interval $[0, 1]$ defines the control landscape. This control problem possesses two characteristic durations of the quantum evolution, T_c and T_{QSL} , associated with different types of instabilities leading to control phase transitions in the optimal level set $\{\arg \min_s I_T[s]\}$ [28]. At T_c , the optimal control protocol becomes singular [3,4,28]. T_{QSL} is the critical duration, past which one can prepare the target with arbitrarily small infidelity via numerical optimization (QSL: quantum speed limit). The specific choice of parameters h_z, h_x, J affects the precise location of these

CLPTs but does not lead to a qualitative change in the control phase diagram. In addition, this control problem possesses a $\mathbb{Z}_2$ symmetry-breaking transition: at $T = T_{\text{sb}}$ optimal protocols break the control symmetry $s(t) \rightarrow -s(T-t)$; this CLPT depends on the choice of initial and target states $|\psi_{0,*}\rangle$, respecting the cost function symmetry $I_T[s(t)] = I_T[-s(T-t)]$ [27].

Methods—Control phase transition are a property of the control landscape and do not depend on the specific algorithm used to probe the landscape. We now generalize this concept to include topological CLPTs, which occur as discontinuous changes in the topological invariants of the optimal level set $\{\arg \min_s I_T[s]\}$ [Fig. 1(c)]. Topological invariants capture global properties of topological spaces. For this reason, they are in general inaccessible by analysis limited to the local properties.

In this Letter, we focus on changes in the number of connected components $b_0 = b_0(T) \in \mathbb{N}$ (so-called 0th Betti number). To detect topological transitions, we use the Langevin-Monte Carlo algorithm (LMC) [28], which we now briefly recap. First, the time interval $[0, T]$ is divided in L equidistant steps $\{t_1, \dots, t_L\}$ of length $\delta t = T/L$ (Fig. 1(a)). LMC explores the control landscape using purely diffusive stochastic dynamics which updates the piecewise-constant protocol $s = (s(t_1), \dots, s(t_L))$. The optimization dynamics depends on two parameters: σ sets the typical length of the local update $s_{n+1}(t) = s_n(t) + \xi(t)(\xi(t)$ is a Gaussian-distributed random variable, with zero mean and variance σ^2); β controls the Monte Carlo acceptance probability $e^{-\beta\Delta I_{n+1}}$ which depends on the infidelity increment $\Delta I_{n+1} = I(s_{n+1}) - I(s_n)$ following the update. We then run R independent LMC simulations, starting from different initial protocols. We collect R independent sets, $\{\mathcal{S}_\alpha\}_{\alpha=1}^R$, each containing M sampled protocols: $\mathcal{S}_\alpha = \{s_1^\alpha, \dots, s_M^\alpha\}$. Between two sampled protocols, we wait for $\Delta n = 2^{14}$ LMC iterations, corresponding to $\Delta n L$ attempted local updates [28]. As β is increased, the LMC dynamics approximates a continuous deformation (i.e., a homotopy) of the protocol s within the optimal level set. In the limit $\beta \rightarrow \infty$, each LMC run is strictly confined to a single connected component after waiting for sufficiently many iterations. We extract global information about the optimal level set by comparing properties of the independent sets $\{\mathcal{S}_\alpha\}$.

First, we study the distance distribution between pairs of sets in $\{\mathcal{S}_\alpha\}$. The squared L^2 distance between two protocols $s_1, s_2: [0, T] \rightarrow [-1, 1]$ is defined as $d^2(s_1, s_2) = T^{-1} \int_0^T dt (s_1(t) - s_2(t))^2$. The average distance $d_{\text{avg}}(\mathcal{S}_\alpha, \mathcal{S}_\beta)$ between two sets $\mathcal{S}_\alpha, \mathcal{S}_\beta$ is defined as

$$d_{\text{avg}}(\mathcal{S}_\alpha, \mathcal{S}_\beta) = \frac{1}{M^2} \sum_{i,j=1}^M d(s_i^\alpha, s_j^\beta). \quad (2)$$

The distance $d(s_1, s_2)$ is a central concept in topological data analysis [37,38], closely related to the protocol overlap, $s_1 \cdot s_2 = T^{-1} \int_0^T dt s_1(t)s_2(t)$, routinely used to investigate

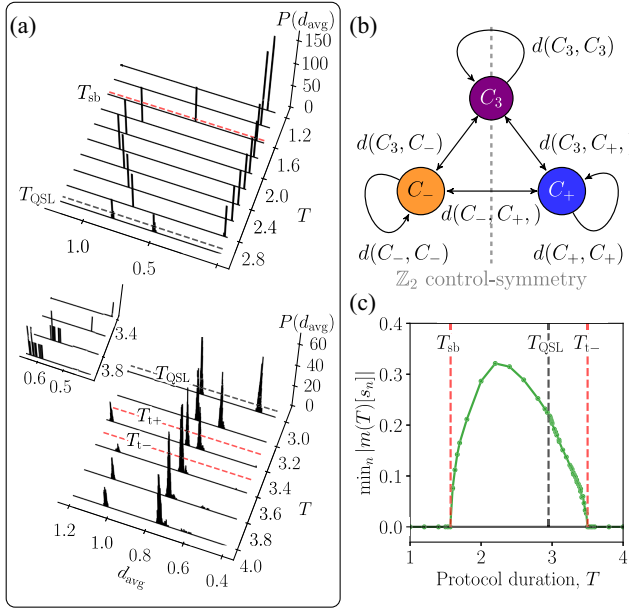


FIG. 2. Analysis of the connected components of the optimal level set $\{\arg \min_s I_T[s]\}$, before and after the quantum speed limit T_{QSL} (black dashed line). (a) Density distribution $P(d_{\text{avg}})$ of the average distance d_{avg} in Eq. (2), for different protocol durations T . Changes in the number of peaks in $P(d_{\text{avg}})$ mark topological transitions (red dashed lines). (b) Graph summarizing structure of optimal level set for $T_{\text{t+}} \leq T \leq T_{\text{t-}}$. Connected components C_+ , C_- , C_3 are represented by nodes while edges represent the relative distances $d(C_i, C_j)$, $i = +, -, 3$. A peak in the distance distribution $P(d_{\text{avg}})$ in (a) corresponds to at least one edge in the graph. (c) The order parameter $\min_n |m(T)[s_n]|$ in Eq. (3) identifies with high precision the location $T_{\text{t-}}$, associated with the merging $C_+ C_- \rightarrow C_{\pm}$. LMC parameters: $L = 64$, $\beta = 10^6$, $\sigma = 10^{-2}$.

spin glass physics [39]; below, we leverage its properties to detect topological CLTPs. For alternative definitions of the distance between the sets $\{\mathcal{S}_\alpha\}$ see Ref. [40].

From $\{\mathcal{S}_\alpha\}$, we obtain a total of $R(R-1)/2$ values, representing the distance d_{avg} between the pairs of sets. The information about the different connected components $\{C_1, C_2, \dots\}$ of the optimal level set is contained in the distribution $P(d_{\text{avg}})$ (Fig. 2(a)). Notice that the distribution is a function of the protocol duration T ; as T increases, the optimal level set deforms, and the number of connected components changes: $b_0(T)$ exhibits discontinuous jumps [Figs. 1(c) and 1(d)].

Topological control landscape phase transitions—The distribution $P(d_{\text{avg}})$, shown in Fig. 2(a) for $T \in [0, 4]$, depends on the LMC parameters used to sample the sets $\{\mathcal{S}_\alpha\}$. Here, we are interested in the features exhibited by $P(d_{\text{avg}})$ surviving the limit $M \rightarrow \infty$, $\beta \rightarrow \infty$, in this order [41]. Each peak in $P(d_{\text{avg}})$ sharpens into a delta function for $M \rightarrow \infty$, as the distance d_{avg} between each pair $\{C_i, C_j\}$ is unique. Hence, in the case of a finite number of samples, we identify the location of each peak with the average distance $d_{\text{avg}}(C_i, C_j)$ between two connected components

C_i, C_j (possibly $i = j$). For $T \in [0, T_{\text{sb}}]$, $P(d_{\text{avg}})$ has a single peak located at $d_{\text{avg}} \approx 0$ so that the optimal level set consists of a single connected component, which we call C_0 ; C_0 consists of an isolated protocol since the peak moves to $d_{\text{avg}} = 0$ as $\beta \rightarrow \infty$ [40].

This behavior changes past T_{sb} , where $P(d_{\text{avg}})$ acquires a second peak centered at $d_{\text{avg}} \approx 0.6$ for $T = 1.6$ (cf. Fig. 2): the optimal level set undergoes a topological transition and now possesses two distinct connected components, called C_+ , C_- . The location of the first peak converges to $d_{\text{avg}} = 0$ as β is increased [40]: the two connected components are two isolated optimal protocols, called s_+ , s_- . This bifurcation is associated with the breaking of the control symmetry, i.e., $s_{\pm}(t) \neq -s_{\pm}(T-t)$ [27,42]. Here, the topological nature of the T_{sb} CLPT is apparent: the number of connected components in the optimal level set jumps from one to two. Coincidentally, at the symmetry-breaking transition, the optimal level set consists of a single control protocol, so that the local geometry provides information about a topological change. More generally, the identification of topological transitions requires the study of non-local properties of the optimal level set.

The optimal level set changes dimensionality for $T \geq T_{\text{QSL}}$ [28]. Two connected components, containing an infinite number of optimal protocols, originate from s_+ , s_- (Fig. 2(a)): the peak located in $d_{\text{avg}} \approx 0$ for $T < T_{\text{QSL}}$, suddenly moves to the finite value $d_{\text{avg}} \approx 0.5$ for $T > T_{\text{QSL}}$ and its location is unaffected by increasing β .

Interestingly, the peaks corresponding to $d_{\text{avg}}(C_+, C_-)$ and $d_{\text{avg}}(C_+, C_+) = d_{\text{avg}}(C_-, C_-)$ are well-separated for $T_{\text{QSL}} \leq T \leq 3.4$ but indistinguishable for $T \geq 3.6$. We deduce that the components C_+ , C_- merge into a single connected component, called $C_{\pm}$, between $T = 3.4$ and $T = 3.6$. We define $T_{\text{t-}} \in (3.4, 3.6)$ as the critical duration associated with the merging transition $C_+ C_- \rightarrow C_{\pm}$. This marks a new kind of CLPT, involving topological properties of the optimal level set, where the number of connected components $b_0(T)$ decreases by one unit [43].

Rather unexpectedly, for $T \geq 3.2$ we observe two extra peaks appearing in $P(d_{\text{avg}})$: the first one is well visible at $d_{\text{avg}} \approx 1.2$ while the second one is relatively smaller and located at $d_{\text{avg}} \approx 0.4$ [44]. We associate these additional peaks with the appearance of a third connected component in the optimal level set, called C_3 . Hence, we find a second type of topological CLPT at $T_{\text{t+}} \in (3.2, 3.4)$, associated with incrementing the number of connected components in the optimal level set. In [40] we demonstrate that the third connected component C_3 originates from an isolated suboptimal protocol $s_3(t)$ (cf. Fig. 1); s_3 is suboptimal for $T < T_{\text{t+}}$ but turns into a global minimum at $T = T_{\text{t+}}$, consistent with Fig. 2(a).

Even though one might expect a total of six peaks marking the average distances between the three connected components $\{C_+, C_-, C_3\}$, the $s(t) \leftrightarrow -s(T-t)$ control symmetry reduces the number of distinct peaks in $P(d_{\text{avg}})$.

In Fig. 2(c) we visualize the three connected components C_+ , C_- , C_3 as nodes, and their relative distances $d_{\text{avg}}(C_i, C_j)$ as edges, $i, j \in \{+, -, 3\}$. The control symmetry transforms $C_+ \leftrightarrow C_-$ and $C_3 \leftrightarrow C_3$ [45], and hence we find the relations $d_{\text{avg}}(C_+, C_+) = d_{\text{avg}}(C_-, C_-)$, $d_{\text{avg}}(C_+, C_3) = d_{\text{avg}}(C_-, C_3)$ identifying two among the six relative distances.

In Ref. [27] the symmetry-breaking transition occurring at $T = T_{\text{sb}}$ has been related to a cusp in the protocol “magnetization” order parameter $m(T)[s] = T^{-1} \int_0^T dt s(t)$. Now, we adapt this observable to investigate the topological phase transition at $T_{\text{t-}}$: the $s(t) \leftrightarrow -s(T-t)$ symmetry of the optimal protocols, enforcing a vanishing $m(T)[s]$, is broken for $T > T_{\text{sb}}$ and can be used to classify the two connected components C_+ , C_- . We already know from the distribution $P(d_{\text{avg}})$ that the two symmetry-breaking components of the optimal level set merge somewhere in $T \in [3.4, 3.6]$; therefore, we anticipate the same merging transition to be visible in the magnetization order parameter, as we now show.

Past the T_{QSL} transition, the magnetization associated with the protocol s_n evolved by LMC, fluctuates in time. This fact motivates us to consider the *minimum* absolute magnetization registered during each LMC run,

$$\min_n |m(T)[s_n]| = \min_n \left| \frac{1}{T} \int_0^T dt s_n(t) \right|. \quad (3)$$

To restrict the study to the two connected components generated by the T_{sb} transition, we exclude the LMC runs sampled from the third disconnected component C_3 present for $T \geq 3.4$ [46].

In Fig. 2(c) we see that the minimum magnetization achieved during each LMC run is nonvanishing only in the interval $[T_{\text{sb}}, T_{\text{t-}}]$. Hence, we can identify with high precision $T_{\text{t-}} \approx 3.5$ as the critical duration at which $\min_n |m(T)[s_n]|$ vanishes. We confirm that, for $T > T_{\text{t-}}$, the two (previously distinct) connected components C_+ , C_- have merged and formed the single $C_{\pm}$; LMC can move between the positively and negatively magnetized regions of $C_{\pm}$ via homotopies while preserving vanishing infidelity [47].

The results presented are obtained at a fixed number of protocol time steps, $L = 64$. We do not observe qualitative changes in the results with varying L [40].

Outlook—We have presented numerical evidence of a new class of control landscape phase transitions (CLPTs), involving topological properties of the optimal level set and independent of the symmetries(s) of the control problem. We have illustrated the underlying mechanism using a constrained two-qubit state-preparation problem. In particular, we infer global information about the topology of the optimal level set by sampling optimal protocols via stochastic homotopic dynamics and considering (i) the distances between sampled sets, and (ii) the control symmetry-breaking order parameter.

The techniques and concepts we have developed are general and can be applied straightforwardly to detect CLPTs in generic control problems. Our novel approach shows that, for protocol durations greater than the quantum speed limit, the optimal level set consists of a few connected components, each containing infinitely many homotopically equivalent optimal protocols. By tracking the number of connected components as the duration of the quantum evolution is changed, we discover two types of topological CLPTs: in the first type, distinct connected components C_i , C_j undergo a merging process $C_i C_j \rightarrow C_{i+j}$; in the second type, a new connected component originates from a sub-optimal protocol (local trap) reaching global optimality.

Our Letter raises the question about the existence of topological CLPTs involving *higher* Betti numbers of the optimal level set; these may be detected, e.g., using more advanced techniques from topological data analysis [37,38]. For instance, we expect a change in higher Betti numbers to occur at the quantum speed limit transition, as a previous work suggests a drastic change in the shape of the optimal level set, from a single point to a high-dimensional truncated hyperboloid [28].

In the End Matter, using the reduced density matrix, we show that topological CLPTs are associated with homotopically inequivalent trajectories in Hilbert space. Hence, modern-day quantum platforms capable of simulating isolated two-qubit systems can detect topological CLPTs by measuring the qubit magnetization or entanglement entropy.

Our Letter marks a step toward an improved characterization of two-qubit quantum control landscapes as we show that the optimal level set consists of a small number of connected components. Each connected component is associated with an average protocol, while fluctuations measure the size of the connected component. Whereas we focused on a two-qubit control problem, our Letter reveals behaviors of optimal level sets that are likely generic.

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Data availability—The data and code associated with this manuscript are available under [48].

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- [39] In our case, s_1, s_2 are not normalized to unity nor restricted to $\{\pm 1\}$ (as in the spherical and Ising model).
- [40] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/dqvj-p6fq> for technical details and the animation visualizing LMC trajectories in the Bloch sphere.

- [41] The order of the two limits is important to guarantee that the distribution of protocols from the region around the optimal level set is converged, at fixed β .
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- [43] If the point(s) of contact between the two connected components at $T = T_{t-}$ is known, an analysis of the local geometry at this point(s) should also reveal the merging topological transition. However, the identification of the minimum cost pathway(s) between two connected components requires a non-local analysis; typically used algorithms are the *string method* [49] or the *nudged elastic band method* [50,51].
- [44] The different heights of the peaks are due to the different number of LMC runs confined in each connected component. In this case, LMC runs that sample the component C_3 are fewer in number than the ones exploring C_+ or C_- .
- [45] Notice $d(C_3, C_-) = d(C_3, C_+)$ since C_3 is left invariant by the $s(t) \leftrightarrow -s(T-t)$ symmetry.
- [46] This can be done, e.g., by excluding LMC runs corresponding to the C_3 peak in the distance distribution $P(d_{\text{avg}})$ (cf. Fig. 2(a)).
- [47] Rigorously, the positively and negatively magnetized regions of $C_{\pm}$ are defined by $\{s \in C_{\pm} : m(T)[s] > 0\}$ and $\{s \in C_{\pm} : m(T)[s] < 0\}$, respectively.
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- [52] Due to the qubit exchange symmetry of the Hamiltonian, tracing out the second or first qubit yields the same reduced density matrix.
- [53] In general, the choice of the quantum observable(s) for the detection of topological CLPTs depends on the trajectories followed by the system along optima in different connected components. The fact that the single-qubit reduced density matrix contains partial information about the system's state is not limited to two-qubit systems. For this reason, we expect topological CLPTs to affect expectation values of single-particle observables even for controlled systems with many interacting qubits. Nevertheless, if this is not the case, quantum observables involving more qubits are required.

End Matter

Observing topological control phase transitions in quantum experiments—Topological CLPTs give rise to distinct signatures, readily observable in modern-day experiments. We can directly visualize the topological transitions by considering the single-qubit reduced density matrix $\rho_1(t) = \text{Tr}_2[\rho(t)]$, where $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ and Tr_2 is the partial trace over the second qubit [52]. Because the reduced density matrix contains information about the system's state, we expect topological CLPTs to affect expectation values of single-particle observables [53]. Here, we write $\rho_1(t) = \mathbb{I}/2 + \vec{n}_1 \cdot \hat{S}_1$, and plot the trajectory of $\vec{n}_1(t)$ inside the Bloch sphere, $|n_1| \leq 1$.

Figure 3 shows three optimal trajectories of $\vec{n}_1(t)$ generated by three distinct protocols, for $T = 2.6, 3.0, 3.4, 3.6$. Each solid line shows the *average* trajectory over a single LMC run, sampled from a single connected component among C_+, C_-, C_3 (blue, orange, and purple, respectively). Notice that, for $T < T_{t+}$, C_3 consists of an isolated local trap (i.e., its trajectory does not reach the target state) and hence it does not belong to the optimal level set. To visualize fluctuations within each connected component, we plot a subset of protocols in each LMC run using half-transparent dots; in this way, fluctuations within the same

connected component generated by the LMC dynamics are visible as a “cloud” of trajectories distributed around the solid line of the same color.

For $T = 2.60 < T_{\text{QSL}}$, the dotted trajectories coincide with the solid line, in agreement with optimal protocols being isolated. For $T = 3.00 \approx T_{\text{QSL}}$, trajectories from the C_+, C_- connected components (blue and orange curves) connect the initial state (green arrow) with the target state (red arrow), in accord with the vanishing infidelity associated with C_+, C_- past T_{QSL} . Here, dotted trajectories start deviating from the solid line, and a semi-transparent cloud of trajectories in the Bloch sphere is visible around it. For $T > T_{t+}$, component C_3 has vanishing infidelity, and the purple trajectories now connect the initial and final states. Nevertheless, the three solid lines and the surrounding clouds of trajectories are distinguishable from each other: LMC cannot jump between the different connected components for large enough β . Finally, for $T > T_{t-}$ the blue and orange curves are indistinguishable from one another: C_+, C_- have merged into the single connected component $C_{\pm}$. For $T \geq T_{t-}$, we assign to LMC trajectories in the $C_{\pm}$ connected components orange or blue colors depending on the average magnetization of protocols in each run.

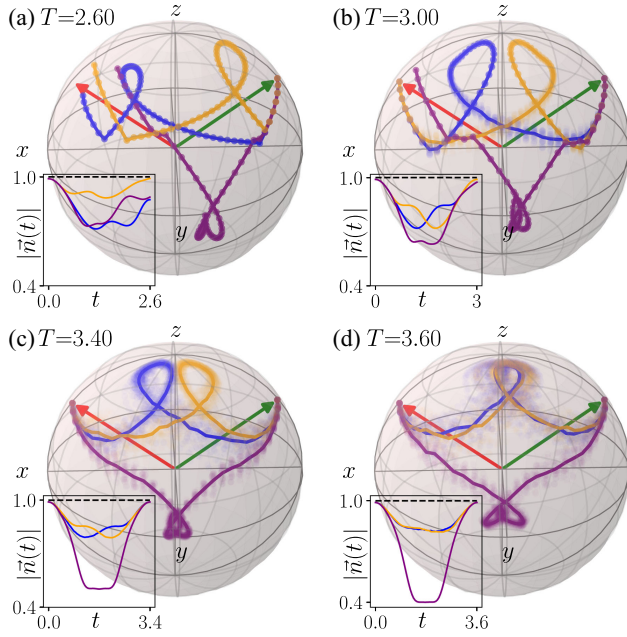


FIG. 3. Topological transitions are experimentally accessible in modern-day experiments. The figure shows Bloch sphere trajectories in the reduced density matrix obtained from protocols sampled by the Langevin-Monte Carlo (LMC) algorithm. The green (red) arrow indicates the initial (target) state. Solid lines represent *averages* over three LMC runs, each sampling one of the three connected components C_+ , C_- , C_3 of the optimal level set (colored in blue, orange and purple, respectively); low-opacity dotted trajectories represent *single* protocols and serve to visualize protocol fluctuations. (a) $T = 2.6 > T_{sb}$: blue and orange trajectories correspond to the isolated optimal protocols s_+ , s_- ; the purple curve s_3 is a local trap. (b) $T = 3.0 > T_{QSL}$: optimal level set consists of two connected components C_+ , C_- , each containing infinitely many optimal protocols (cf. blue and orange clouds surrounding the solid lines). (c) $T = 3.4 > T_{t+}$: the orange and blue clouds are still well-separated; the purple protocol, now also optimal, gives rise to the third connected component C_3 . (d) $T = 3.6 > T_{t-}$: orange and blue trajectories overlap and the corresponding clouds are indistinguishable: the previously distinct connected components C_+ , C_- have merged and formed $C_{\pm}$. Insets: the purple trajectories reach the largest entanglement entropy as the associated norm $|n(t)|$ drops to ≈ 0.5 during the evolution.

Different connected components may possess substantially different physical features. For example, in the insets of Fig. 3 we show the norm $|n(t)|$ of the Bloch sphere vectors. The entanglement entropy $S_E = -q \log q - (1 - q) \log(1 - q)$ with $q = (|n| + 1)/2$ is different in the three cases. In particular, the minimum value $|n| \approx 0.5$ reached by C_3 trajectories (purple), is half the value reached by C_+ , C_- trajectories (orange and blue). We expect that, if present, noise in the form of decoherence or depolarization may reveal some of these trajectories as more robust than others.

In Supplemental Material [40], we provide an animation showing three optimal trajectories in the Bloch sphere, which represents the evolution of the one-qubit reduced density matrix. The blue and orange trajectories are generated by optimal protocols with total duration $T = 3.4$. They belong to the different connected components C_+ , C_- of the optimal level set (they cannot be continuously deformed into each other while preserving vanishing infidelity). The black trajectory has a total duration $T = 3.6$ and its dynamics is governed by the LMC algorithm. The movie demonstrates that for $T = 3.6$ the blue and orange connected components have merged into a single connected component: $C_+ C_- \rightarrow C_{\pm}$. This is proven by the black trajectory that, through approximate continuous deformations (LMC dynamics with $\beta = 10^6$), moves between the “orange” and “blue” regions of $C_{\pm}$.