

Uniqueness of Purifications Is Equivalent to Haag Duality

Lauritz van Luijk[✉], Alexander Stottmeister[✉], and Henrik Wilming[✉]

Leibniz Universität Hannover, Institut für Theoretische Physik, Appelstraße 2, 30167 Hannover, Germany



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The uniqueness of purifications of quantum states on a system A up to local unitary transformations on a purifying system B is central to quantum information theory. We show that, if the two systems are modeled by commuting von Neumann algebras M_A and M_B on a Hilbert space $\mathcal{H}$, then uniqueness of purifications is equivalent to *Haag duality* $M_A = M'_B$. In particular, the uniqueness of purifications can fail in systems with infinitely many degrees of freedom—even when M_A and M_B are commuting factors that jointly generate $B(\mathcal{H})$ and hence allow for local tomography of all density matrices on $\mathcal{H}$. We present a simple argument showing that the uniqueness of purifications, and, hence, Haag duality, fail in ground state sectors of topologically ordered models at renormalization group fixed points on infinite two-dimensional lattices, partitioned into the union of two spatially separated cones and its complement.

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Motivated by many-body physics, quantum field theory, and quantum gravity, rigorous results on entanglement in systems with infinitely many degrees of freedom have been receiving increased interest [1–14]. When describing entanglement in such systems, an operator algebraic approach offers a natural, model-free perspective. In this setting, there is, in general, no tensor product decomposition of the Hilbert space describing subsystems. Instead, subsystems $A, B, \dots$ correspond to von Neumann algebras $M_A, M_B, \dots$ on the full system's Hilbert space $\mathcal{H}$. Importantly, the von Neumann algebras M_A and M_B commute if the corresponding subsystems are distinct.

If both subsystems are purely quantum, i.e., do not contain degrees of freedom that can be measured without being perturbed, then M_A and M_B are *factors*, see [15] or Sec. 4.4 in [16]. A factor is a von Neumann algebra $M \subset B(\mathcal{H})$ such that $M \cap M' = \mathbb{C}$, where M' denotes the commutant: $M' = \{x \in B(\mathcal{H}) : [x, y] = 0 \ \forall y \in M\}$.

The negative resolution of Tsirelson's problem [17–20] shows that commuting operator models can give rise to correlations that cannot even be approximated by correlations of systems modeled with tensor products [21,22] (see also [23,24]). However, this limitation typically does not appear in physical systems like quantum field theory or quantum many-body systems in the thermodynamic limit, where the von Neumann algebras describing subsystems are hyperfinite (approximately finite dimensional) [12,17]. It has also been shown that commuting operator models

allow for operationally distinct kinds of infinite entanglement [3,11,13], even allowing the embezzling of arbitrary entangled states [10,25–28]. Such effects motivate us to reexamine the foundational results of finite-dimensional quantum information theory in commuting operator models.

A typical situation that one considers is that of a bipartition: the full system is divided into the two parts A and B so that no degrees of freedom are left out. This immediately leads to our central question: how does one model that no degrees of freedom are left out?

Below we will discuss three different ways to model this requirement: (1) local tomography, (2) Haag duality, and (3) the *Uhlmann property*, encoding the uniqueness of purifications. In finite dimensions, more generally in a tensor product framework, all of these are equivalent. In systems with infinitely many degrees of freedom, this equivalence generally fails, motivating us to clarify the relation between different models. For purely quantum systems, Haag duality implies local tomography, but the converse is false. Our main result is that Haag duality and the Uhlmann property are equivalent.

We discuss below how these different ways of modeling bipartite systems play out in spatial bipartitions of quantum many-body systems described by *pure states*. The upshot of this discussion is that local tomography is automatic, whereas Haag duality is not. Instead, the validity of Haag duality depends on the physics of the systems and on the geometry of the bipartition ([29], Sec. III.4.2). Verifying it in concrete models is a notoriously difficult task [1,3,5,14,30–32], even for linear quantum fields, see, e.g., [29,33–35], although it is expected to hold for simple geometries on general grounds [36–38].

Local tomography—A (seemingly) minimal requirement that A and B together constitute the full system is that every

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density operator ρ on $\mathcal{H}$ should allow for *local tomography* by Alice and Bob: it can be uniquely identified by all correlation functions of the form

$$(a, b) \mapsto \text{tr}(\rho ab), \quad a \in M_A, b \in M_B. \quad (1)$$

The local tomography condition is equivalent to $M_A \vee M_B = B(\mathcal{H})$, where $M_A \vee M_B = (M_A \cup M_B)''$ is the von Neumann algebra generated by M_A and M_B . Since M_A and M_B commute, local tomography requires that M_A and M_B are factors:

$$M_A \cap M'_A \subseteq M'_B \cap M'_A = (M_B \vee M_A)' = B(\mathcal{H})' = \mathbb{C}, \quad (2)$$

and similarly $M_B \cap M'_B = \mathbb{C}$. Indeed, if M_A is not a factor, there is a nontrivial unitary operator u that commutes with both M_A and M_B . Hence, there are vectors $\Psi \neq u\Psi$ which cannot be distinguished by local tomography.

To summarize: local tomography holds if and only if M_A and M_B are factors such that $M_A \vee M_B = B(\mathcal{H})$ (or, equivalently, $M'_A \cap M'_B = \mathbb{C}$).

Haag duality—Instead of the local tomography condition, one could also consider a different way to model that no degrees of freedom are left out: demanding that Alice's observables are exactly those that are jointly measurable with *each* of Bob's observables. Since joint measurability is equivalent to commutativity if one observable is sharp (i.e., projection valued) [39], this can be expressed as

$$M_B = M'_A. \quad (3)$$

This condition is known as *Haag duality*, after a related property in algebraic quantum field theory [2,29].

Haag duality (or approximate versions of it) is of vital importance to derive rigorous results on entanglement in systems with infinitely many degrees of freedom [2,4,11,13,26,40,41] and in the rigorous treatment of topologically ordered systems in the thermodynamic limit [30,31,42–49]. For this reason, it is sometimes regarded as a technical condition [3,47,49–51]. But, it is also intimately related to the presence of global and local symmetries as well as additivity properties of the local algebras [38]; see also [52–55] for recent discussions. Our main result will provide a new operational interpretation of Haag duality.

If M_A is a factor and Haag duality holds, then M_B is a factor, too. In this case, Haag duality implies local tomography (and only in this case, since local tomography requires factors). But the converse is false in general: there exist pairs of factors M_A, M_B that allow for local tomography, but the inclusion $M_A \subset M'_B$ is strict. Notably, there is no nontrivial operator in M'_B that commutes with M_A : since $M_A \vee M_B = B(\mathcal{H})$ by local tomography we have [see Eq. (2)]

$$M'_A \cap M'_B = \mathbb{C}. \quad (4)$$

Such a situation is termed an “irreducible subfactor inclusion” in mathematics and is a rich topic of research [56–64]. We will discuss a physical example below. We conclude that Haag duality cannot be given an operational interpretation in terms of local tomography. Of course, Haag duality is trivially true in the tensor product framework, where $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, $M_A = B(\mathcal{H}_A) \otimes 1$, and $M_B = 1 \otimes B(\mathcal{H}_B)$.

Note that a failure of Haag duality may separate the *symmetries* of a subsystem A (unitary operators in M'_A) from the unitary operators generated by observables in the complementary subsystem (unitary operators in M_B). Even in finite-dimensional systems, Haag duality may be violated if one imposes symmetry constraints (e.g., restricts to the invariant subspace of a global symmetry).

Uhlmann property—It is of fundamental importance to quantum information theory that every density matrix has a purification on a larger Hilbert space and that all such purifications are related by a (partial) isometry between the purifying Hilbert spaces. Purifications are unique up to (partial) isometries. Operationally, this means that if ρ on $\mathcal{H}_A$ describes Alice's subsystem and Ψ_ρ is a purification on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, then Bob, acting on $\mathcal{H}_B$ by local operations, can prepare any other purification of Alice's state.

An important consequence of the uniqueness of purifications is Uhlmann's theorem [65–68], which characterizes the fidelity $F(\rho, \sigma) = \|\rho^{1/2} \sigma^{1/2}\|_1$ of a pair of density operators ρ, σ on a Hilbert space $\mathcal{H}$ as the maximal overlap of their purifications

$$F(\rho, \sigma) = \sup_U |\langle \Psi_\sigma, (1 \otimes U) \Psi_\rho \rangle|, \quad (5)$$

where Ψ_ρ and Ψ_σ are two arbitrary purifications in a common extension $\mathcal{H} \otimes \mathcal{K}$ and where the optimization is over unitaries U on $\mathcal{K}$. Uhlmann's theorem is central to quantum information theory. Recently, it has received renewed interest from the point of view of the quantum computational complexity [69–71] as well as from the point of view of its rigidity [72].

As stated above, Uhlmann's theorem only applies to systems with finitely many degrees of freedom. The last condition we consider to formalize that no degree of freedom is left out is whether uniqueness of purifications (and, as a consequence, Uhlmann's theorem [16,65–67]) is fulfilled: starting from any pure state Ψ on $\mathcal{H}$, is it true that any other $\Phi \in \mathcal{H}$ with the same reduced state on M_A as Ψ may be reached by acting with local operations by Bob? We formalize this property as follows:

Definition 1 (Uhlmann property)—A pair of commuting von Neumann algebras M_A, M_B on $\mathcal{H}$ has the Uhlmann property if for any two pure state vectors $\Phi, \Psi \in \mathcal{H}$ such that $\langle \Psi, a \Psi \rangle = \langle \Phi, a \Phi \rangle$ for all $a \in M_A$ it follows

that for any $\varepsilon > 0$ there exists a unitary $u \in M_B$ such that $|\langle \Psi, u\Phi \rangle| \geq 1 - \varepsilon$.

We emphasize that the Uhlmann property only makes a statement about those states on M_A that admit a purification on the joint system, but does not require that all reduced states of one of the subsystems have purifications.

In contrast to local tomography and Haag duality, the Uhlmann property is *a priori* not symmetric relative to the subsystems A and B . It will be a consequence of our main result that it is, in fact, symmetric.

In the definition of the Uhlmann property, we already anticipate that in systems with infinitely many degrees of freedom, one can generally only approximate a given purification Φ up to arbitrary accuracy using unitary operators from M_B acting on Ψ . However, we will see below that if this is possible, then one can in fact find a local partial isometry, and hence a local quantum channel, that deterministically maps Ψ to Φ .

The approximate nature of Definition 1 has the important consequence that we can freely restrict the unitaries to any weakly dense subgroup $G_B \subset \mathcal{U}(M_B)$ of the unitary group $\mathcal{U}(M_B)$, without changing the validity of the Uhlmann property. This is, for instance, useful when discussing quantum many-body systems, where we can take G_B to be the group of finitely supported unitaries in the region B (see below).

Bipartitions of quantum many-body systems in the thermodynamic limit—An important motivation for studying entanglement in operator algebraic settings is given by many-body physics at zero temperature and in the thermodynamic limit. We briefly review their description; see, for example, [50] for details. Consider an infinite lattice Γ of spins with local Hilbert space dimension d . Let $\mathcal{A}_\Gamma$ be the $*$ -algebra of operators that have finite support on Γ . For any subset $A \subset \Gamma$, let $\mathcal{A}_A \subset \mathcal{A}_\Gamma$ be the $*$ -subalgebra of operators with finite support in A . If $A_1 \subset A_2$, then $\mathcal{A}_{A_1} \subset \mathcal{A}_{A_2}$ and if $A_1 \cap A_2 = \emptyset$, then $\mathcal{A}_{A_1}$ and $\mathcal{A}_{A_2}$ commute. If $\Gamma = A \cup B$ is a bipartition of the lattice, $\mathcal{A}_\Gamma$ is isomorphic to the algebraic tensor product of $\mathcal{A}_A$ and $\mathcal{A}_B$. Thus, the analog of Haag duality always holds on the level of operators with finite support: if $x \in \mathcal{A}_\Gamma$, then

$$x \text{ commutes with } \mathcal{A}_A \Leftrightarrow x \in \mathcal{A}_B. \quad (6)$$

A ground state of a Hamiltonian is an algebraic state ω on $\mathcal{A}_\Gamma$. The ground state sector $\mathcal{H}$ arises from its Gelfand-Naimark-Segal (GNS) representation π , and for any bipartition $A \cup B = \Gamma$ we obtain von Neumann algebras $M_A = \pi(\mathcal{A}_A)''$ and $M_B = \pi(\mathcal{A}_B)''$. If ω is a pure state, π is irreducible and we automatically have that $M_A \vee M_B = B(\mathcal{H})$. Thus, for any bipartition $\Gamma = A \cup B$, the factors (M_A, M_B) satisfy the local tomography criterion. However, in general, Haag duality $M_A = M'_B$ fails—although the analog of it holds at the level of finitely supported operators, as discussed above.

We will see an explicit example of such a failure below. As mentioned in the introduction, proving Haag duality in specific situations is generally difficult and no general method is available; see [1,3,5,14,30–32] for examples.

The unitary group of $\mathcal{A}_B$ is weakly dense in $\mathcal{U}(M_B)$ [73]. Hence, by the comments at the end of the previous section, Uhlmann's property can be completely formulated in terms of unitary operators with finite support. Thus, if Haag duality holds, B can approximately map any purification to any other purification with unitary operators from $\mathcal{A}_B$.

Main result—We are now in the position to state our main result.

Theorem 2—A pair of commuting von Neumann algebras M_A, M_B on $\mathcal{H}$ has the Uhlmann property if and only if Haag duality holds. In this case, if state vectors $\Psi, \Phi \in \mathcal{H}$ have the same A marginal, there exists a partial isometry $v \in M_B$ with $v\Psi = \Phi$.

Before we come to the proof of the theorem, which can also be found in the Ph.D. thesis of the first-named author [16], let us briefly discuss its consequences. Recall that every irreducible subfactor $M_A \subset M'_B$ gives rise to a bipartite system where local tomography holds, but Haag duality does not. Since irreducible subfactors exist, see, e.g., [57], we conclude the following: if M_A, M_B are commuting factors on $\mathcal{H}$, then

$$\text{local tomogr.} \stackrel{\Leftarrow}{\neq} \text{Haag duality} \Leftrightarrow \text{Uhlmann property.}$$

Thus, in general, local tomography does not imply the uniqueness of purifications.

Let us illustrate how severe the failure of the Uhlmann property can be: there exist irreducible subfactor inclusions $M_A \subset M'_B$ together with a state on M_B and two purifications Ψ, Φ such that

$$\langle \Psi, u\Phi \rangle = 0 \quad \text{for all } u \in M_A. \quad (7)$$

We discuss a physical example below. Note that, as a corollary, we also see that Uhlmann's theorem fails in the strongest possible form: the fidelity between any state on M_B and itself is 1 (one can define fidelity for general von Neumann algebras [75]), but the maximum overlap one can achieve with local unitaries of Alice between the two purifications Ψ and Φ is zero.

Example: Surface code—We now illustrate the points that we made above with a physical example observed by Naaijken [43]. We will restrict ourselves to physical arguments; the mathematical arguments can be found in [43,44,47]. Consider the (unique) translation-invariant ground state of Kitaev's surface code [50,76,77] on an infinite square lattice Γ . We divide the lattice into three regions A_1, A_2 , and $B = (A_1 \cup A_2)'$, where the prime denotes the complement of a set of lattice sites, as shown in Fig. 1. Consider the Hilbert space $\mathcal{H}$ arising from the GNS representation of the local algebra $\mathcal{A}_\Gamma$ relative to the

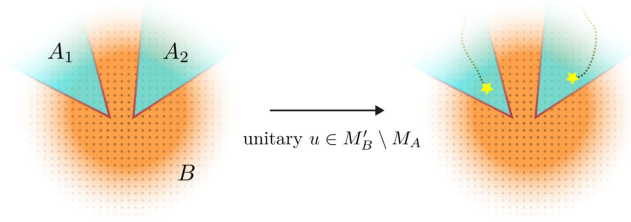


FIG. 1. Surface code on the infinite lattice: a pair of anyons can be created by a unitary that is an element of the commutant of M_B , but not part of M_A , because the two regions A_1 and A_2 are not connected.

translation invariant ground state of the surface code. To every region Λ of the lattice, we associate the von Neumann algebra M_Λ generated by operators with finite support in Λ .

First, consider the bipartition A_1 vs A'_1 . In this case Haag duality $M_{A'_1} = M'_{A_1}$ holds [30,31]. Hence, by Theorem 2 all purifications of the marginal on A'_1 are related by operators in M_{A_1} .

Now consider instead the bipartition $A := A_1 \cup A_2$ vs $B = A'$. Instead of the ground state $\Psi \in \mathcal{H}$, we can also consider a state $\Phi \in \mathcal{H}$ where a pair of nontrivial anyons is created on top of the ground state, one located in A_1 and one located in A_2 (the precise location within the regions is irrelevant), see also Fig. 1.

The state $\Phi \in \mathcal{H}$ is clearly orthogonal to the ground state, but it has the same marginal on M_B : for every operator b with finite support in B , we can use a unitary ribbon operator with support away from b to create Φ from Ψ . Since the operators with finite support are (weakly) dense in M_B we conclude

$$\langle \Psi, b\Psi \rangle = \langle \Phi, b\Phi \rangle, \quad b \in M_B. \quad (8)$$

By Theorem 2, there exists a partial isometry $v \in M'_B$ such that $\Phi = v\Psi$. It can be seen to arise from a limiting process, where a ribbon operator connecting the two anyons is pushed to infinity (see Fig. 1) [44,47].

On the other hand, with unitary operators from M_A , one can arbitrarily move around the anyons within A_1 and A_2 , respectively, but one cannot create or remove them, i.e., we have

$$\langle \Psi, u\Phi \rangle = 0, \quad u \in M_A. \quad (9)$$

By construction of M_A , it suffices to check (9) for unitaries u with finite support $F \subset A$. Orthogonality holds because we can perfectly distinguish $u\Phi$ from Ψ by measuring a closed, topologically trivial ribbon operator that separates $F \cap A_1$ from $F \cap A_2$.

Since we can create three different kinds of anyons (electric, magnetic, or a combination of both), there are in fact four classes of pairwise orthogonal purifications of the ground state marginal on M_B . We emphasize that the above

argument only uses the existence of deformable ribbon operators that are supported on strips of finite width, the end points of which carry nontrivial anyons that are separately detectable, e.g., by another closed, topologically trivial ribbon operator. No further specific properties of the surface code are required. Hence, the argument generally applies to two-dimensional topological ordered models at renormalization group fixed points. We stress that the argument depends on the nontrivial geometry at hand, as it is known that Haag duality holds for single cones at said fixed points [30–32].

Proof of Theorem 2—We first show “Haag duality $\Rightarrow$ Uhlmann property” (see also [67]). Consider two states Ψ and Φ that purify the same normal state ω on M_A . The subspaces $\mathcal{H}_\Psi = \overline{M_A\Psi} \subset \mathcal{H}$ and $\mathcal{H}_\Phi = \overline{M_A\Phi} \subset \mathcal{H}$ both provide GNS representations of ω . By the uniqueness of the GNS representation up to unitary equivalence [74], Theorem I.9.14, there exists a unitary $w: \mathcal{H}_\Psi \rightarrow \mathcal{H}_\Phi$ that intertwines the two representations, i.e., fulfills

$$wa\Psi = aw\Psi = a\Phi, \quad a \in M_A. \quad (10)$$

Extending w to a partial isometry $v: \mathcal{H} \rightarrow \mathcal{H}$, we find that v must commute with all elements of M_A . Hence, there is a partial isometry $v \in M'_A = M_B$ with $v\Psi = \Phi$. This shows the second claim. We finally show that v may be replaced by a unitary up to arbitrarily small error, using the argument in [78], Lemma 2.4. By the Russo-Dye-Palmer theorem [79,80], for any $\varepsilon > 0$ we have $(1 - \varepsilon)v = \sum_i p_i u_i$ for some finite probability distribution p_i over unitaries $u_i \in M_B$. Thus, by the triangle inequality, we have

$$1 - \varepsilon = (1 - \varepsilon)|\langle \Phi, v\Psi \rangle| \leq \sum_i p_i |\langle \Phi, u_i\Psi \rangle|. \quad (11)$$

Thus, $|\langle \Phi, u\Psi \rangle| \geq 1 - \varepsilon$ with $u = u_i$ for some index i .

We now show “Uhlmann property $\Rightarrow$ Haag duality.” Let $\mathcal{U}(M)$ denote the unitary group of a von Neumann algebra M . By the argument in the previous section, the Uhlmann property implies $\overline{\mathcal{U}(M_A)\Psi} = \overline{\mathcal{U}(M'_B)\Psi}$ for all $\Psi \in \mathcal{H}$. Since a von Neumann algebra is spanned by its unitary group (Prop. I.4.9 in [74]) we thus have $[M_A\Psi] = [M'_B\Psi]$ for all $\Psi \in \mathcal{H}$, where $[V]$ denotes the orthogonal projection onto the closed subspace generated by a subset $V \subset \mathcal{H}$. Since a projection $p \in B(\mathcal{H})$ is in a von Neumann algebra $M \subset B(\mathcal{H})$ if and only if its range $p\mathcal{H}$ is M' invariant, we have $[M_A\Psi] \in M_B$ for all $\Psi \in \mathcal{H}$. Now consider some projection $p \in M'_A$ and let $\{\Omega_n\}$ be a basis of $p\mathcal{H}$. Then $p = [\cup_n M_A\Omega_n] = \vee_n [M_A\Omega_n]$, where $\vee_n p_n$ denotes the smallest projection $q \in B(\mathcal{H})$ such that $q \geq p_n$ for all n . Since $[M_A\Omega_n] \in M_B$ by our previous observation it follows that $p \in M_B$ ([81], Corollary 3.7). Hence, every projection $p \in M'_A$ is also an element of M_B and therefore $M_B \subset M'_A \subset M_B$.

Discussion and outlook—We have shown that the uniqueness of purifications in a bipartite system modeled by commuting von Neumann algebras M_A, M_B is equivalent to Haag duality $M_A = M'_B$. This finding shows that Haag duality may be seen as the Heisenberg-picture formulation of the uniqueness of purification and provides a quantum information-theoretic explanation for the importance of Haag duality. Conversely, it shows that in systems where Haag duality fails, many results of quantum information theory are not applicable. In the case of factors, the failure of both conditions is measured by the nontriviality of the subfactor inclusion $M_A \subset M'_B$, which is irreducible if and only if local tomography holds. It is well known that there exists at most one normal conditional expectation $E: M'_B \rightarrow M_A$ in such a case. Associated with E is an *index*, which can be seen to characterize how much bigger M'_B is compared to M_A [56,58,59,63]. The index can also be understood as measuring how much more classical information can be encoded into E -invariant states when having access to M'_B instead of M_A [44,47]. Thus, if E has operational meaning, then the index carries operational meaning. In upcoming work, we show that the existence of a conditional expectation that preserves a given marginal state can be given an operational meaning itself.

An interesting open question is whether the uniqueness of purifications and Haag duality can also be related in a *quantitative* rather than a qualitative way. It is not difficult to see that situations as discussed in the example above, where purifications remain orthogonal under arbitrary operations by Alice, exist for arbitrarily large index [82]. It is therefore not clear whether there is a general and meaningful quantitative connection that the index measures.

Our work exemplifies that foundational results from quantum information theory do not always transfer to systems with infinitely many degrees of freedom. In simulations, one typically mimics the effect of an infinite system by restricting to observables with support much smaller than the total system size, i.e., separating the observation scale from the system scale. Our results show that separating observational and system scales can lead to different realizations of local observables (M_B vs M'_A). This difference is potentially detectable at finite system size, but we leave the detailed study of finite-size implications of failure of Haag duality for future work. Specifically, this may be relevant to the information-theoretic aspects of simulating quantum field theories. As such simulations need to happen at finite size, the above-mentioned separation of scales is natural.

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