

Critically Slow Hilbert-Space Ergodicity in Quantum Morphic Drives

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The maximum entropy principle is foundational for statistical analyses of complex dynamics. This principle has been challenged by the findings of a previous work [Phys. Rev. X **7**, 031034 (2017)], where it was argued that a quantum system driven in time by a certain aperiodic sequence without any explicit symmetries, dubbed the Thue-Morse drive, gives rise to emergent nonergodic steady states which are underpinned by effective conserved quantities. Here, we resolve this apparent tension. We rigorously prove that the Thue-Morse drive achieves a very strong notion of quantum ergodicity in the long-time limit: The time evolution of any initial state uniformly visits every corner of its Hilbert space. On the other hand, we find the dynamics also approximates a Floquet drive for arbitrarily long, albeit finite periods of time with no characteristic timescale, resulting in a scale-free ergodic dynamics we call critically slow complete Hilbert-space ergodicity. Furthermore, numerical studies reveal that critically slow complete Hilbert-space ergodicity is not specific to the Thue-Morse drive and is, in fact, exhibited by many other aperiodic drives derived from morphic sequences, i.e., words derived from repeatedly applying substitution rules on basic characters. Our Letter presents a new class of dynamics in time-dependent quantum systems where full ergodicity is eventually attained but only after astronomically long times.

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The maximum entropy principle (MEP) is a powerful tenet in physics, guiding predictions about the limiting behavior of a complex system without the need for a full, microscopic solution [1]. It posits that, at equilibrium, a system is well described by a state which maximizes a suitable notion of entropy, constrained only by globally conserved quantities. For example, the MEP predicts that the steady state achieved in thermalization is a Gibbs state, characterized solely by a few macroscopic properties like temperature and chemical potential.

The MEP has also guided recent studies of quantum ergodicity concerning the temporal ensemble of wave functions of a closed quantum system [2–9]. These works inquire about the limiting *distribution* formed by the collection of pure states arising in evolution and go beyond traditional notions of quantum ergodicity involving statistics of energy eigenvalues or eigenstates [10]. For example, when dynamics possesses a time-translation symmetry (e.g., generated by a time-independent or periodic Hamiltonian), the MEP predicts that the *random phase ensemble* emerges—a collection of states wherein energy

eigenstates are superposed with random phases [5,6]. In contrast, in the absence of time-translation and other internal symmetries, the MEP suggests the emergence of the maximally ergodic, uniform (Haar) distribution, a phenomenon dubbed *complete Hilbert-space ergodicity* (CHSE) [3,4]. Indeed, Ref. [3] rigorously showed that CHSE arises in the discrete time-quasiperiodic Fibonacci drive, which is derived from the Fibonacci word [11–17]. More generally, the same universal maximally ergodic behavior is expected to hold for generic quantum systems driven *aperiodically* in time, which do not feature any explicit conserved quantities.

However, a previous study [18] of the equilibration dynamics in the quantum Thue-Morse drive (TMD) [15,17,19–25], derived from the aperiodic Thue-Morse word [Fig. 1(a)], challenges this expectation. Reference [18] argued that the TMD has a self-similar structure and, thus, exhibits an emergent (Floquet) time-translation symmetry, precluding CHSE—see Fig. 1(b) for an example of seemingly non-ergodic dynamics. If these predictions are correct, this represents a surprising exception to the MEP and potentially presents new routes to stabilize driven quantum systems against the deleterious effects of heating.

In this Letter, we carefully analyze the dynamics of the TMD and through it unravel a new class of quantum ergodic dynamics which we dub *critically slow complete Hilbert-space ergodicity* (CS-CHSE). Specifically, we rigorously prove that a qubit driven by the TMD generically

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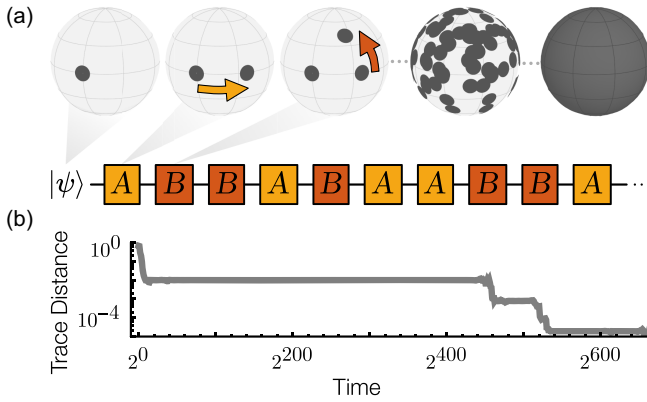


FIG. 1. (a) The Thue-Morse drive involves applying a pair of unitaries A and B according to the Thue-Morse word. We ask if, under such dynamics, a quantum state uniformly covers the Hilbert space over time. (b) Trace distance between the temporal distribution [dotted spheres in (a)] and the uniform distribution (dark sphere) for a representative initial state. One sees a nonsmooth decay involving many intervening long-lived plateaus, each lasting for astronomically long times. It is not clear whether the trace distance keeps decreasing (CHSE) or eventually saturates at a nonzero value.

displays emergent time-translation symmetries with arbitrarily good accuracy over arbitrary durations of time; however, these symmetries are finitely lived, resulting in the system achieving CHSE in the strict infinite time limit. Our finding resolves the apparent tension between the purported nonergodicity claimed by Ref. [18] in the TMD and the maximally ergodic predictions of the MEP. Furthermore, our analysis uncovers that the exploration of the full Hilbert space occurs in a scale-free manner—precisely, the distribution of timescales that the emergent time-translation symmetries emerge for obeys a power law and has an unbounded mean, hence the term “critically slow.”

Motivated by these findings, we numerically explore other aperiodic driving sequences and find that the phenomenology of CS-CHSE holds similarly for drives derived from certain morphic words—words derived from repeatedly applying substitution rules on basic characters [26]. Thus, CS-CHSE represents a new class of quantum ergodic dynamics in which full ergodicity is eventually achieved—as predicted by the MEP—but over timescales astronomically larger than those natural to the system. This adds to recent studies on quantum dynamics emphasizing how there can be a large separation of timescales between the asymptotic *late-time* dynamics and a transient but potentially physically more-relevant *long-time* dynamics, such as two-step [27] and slow thermalization [28–32].

Thue-Morse drive and CHSE—We begin by defining the quantum TMD. It is derived from the so-called Thue-Morse word (TMW) [33], an infinite word on two characters $\{0, 1\}$ constructed by the following simple concatenation rules [26]. Starting from the base word $W_0 = 0$, form the

word W_{n+1} at the $(n + 1)$ th level by concatenating W_n with its bitwise negation $\overline{W_n}$ (i.e., interchanging 0 and 1):

$$W_{n+1} = W_n \overline{W_n}. \quad (1)$$

For example, $W_1 = 01$ yields $W_2 = 0110$, then $W_3 = 01101001$, etc. The infinite-level word W_∞ is the TMW, which is known to be aperiodic, i.e., never eventually repeats, but is not quasiperiodic [34,35].

The quantum TMD for a d -dimensional quantum system is realized by applying two fixed unitaries $A, B \in \text{SU}(d)$ according to the characters of W_∞ . Specifically, at integer time $t = n$, apply A (B) if the n th character of W_∞ is 0 (1). Thus, time evolution of an initial state $|\psi(0)\rangle$ is $|\psi(n)\rangle = U(n)|\psi(0)\rangle$, where

$$U(n) = \prod_{j=1}^n \begin{cases} A & \text{if the } j\text{th character of } W_\infty \text{ is 0,} \\ B & \text{if it is 1} \end{cases} \quad (2)$$

is the unitary time-evolution operator. Above, the product is ordered from right to left. For example, $|\psi(8)\rangle = BAABABBA|\psi(0)\rangle$.

Recent previous works have studied the TMD, for instance, on prethermalization timescales in the high-frequency regime [20,22,23] and in defining so-called “time-rondeau crystals” [24]. As mentioned, a work most relevant for us is Ref. [18], which studied the equilibration dynamics of free-fermionic spin chains under the TMD and argued for the emergence of nontrivial steady states. In our context, this amounts to a violation of CHSE, which we define next.

CHSE refers to the property of a sequence of pure states $\{|\psi(t)\rangle\}_{t \in \{0,1,2,\dots\}}$ generated in dynamics, called the *temporal ensemble* [6], uniformly covering the Hilbert space [3,4,36]. Formally, we mean that the empirical distribution of the temporal ensemble asymptotically follows the *Haar ensemble*, a collection of states $\{V|\psi(0)\rangle\langle\psi(0)|V^\dagger\}_V$, where V is a Haar-distributed unitary on $\text{SU}(d)$. This can be probed through the trace distance [37]

$$\Delta_T^{(k)} := \frac{1}{2} \left\| \rho_T^{(k)} - \rho_{\text{Haar}}^{(k)} \right\|_1, \quad (3)$$

of the k th temporal moment of the finite-time ensemble $\rho_T^{(k)} = (1/T) \sum_{t=0}^{T-1} [U(t)|\psi\rangle\langle\psi|U(t)^\dagger]^{\otimes k}$ to the corresponding Haar moment $\rho_{\text{Haar}}^{(k)} = \int dV (V|\psi\rangle\langle\psi|V^\dagger)^{\otimes k}$, and inquiring if at large times $\lim_{T \rightarrow \infty} \Delta_T^{(k)} = 0$ for all k [38]. In quantum information parlance, one asks if the temporal ensemble forms a *state design*.

Here, we will actually study the stronger notion of ergodicity of the temporal ensemble of unitaries themselves, namely, ask if the statistical properties of the time-evolution operators $\{U(t)\}_t$ of the TMD are

close to those of Haar random unitaries. Specifically, we consider the k th-moment time-averaging channel $\mathcal{N}_T^{(k)}[\cdot] = (1/T) \sum_{t=0}^{T-1} U(t)^{\otimes k}[\cdot] U(t)^{\dagger \otimes k}$ and compare its similarity to the corresponding k th-moment Haar averaging channel $\mathcal{N}_{\text{Haar}}^{(k)}[\cdot] = \int dV V^{\otimes k}[\cdot] V^{\dagger \otimes k}$, for example, through the Hilbert-Schmidt distance

$$D_T = \left\| \mathcal{N}_T^{(k)} - \mathcal{N}_{\text{Haar}}^{(k)} \right\|_2. \quad (4)$$

If $\lim_{T \rightarrow \infty} D_T = 0$ for all k , then this property of the unitary ensemble is dubbed *complete unitary ergodicity* (CUE) [4]. In quantum information parlance, this is asking whether the ensemble of time-evolution operators forms a *unitary design*. Since the k th moment of the finite-time state ensemble can be recovered via $\rho_T^{(k)} = \mathcal{N}_T^{(k)}[|\psi(0)\rangle\langle\psi(0)|^{\otimes k}]$ and similarly for the Haar moment, it is clear CUE implies CHSE.

Transient time-translation symmetries in the TMD—First, we illustrate the basic phenomenology seen in numerical simulations of the TMD [39] where the basic unitaries $A, B \in \text{SU}(d)$ are chosen Haar randomly. Figure 2(a) shows the trace distance $\Delta_T^{(k)}$ for $k = 2$. For larger dimension d , there is a power-law behavior $\sim 1/\sqrt{T}$, which provides strong numerical evidence for CHSE. However, for small d and particularly for $d = 2$ [Figs. 2(b)–2(d)], the decay is not smooth, and, in fact, there are surprisingly many intermediate long-lived plateaus, some up to times 2^{600} , the limit of our finite-time numerics. Despite the astronomically large times reached by our simulations (see Ref. [39] for details), it is unclear if these plateaus survive indefinitely or if there is eventual decay at even later times, such that CHSE is achieved.

Dynamics of unitaries—We now turn to our rigorous analysis of the asymptotic behavior of the TMD. A starting observation is that we may simplify the analysis of the TMD by concentrating on exponential times $t = 2^n$, i.e., study the sequence of unitaries $A_n := U(2^n)$. We find A_n may be generated by repeated applications of a dynamical map $(A_n, B_n) = \Phi(A_{n-1}, B_{n-1})$, where B_n is a partner sequence with all A 's swapped with B 's, which we call the “complementary TMD.” Specifically, define the map $\Phi: \text{SU}(d) \times \text{SU}(d) \rightarrow \text{SU}(d) \times \text{SU}(d)$, where $\Phi(A, B) := (BA, AB)$; then

$$(A_n, B_n) = \Phi^n(A, B), \quad (5)$$

which is equivalent to the recursion relations $A_{n+1} = B_n A_n$ and $B_{n+1} = A_n B_n$ [18,39].

Using this map, the origin of the plateauing structure of the trace distance can be intuitively understood by analyzing the iterates A_n and B_n . Suppose (for some reason) $A_n \approx B_n$ within a range $n \in [n_*, N_*]$. Then, defining $V_* = A_{n_*}$, we have $\Phi(A_{n_*}, B_{n_*}) \approx (V_*^2, V_*^2)$ and,

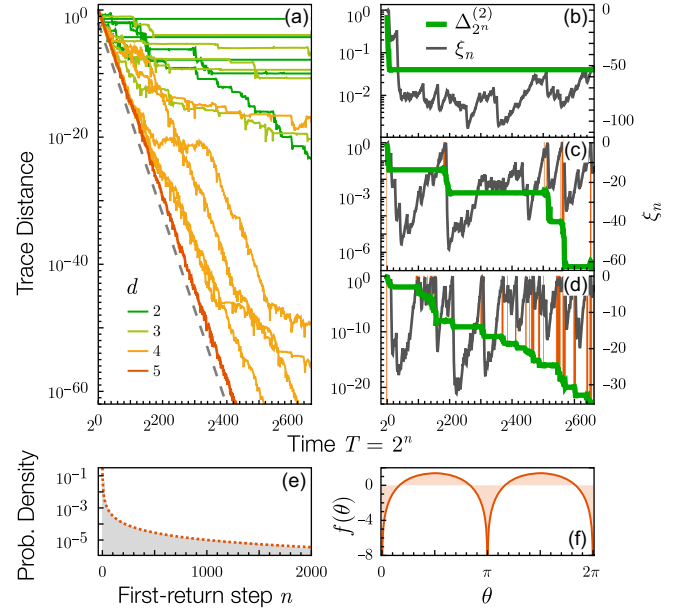


FIG. 2. Trace distance $\Delta_{T=2^n}^{(k)}$ for $k = 2$ between temporal and Haar distribution for the TMD (each line is a different realization of Haar-randomly sampled basic unitaries A, B). (a) Trace distances for various d -level systems. A power-law decay $T^{-1/2}$ is shown in a dashed line for reference, which is tracked for large d . However, plateaus appear for small d . (b)–(d) Different instances of trace distance for $d = 2$ (green line) and distance ξ_n (gray line). Orange vertical lines indicate times at which $\xi_n > -1$. (e) Distribution of first-return steps of the quasirandom walk on ξ_n , over random initial θ_1 and $\xi_1 = -1/2$. The orange dotted line is $\propto n^{-3/2}$. (f) Plot of step-length function $f(\theta) = \log(4 \sin^2(\theta/2))$.

subsequently, the evolution operator satisfies $U(T = 2^n) = A_n \approx V_*^{2^n}$. In other words, between times 2^{n_*} and 2^{N_*} , the system behaves like a Floquet system, evolving with the repeated application of V_* . This emergent discrete time-translation symmetry would halt the further exploration of the Hilbert space, due to quasienergy conservation [4].

This preliminary analysis suggests that the emergent Floquet dynamics seen in numerics can be understood by showing $A_n \approx B_n$ for a large range of n . Thus, the distance $\xi_n := \xi(A_n, B_n) < 0$ between the iterates, where

$$\xi(A, B) := \log \left(\frac{1}{2} \left| 1 - \frac{1}{d} \text{tr}(A^\dagger B) \right| \right), \quad (6)$$

is a key quantity to consider: It governs the accuracy of the emergent time-translation symmetries in exponential precision (we will find that it is natural to measure the distance in logarithmic scale). Indeed, ξ_n for a qubit system is plotted in Figs. 2(b)–2(d), where two salient features are observed: When ξ_n is close to 0 (vertical orange lines), the trace distance $\Delta_{2^n}^{(k)}$ appears to drop; while when ξ_n is large, the trace distance plateaus, as expected, confirming its

important role in dynamics. However, ξ_n is also seen to behave rather erratically, as if driven by a random process: Over time, there are multiple large excursions from the origin but also multiple recurrences back to it. Understanding which process dominates (if any) is paramount to understanding whether CHSE is achieved.

Quasirandom walk of ξ_n and CHSE—We find that, for a system consisting of a single qubit, the behavior of the distance ξ_n , which we interpret as the position of a “particle” living on the semi-infinite real line $(-\infty, 0]$, can be exactly captured by a dynamical map in which the position and an angular variable $\theta_n \in [0, 2\pi)$ are coupled:

$$\begin{aligned}\xi_{n+1} &= \xi_n + \log(1 - e^{\xi_n}) + f(\theta_n), \\ \theta_{n+1} &= \arg[e^{i2\theta_n} + (1 - e^{i2\theta_n})e^{\xi_n}],\end{aligned}\quad (7)$$

where $f(\theta) = \log(4 \sin(\theta)^2)$. This map is derived from the map on unitaries Φ [39]. The initial points ξ_1, θ_1 are related to the starting choice of unitaries A, B whose precise expressions and values are not important in determining the generic long-time behavior of the map. Alternatively, the dynamics can equivalently be represented as a nonlinear map $c_{n+1} = c_n^2 + 1 - |c_n|^2$ of a single complex variable in the unit disk $|c_n| \leq 1$, which accords a complementary geometrical viewpoint of the TMD in terms of motion within the disk (see Ref. [39] for details).

One may gain more physical insight into Eq. (7) when the particle is far from the origin, so that the factor e^{ξ_n} is small and can be ignored; then the angular part decouples:

$$\xi_{n+1} \approx \xi_n + f(\theta_n), \quad (8a)$$

$$\theta_{n+1} \approx 2\theta_n \mod 2\pi. \quad (8b)$$

Equation (8b) is the angle doubling map, which is well known to be chaotic, in the sense that the angles θ_n behave like random numbers for a generic initial condition. These random numbers, in turn, drive the position of the particle with the length of each step given by $f(\theta_n)$; thus, the iteration in Eq. (8a) describes a quasirandom walk, explaining qualitatively the behavior seen in Figs. 2(b)–2(d). Of course, the exact equations of motion in Eq. (7) feature an additional “soft-wall” constraint $\log(1 - e^{\xi_n})$ preventing the particle from crossing the origin.

The equations of motion (7) also provide a quantitative understanding of the steplike structure seen in the trace distance in Figs. 2(b)–2(d). Key is the function $f(\theta)$ [Fig. 2(f)]: It is upper bounded by $\log(4)$ —implying a finite maximum increase in ξ_n per step—but at the same time is unbounded from below, diverging to $-\infty$ at $\theta = 0, \pi$, implying a potentially unbounded decrease in ξ_n per step. Thus, upon incurring a single large negative jump in ξ_n , the system becomes almost “Floquet-like” so that $\Delta_{2^n}^{(k)}$ plateaus and requires many small positive steps to

conspire together to increase the value of ξ_n again so that $\Delta_{2^n}^{(k)}$ drops. One can estimate the *distribution* of the lengths of these plateaus τ by analyzing the distribution of the first-return step of ξ_n under Eq. (7) (i.e., the minimum $n > 1$ such that $\xi_n \geq \xi_1$). We numerically find a power-law distribution $P(n) \sim n^{-3/2}$ [Fig. 2(e)], the same as for a random walk in 1D [42]. Taking $\tau = 2^n$, the distribution of plateau lengths is, thus, $P(\tau) \sim \tau^{-1} \log(\tau)^{-3/2}$, a heavy-tailed distribution with unbounded mean.

Turning now to the important question of whether the random walker wanders off to (negative) infinity or remains near its (finite) starting position at late times, interestingly, because the function $f(\theta)$ governing the step size has zero mean and finite variance, the random walk belongs to the Gaussian universality class, and so *both* generically happen infinitely often, rigorously captured by the following lemma.

Lemma 1—For almost any initial pair (ξ_1, θ_1) [43], the dynamical map Eq. (7) has different subsequences $\{n_m\}$ which (i) converge back to the initial point $\lim_{m \rightarrow \infty} (\xi_{n_m}, \theta_{n_m}) = (\xi_1, \theta_1)$ and (ii) wander off to infinity $\lim_{m \rightarrow \infty} \xi_{n_m} = -\infty$.

The proof, taken from Ref. [44], is presented in Supplemental Material [39].

Statement (ii) of Lemma 1 formalizes the long timescales of emergent time-translation symmetry and nonergodicity. Conversely, statement (i) guarantees that, despite these long timescales, eventual uniform ergodicity is achieved: We further show that each recurrence near the initial point (ξ_1, θ_1) is accompanied by a *strict decrease* in the degree of dissimilarity between the ensemble of time-evolution unitaries of the TMD to Haar random ones (proof in Supplemental Material [39]).

Lemma 2—Let $M_n := D_{2^n}^2 + \bar{D}_{2^n}^2$, where $\bar{D}_T$ is the distance measure Eq. (4) but for the complementary TMD (i.e., with A interchanged with B). Then, M_n is a monotone and strictly decreases by a constant multiplicative factor every time (ξ_n, θ_n) is sufficiently close to (ξ_1, θ_1) , for almost all choices of starting unitaries (A, B) .

Since Lemma 1 (i) implies these strict decreases happen infinitely often, $\lim_{n \rightarrow \infty} M_n = 0$ and, hence, $\lim_{T \rightarrow \infty} D_T = 0$ [39], yielding the expected result from the MEP that the TMD ultimately achieves maximal ergodicity.

Theorem 1—The TMD for a qubit system exhibits CUE and, thus, CHSE for almost any choice of $A, B \in SU(2)$ (with respect to the Haar measure).

We note that the technical reason Ref. [18] did not identify the CHSE of the TMD is their claim that $\xi_n \rightarrow -\infty$ ($\phi_m \rightarrow 0$ in their notation), which does not hold generically [statement (i) of Lemma 1].

Discussion and outlook—Our Letter studying the aperiodic TMD has confirmed the applicability of the MEP: It correctly predicts that the drive, which lacks any explicit symmetries, ultimately produces an ergodic temporal ensemble of states, uniformly distributed over the Hilbert

space. At the same time, our Letter has also explained how this maximal ergodicity is attained dramatically slowly and in a scale-free manner (at least for a qubit), a dynamical phenomenon we dub CS-CHSE. Our Letter has direct implications for experiments probing CHSE (such as in a nitrogen-vacancy center setup, recently performed [9]), in correctly interpreting the apparent nonergodic behavior they would find at short times if they were to implement the TMD. We leave as an open question why the long periods of nonergodicity seem to vanish with increasing system size [Fig. 1(a)], for which it would be useful to understand the invariant measure of the map Φ in arbitrary dimension.

Beyond the TMD, it is natural to inquire about the generality of CS-CHSE. To that end, we perform a preliminary numerical study of aperiodic discrete quantum drives constructed by applying basic unitaries A (B) according to the characters 0 (1) appearing in a binary morphic word—an infinite word generated by iterating substitution rules $0 \rightarrow w_0, 1 \rightarrow w_1$ starting from 0, where w_0 and w_1 are some fixed subwords [26]. For example, the TMW studied in this Letter is the morphic word defined by $0 \rightarrow 01, 1 \rightarrow 10$.

In Fig. 3, we plot the trace distances between the temporal and Haar ensembles for some morphic drives constructed from words with choices of w_0 and w_1 of length at most 3. We find three distinct behaviors: first, drives which display “regular” CHSE (green lines), in the form of a power-law decay, regardless of dimension; these include the generalized Fibonacci drives ($0 \rightarrow 0^m 1, 1 \rightarrow 0$) studied in [3] and generalizations like the Octonacci drive ($0 \rightarrow 010, 1 \rightarrow 0$) [45]. Second, drives which appear to display CS-CHSE (yellow-red lines) for a qubit [Fig. 3(a)], which include the TMD, but also drives from other sequences, like the period-doubling word ($0 \rightarrow 01, 1 \rightarrow 00$) [46]. This shows that CS-CHSE is not particular to the TMD. While in the present examples CS-CHSE is seen to vanish with increasing Hilbert-space dimension [Fig. 3(b)], understanding if there are more general morphic drives wherein CS-CHSE survives in the many-body regime would be an important direction for future work. Third, there is a small proportion of drives (black and gray lines) which exhibit a yet slower decay of the trace distance,

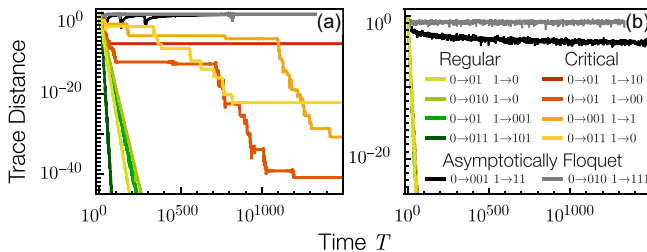


FIG. 3. Trace distance $\Delta_T^{(k)}$ between the Haar distribution and temporal ensemble generated by quantum morphic drives associated with substitution rules labeled in (b), for Haar-randomly chosen starting unitaries A, B (the same for all drives). (a) $d = 2$, $k = 2$; (b) $d = 5$, $k = 1$.

present for any dimension. However, these have arguably a trivial origin, as the underlying words have arbitrarily long streaks of repeated characters (e.g., for $0 \rightarrow 001, 1 \rightarrow 11$, each substitution doubles the length of the blocks of 1’s) [47]; hence, we term them “asymptotically Floquet drives.” Nevertheless, given that asymptotically Floquet drives are strictly speaking aperiodic, it is an open and nontrivial question whether CHSE is ultimately achieved or not in their dynamics. We see that there is a rich interplay between the combinatorial structure and complexity of the aperiodic morphic words and the degree of ergodicity in the quantum drives they generate, which deserves to be better studied.

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Data availability—The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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