

Quantum Hall Effect without Chern Bands

Benjamin Michen^{1,*} and Jan Carl Budich^{1,2,†}

¹*Institute of Theoretical Physics, Technische Universität Dresden and Würzburg-Dresden Cluster of Excellence ct.qmat, 01062 Dresden, Germany*

²*Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Straße 38, 01187 Dresden, Germany*

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The quantum Hall effect was originally observed in a two-dimensional electron gas forming Landau levels when exposed to a strong perpendicular magnetic field and was later generalized to Chern insulators without net magnetization. Here, further extending the realm of the quantum Hall effect, we report on the robust occurrence of an integer quantized transverse conductance at the onset of disorder in a microscopic lattice model, all bands of which are topologically trivial (zero Chern number). We attribute this phenomenon to the energetic separation of nonquantized Berry fluxes within those bands. Adding disorder then nudges the system into a quantum Hall phase from an extended critical regime obtained by placing the Fermi energy within a broad window inside a trivial band. This natural integer-rounding mechanism manifests as the mobility-gap-induced quantization of a nonuniversal Hall conductance. Our results are corroborated by numerical transport simulations and the analysis of two complementary topological markers.

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The quantum Hall effect, one of the most remarkable phenomena occurring in nature, is of foundational importance to the broad research field of topological matter [1–12]. Establishing topology as the rationale behind physical robustness, the experimental observation of a transverse conductance σ_{xy} quantized in units of e^2/h has been explained with theory in terms of a topological invariant known as the first Chern number [13–16]. This represents a pioneering example of the by now well-established paradigm of topological Bloch bands characterized by global properties, the discrete value of which remains unchanged under continuous perturbations [17–20]. Adiabatic continuity is in this context largely synonymous with the existence of a finite energy gap that must close at a topological quantum phase transition for a topological property to change [21,22].

Surprisingly, in this Letter we provide strong evidence that the topological quantization of the Hall conductance can occur with the onset of disorder in an extended parameter regime of a microscopic lattice model, both Bloch bands of which are topologically trivial. The physical

mechanism behind this phenomenology may be seen as a natural integer rounding effect: when placing the Fermi energy E_F within either band of our model, the nonuniversal value of σ_{xy} is quantized to the closest integer value in units of e^2/h [15] by the continuous opening of a mobility gap [23–27] upon switching on an on-site disorder potential $\hat{W}$. The underlying clean model exhibits a wide energy window where $\sigma_{xy} > 0.5e^2/h$ (see Fig. 1) due to the energetic separation of sizable though nonquantized Berry

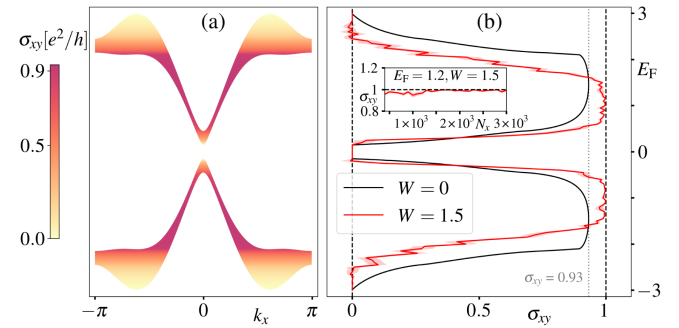


FIG. 1. (a) Band structure of the free Hamiltonian $\hat{H}_0$ [see Eq. (1)]. Color indicates Hall conductance $\sigma_{xy}^{W=0}(E_F)$ of the clean system with Fermi energy E_F , which is proportional to the Berry curvature accumulated up to E_F [cf. Eq. (2)]. (b) $\sigma_{xy}^W(E_F)$ at disorder strength $W = 0$ ($W = 1.5$) in black (red), and system size $N_x = 2000$, $N_y = 500$, averaged over $S = 40$ disorder realizations, where the shaded corridor indicates the sample standard error. Inset: finite size scaling at $E_F = 1.2$ as a function of N_x at fixed $N_x/N_y = 4$. Other parameters in all plots are $r = 1.5$, $\epsilon_1 = 0.3$, $\epsilon_2 = 2$, $\gamma = 2$, and $\gamma_2 = 0.3$.

* Contact author: benjamin.michen@tu-dresden.de

† Contact author: jan.budich@tu-dresden.de

fluxes within its topologically trivial bands. In this light, the clean metallic situation with E_F inside the eligible window may be seen as an extended critical region that is immediately nudged into a quantum Hall phase by a generic disorder potential.

On the other hand, we emphasize that the underlying clean band structure of our model may be continuously deformed into a trivial atomic insulator because $\sigma_{xy} = 0$ inside the band gap. That way, all structure is removed from the entire microscopic lattice system without a topological quantum phase transition, which clearly makes the phenomena reported in this Letter quite different from the familiar quantum Hall scenarios based on Chern bands and Landau levels, respectively. As we detail below, our findings also differ qualitatively from several other settings in which topological phenomena have been identified in unconventional relation to Chern numbers [28–31], in particular from the topological Anderson insulator [32–35] where a trivial bulk band gap is closed and reopened with increasing disorder strength, thus inducing an effective Chern insulator band structure.

Besides performing extensive numerical simulations on the transport properties of our model system (see Figs. 1–3) based on the software package *Kwant* [36–38], we verify that a topological invariant defined in the mobility gap [39–41] characterized by a finite localization length of bulk states indeed confirms our predictions regarding the quantization of Hall conductance. A careful analysis of the spectral localizer gap structure [42–47] in the considered system provides further complementary evidence corroborating our results (see Fig. 4).

Minimal two-banded lattice model—We introduce a minimal lattice model specified by the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{W}, \quad (1)$$

where $\hat{H}_0$ is a free translation-invariant two-band Hamiltonian and $\hat{W} = W \sum_{j_x, j_y} \sum_{\alpha=a,b} f_{j, \alpha} c_{j, \alpha}^\dagger c_{j, \alpha}$ is the disorder potential with strength $W \geq 0$ and random amplitudes $f_{j, \alpha}$ drawn independently from the uniform distribution on the interval $[-1, 1]$. The translation-invariant part is specified by the Bloch Hamiltonian $h_0(\mathbf{k}) = \mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\sigma}$, with $\boldsymbol{\sigma}$ being the vector of Pauli matrices and the coefficient vector $d_x(\mathbf{k}) = \gamma \sin(k_x)$, $d_y(\mathbf{k}) = \lambda(k_x) \sin(k_y)$, $d_z(\mathbf{k}) = \gamma_2[r - \cos(2k_x)] - \lambda(k_x) \cos(k_y)$, and $\lambda(k_x) = \epsilon_1 + \epsilon_2[1 - \cos(k_x)]/2$, which corresponds to a tight-binding model on a two-dimensional (2D) square lattice with a unit lattice constant. Containing only nearest-neighbor, plaquette diagonal, and next-nearest-neighbor in x -direction hopping terms, our model is local and fully microscopic; see Supplemental Material [48]. In the following, we discuss its design principles and exemplify our findings using the parameters $r = 1.5$, $\epsilon_1 = 0.3$, $\epsilon_2 = 2$, $\gamma = 2$, and $\gamma_2 = 0.3$, where both bands are topologically trivial (zero Chern number) and there is a finite band gap [see Fig. 1(a)].

Berry flux and transport without disorder—To allow a finite transverse conductance, the topologically trivial Bloch bands of our model accommodate nonquantized momentum-local Berry fluxes that, however, compensate each other within each band when integrated over the first Brillouin zone. To get a feeling for the transport properties of the clean system ($W = 0$), we express the zero temperature Hall conductance as a function of Fermi energy in terms of the accumulated Berry flux up to the Fermi energy, i.e.,

$$\sigma_{xy}^{W=0}(E_F) = \frac{e^2}{h} \frac{1}{2\pi} \int_{E_k < E_F} d^2k \mathcal{F}, \quad (2)$$

where $\mathcal{F}$ is the Berry curvature [48]. The color code in Fig. 1(a) and the $W = 0$ plot line in Fig. 1(b) visualize the landscape of $\sigma_{xy}^{W=0}(E_F)$.

We would like to emphasize two points. First, the zero value of $\sigma_{xy}^{W=0}(E_F)$ in the band gap hallmarks the topologically trivial nature of the bands. Second, the approximate plateau of $\sigma_{xy}^{W=0}(E_F)$ within either band has a nonquantized (and nonuniversal with model parameters) value, reflecting the energetic separation of the aforementioned Berry fluxes within the bands. Since there is no spectral gap or mobility gap in the energy windows of finite Hall conductance in the clean case, the value of $\sigma_{xy}^{W=0}$ is not quantized and can be tuned with the hopping parameters. Clearly, for E_F inside the bands, the metallic nature is reflected in an extensive value of $\sigma_{xx}^{W=0}$.

Transport properties at finite disorder—Now, consider switching on disorder, i.e., $W > 0$. In 2D, the theory of Anderson localization tells us that even a small random potential localizes (almost) all states and introduces a mobility gap, thus forcing the Hall conductance to a quantized value. At small W , it seems natural that σ_{xy} may jump to the closest possible integer value, which is then retained on further increasing disorder strength. This intuition agrees with field-theoretical studies of disordered models with nontrivial Chern bands [6–9, 51, 52], which study plateau broadening finding that integer quantum Hall (IQH) transitions concur with half-integer values of the accumulated Berry flux [cf. Eq. (2)]. Here, we extend this phenomenology to systems without Chern bands. Importantly, once protected by a mobility gap, a quantized value of σ_{xy} is topologically stable against any small perturbation, just as for the conventional Quantum Hall scenario with an underlying Chern band structure or Landau level structure. Remarkably, extensive numerical analysis of our model system fully supports this rationale of a robust quantum Hall effect without Chern bands, as we detail in the following [see $W = 1.5$ line and inset in Fig. 1(b) for a first impression].

Using the software package *Kwant* [36–38], we start our numerical simulations with a two-terminal linear response calculation of the longitudinal conductance σ_{xx} as a

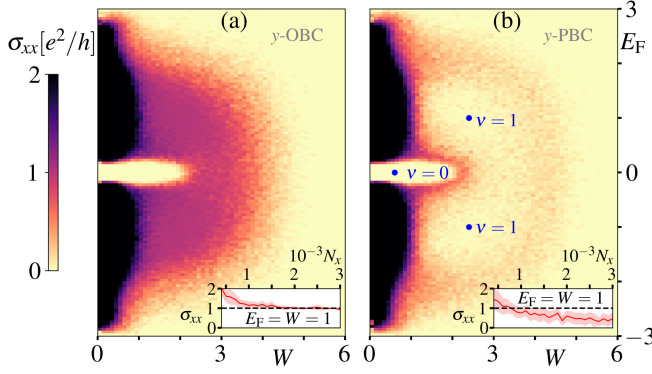


FIG. 2. (a) Two-terminal conductance σ_{xx} for a system with $N_x = 600$, $N_y = 150$ sites, and OBC in the y direction. (b) Same as (a) but with PBC in the y direction. Winding number ν [see Eq. (4)] indicated at the blue points in parameter space. Insets: size scaling of σ_{xx} for $E_F = 1$ and $W = 1$ over N_x at constant $N_x/N_y = 4$ with standard deviation indicated as a shaded corridor. Parameters are $r = 1.5$, $\epsilon_1 = 0.3$, $\epsilon_2 = 2$, $\gamma = 2$, $\gamma_2 = 0.3$, disorder realizations $S = 20$.

function of Fermi Energy E_F and disorder strength W for a mesoscopic system of $N_x = 600$, $N_y = 150$. We note that the aspect ratio of $N_x/N_y = 4$ in this and the following simulations is chosen due to the anisotropic nature of our model. The leads are modeled with a width of 50 sites attached to the center of each vertical face, i.e., at $x = 1$ and $x = N_x$ around $y = N_y/2$. The lead dispersion is set to $t_L[\cos(k_x) + \cos(k_y)]\sigma_0$ with $t_L = 10$. In Fig. 2, we compare the transport properties for open boundary conditions (OBC) and periodic boundary conditions (PBC) to highlight the effect of edge states. For OBC [see Fig. 2(a)], we find an extended region where the conductance is quantized to one, indicating a topologically protected edge state. By contrast, for PBC [see Fig. 2(b)], any edge states disappear, and the sample is found to be insulating in the parameter regime where conductance quantization is most precise for OBC. We also note a strong $\sigma_{xx} > (e^2/h)$ at weak disorder $W < 1$ for any E_F inside the bands. We attribute this to an expected finite size effect: when the average localization length becomes comparable to system size, an extensive number of evanescent modes contributes to the longitudinal conductance. The inset of Fig. 2(a) quantifies the dependence of the conductance on N_x . All data in Fig. 2 are the average of $S = 20$ independent disorder realizations.

We note that Fig. 2(b) also provides a complementary view on the nonquantized conductance of the clean system that is nudged into an IQH phase through the disorder potential, relating to previous work on systems with topological bands [10,30,31]. There, the finite Berry fluxes of opposite sign separate and flow toward mutual annihilation. Importantly, Berry charges do not annihilate within the same band but recombine interband, thus forming an inner arc and an outer arc in the $E_F - W$ parameter plane.

The phase of quantized Hall conductance is enclosed by these two arcs.

We now probe the transverse (or Hall) conductance σ_{xy} at finite $W = 1.5$. This amounts to modeling our transport simulation as a four-terminal setting [53,54]. The system size is set to $N_x = 2000$, $N_y = 500$, and the leads are attached to the center of all four sample faces, with a lead width of 50 sites at the vertical faces and 100 sites at the horizontal faces with larger localization lengths. In Fig. 1(b), numerical data showing the quantization of σ_{xy} to (e^2/h) in an extended window of E_F averaged over $S = 40$ independent disorder realizations are presented. As expected for a finite system with limited self-averaging properties, larger S improves the precision of the quantization. However, we emphasize that already the single measurements clearly fluctuate around the quantized value as opposed to the nonuniversal value of the clean system [48]. The inset of Fig. 1(b) shows how deviations from the quantized value decay with increasing system size for fixed E_F within the plateau region. We note that the quantized Hall plateau at $W = 1.5$ differs from the window of large nonquantized Berry flux at $W = 0$ in agreement with the continuous deformation of the region of quantized transport in Fig. 2 as a function of W . However, we emphasize that there is no bulk band gap closing between $W = 0$ and $W = 1.5$.

Complementary to the above four-terminal setup, we now explore a larger parameter range in the $W - E_F$ plane by computing σ_{xy} through the Kubo-Bastin formula [55–57] for linear response conductivity:

$$\sigma_{xy}(E_F, T) = \frac{ie^2\hbar}{V} \int_{-\infty}^{\infty} dE f(E_F - E, T) \times \text{Tr} \left[v_x \frac{dG^+}{dE} v_y \rho(E) - v_x \rho(E) v_y \frac{dG^-}{dE} \right]. \quad (3)$$

Here, $f(E - E_F, T)$ is the Fermi distribution at temperature T , $v_\alpha = -(i/\hbar)[R_\alpha, H]$ denotes the velocity operator as induced by the position operator R_α , $\rho(E) = \delta(E - H)$ is the density of states (DOS), and $G^\pm = [E \pm i0^+ - H]^{-1}$ is the retarded and advanced Green's function (GF), respectively. In the interest of numerical efficiency, the DOS and GF may be approximated by an expansion in Chebyshev polynomials to order M , with the expansion coefficients determined from a Monte Carlo evaluation of the trace using R random phase vectors [58–60]. Together with S , these parameters control the numerical accuracy in evaluating Eq. (3) [48].

Figure 3 shows σ_{xy} as a function of W and E_F at $T = 0$. These results confirm the edge state picture discussed above in the context of the longitudinal conductance [cf. Fig. 2(a)]. We attribute minor deviations of σ_{xy} from its quantized value in the core IQH parameter regime to the combination of finite-size effects and the approximate

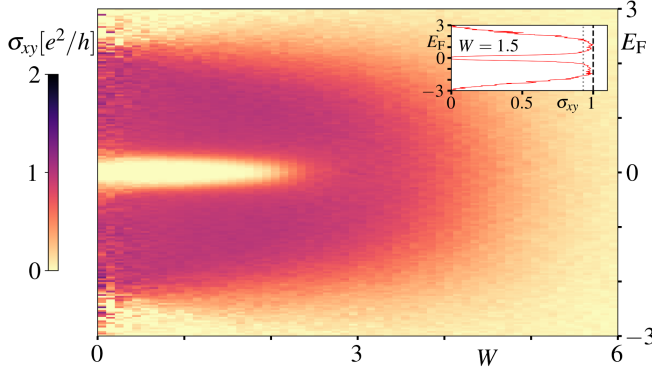


FIG. 3. σ_{xy} as a function of W and E_F for $N_x = 600$, $N_y = 150$ sites with PBC, obtained from the Chebyshev expansion of the Kubo-Bastin formula (3) with $M = 1500$, $R = 5$, and $S = 30$. Parameters are $r = 1.5$, $\epsilon_1 = 0.3$, $\epsilon_2 = 2$, $\gamma = 2$, $\gamma_2 = 0.3$. Inset: calculation with increased accuracy using $M = 2500$ and $S = 50$ for $W = 1.5$. The maximal clean $\sigma_{xy} \approx 0.93$ is shown as a dotted gray line [cf. Fig. 1(b)].

nature of the numerical evaluation. For a quantitative analysis of finite-size effects, we refer the reader to the inset of Fig. 1(b).

Topology of conductance quantization—When the Fermi energy lies inside a mobility gap at finite W , for which zero σ_{xx} is a sufficient condition here, a topological invariant may be defined directly for the reflection part R of the unitary S matrix. Specifically, introducing twisted boundary conditions perpendicular to the direction of transport with a flux angle ϕ , the integer quantized winding number

$$\nu = \frac{1}{2\pi} \oint_0^{2\pi} d\phi \left(\frac{\partial}{\partial \phi} \arg \{ \det[R(\phi, E)] \} \right) \quad (4)$$

of the reflection matrix counts the charge pumped between the cylinder edges for a full twist of the boundary phase, which then corresponds to σ_{xy} for PBC in units of (e^2/h) [39–41]. We calculate ν at multiple points in the phase diagram, indicating the result in Fig. 2(b), which further corroborates the topological nature of the reported IQH phase.

As a complementary diagnostic tool for topology, we analyze the spectral localizer defined as the Hermitian matrix

$$L(x, y, E) = \sigma_z \otimes (H - E) + \kappa \sigma_x \otimes (R_x - x) + \kappa \sigma_y \otimes (R_y - y) \quad (5)$$

for a system in two spatial dimensions and symmetry class A [20,61]. Here, R_α denotes the position operator for coordinate α , and $x, y \in \mathbb{R}$ are real numbers representing a position in real space, while $\kappa > 0$ balances the relative weight of H and R_α [42–46]. A prerequisite of topological robustness is then provided by the bulk localizer gap

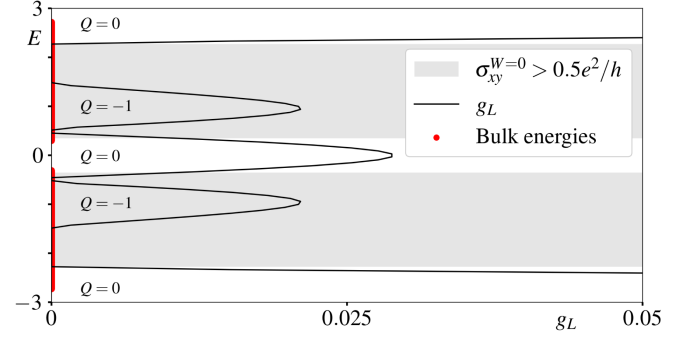


FIG. 4. Localizer gap g_L [cf. Eq. (6)] as a function of E for a clean system of size $N_x = 20$, $N_y = 20$ with $\kappa = 0.4$. The shown g_L is the minimum of 100 reference positions inside the central Wigner-Seitz cell. For finite g_L , we indicate the value of Q [cf. Eq. (7)]. The clean bulk spectrum is shown in red, and the energy window where $\sigma_{xy}^{W=0}(E_F) > 0.5e^2/h$ [cf. Eq. (2)] is marked in gray. Parameters are $r = 1.5$, $\epsilon_1 = 0.3$, $\epsilon_2 = 2$, $\gamma = 2$, $\gamma_2 = 0.3$, $W = 0$.

$$g_L(E) = \min_{x,y \in \text{bulk}} |\text{spec}[L(x, y, E)]|, \quad (6)$$

where $\text{spec}[\dots]$ denotes the set of all eigenvalues. At energy E located in a spectral gap of H , a finite localizer gap $g_L(E)$ can be proven to exist for suitable κ . Given a bulk localizer gap, the index

$$Q(x, y, E) = \frac{1}{2} \text{sig}[L(x, y, E)], \quad (7)$$

where the signature $\text{sig}[L(x, y, E)]$ denotes the difference in the number of positive and negative eigenvalues, is well defined and a constant integer for $x, y \in \text{bulk}$. Physically, Q measures the number of chiral edge states encircling the system at energy E [42–46]. We emphasize that the assumption of a spectral gap does not apply to our present analysis at $W = 0$. However, it has been observed that $g_L(E)$ can remain finite even if there is no spectral or mobility gap at E , which is still an active subject of research in math and physics [31,45].

In our model, choosing $\kappa \approx 0.4$, we find that a localizer gap opens deep inside the bulk band, which is accompanied by a nontrivial index $Q = -1$ (see Fig. 4). Contributing to the ongoing discussion about the role of $g_L(E)$ in metallic situations, we argue that the data in Fig. 4 still predict the formation of a stable IQH phase in the presence of a generic random potential. In particular, at small disorder strength, for which all eigenvalues of $\hat{W}$ are smaller than $g_L(E)$, the localizer gap cannot close by virtue of Weyl’s inequality [31,46,48]. However, in the thermodynamic limit infinitesimal strength of disorder should suffice to cause a mobility gap due to Anderson localization, for which the localizer index predicts a chiral edge state surrounding the system [47], which by bulk-boundary correspondence is tantamount to an IQH phase.

Comparing the accumulated Berry flux with the localizer gap (see Fig. 4), we note that $\sigma_{xy}^{W=0}(E_F) < 0.5e^2/h$ corresponds well to regions of a topologically trivial ($Q = 0$) localizer gap. However, a finite topological localizer gap ($Q = -1$) is only clearly visible in part of the energetic region with $\sigma_{xy}^{W=0}(E_F) > 0.5e^2/h$. While this may to some extent relate to finite size effects [48], we do not exclude that Q contains additional structure beyond rounding $\sigma_{xy}^{W=0}$ to integer values. We emphasize that none of these scenarios contradict our findings regarding σ_{xy} at finite W .

Concluding discussion—We have demonstrated how a topologically stable quantum Hall effect may appear at the onset of disorder without nontrivial Chern bands in the underlying band structure. From a practical computational perspective, we emphasize that the minimal disorder strength $W > 0$ required to observe this effect is constrained only by the numerically accessible system sizes that must exceed the localization length, which diverges for $W \rightarrow 0$. On a more physical note, the subtle interplay between increasing localization length necessitating larger system sizes and the asymptotically discontinuous integer rounding of σ_{xy} within trivial bands represents an intriguing order-of-limits problem worth future analytical study. There, calculating a disorder-averaged real-space Green's function [62,63] and analyzing its topological properties [64] may provide complementary insight. To conclude, we would like to integrate our findings into a broader context by discussing their relation to previous work on nonstandard (from a perspective of topological Bloch bands) topological phenomena.

In periodically driven (Floquet) systems far from equilibrium, a counterpart of the IQH effect relating to Floquet chiral edge states has been reported despite zero Chern number of all bulk bands [28,29]. However, there the underlying band structure is still topological, and the chiral edge states can only be removed by changing a bulk topological invariant unique to Floquet systems, which makes the role of zero Chern numbers very different from our present Letter.

Moreover, in a topological band structure with a trivial band that is energetically sandwiched between two Chern bands and connected to them via topologically protected chiral edge states, a topological fine structure has recently been reported [31]. There, a bulk localizer gap inside the trivial band is connected to a disorder-driven transition where the two Chern bands do not recombine directly but instead with (substructure of) the trivial band. However, this leaves open the main question addressed in our present Letter, namely whether a quantum Hall effect can occur in a system without any topological bands and protected chiral edge states in the underlying band structure, respectively.

Finally, a phase known as the topological Anderson insulator (TAI) has been reported [32–35]. There, starting from a trivial gapped band structure, disorder drives a topological quantum phase transition into a Chern insulator

phase, amounting to a bulk gap closing and reopening in the effective band structure at finite W . While turning a trivial insulator into a topological one by disorder is a powerful and intriguing mechanism, qualitatively this scenario is similar to entering a Chern insulator phase via a bulk gap closing upon tuning a band structure parameter such as a Dirac mass. By contrast, our present finding is that a IQH phase can form in a trivial band structure without a bulk gap closing being responsible. Instead, here the IQH effect occurs with the onset of disorder inside the trivial bands, while a bulk gap closing of TAI type only happens at a quite large disorder strength of $W \approx 2.5$ (cf. Fig. 2).

Broadening the perspective, with the advent of Chern insulators generalizing from the specific setting of Landau levels, it was demonstrated that a magnetic field or net magnetization is not necessary for the occurrence of the quantum Hall effect [65]. In this general context, our present Letter further relaxes the conditions for the generic occurrence of the quantum Hall effect by exemplifying how the energetic separation of substantial but nonquantized Berry fluxes in topologically trivial band structures can be sufficient. Our findings thus further widen the habitat of topologically quantized physical observables.

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Data availability—The data is now available in a Zenodo repository with the following Ref. [66].

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