

Probabilistic Construction of Noncompactified Imaginary Liouville Field Theory

Romain Usciat ¹, Colin Guillarmou ², Remi Rhodes ³, and Raoul Santachiara ¹

¹*LPTMS, CNRS, Université Paris-Sud, Université Paris-Saclay, 91405 Orsay, France*

²*IMO, Université Paris-Saclay, 91405 Orsay, France*

³*Aix Marseille Université, CNRS, I2M, Marseille, France*



(Received 3 June 2025; accepted 22 December 2025; published 21 January 2026)

We propose a probabilistic construction of imaginary Liouville field theory based on a real (noncompactified) Gaussian free field. We argue that our theory represents the first explicit Lagrangian field theory that generates the imaginary Dorn, Otto, Zamolodchikov, and Zamolodchikov (DOZZ) constants without requiring a neutrality constraint. The validity of our theory is backed up by exact results on the behavior of the imaginary multiplicative chaos on a circle, and by numerical simulations on the theory on the sphere. In particular, the three point functions of the theory that are in a very good agreement with the imaginary DOZZ constant.

DOI: [10.1103/d723-gk4m](https://doi.org/10.1103/d723-gk4m)

Two dimensional conformal field theories (CFT) are quantum field theories that can sometimes be solved within the framework of the so-called conformal bootstrap, based on consistency relations, in a somewhat axiomatic fashion [1]. Over the past decade, mathematicians have developed a probabilistic construction of 2D CFTs, thereby providing a rigorous definition of correlation functions in terms of averages over random Gaussian free fields (GFF) [2]. The Liouville field theory (LFT_b) is a one-parameter (b) family of CFT with a continuous family of scaling exponents. When $b \in \mathbb{R}$ it appears as a universal building block for a variety of models for 2D gravity [3]. It also plays a crucial role in recent statistical physics applications, such as the derivation of backbone exponents in percolation [4]. The greatest achievement of the probabilistic approach is the rigorous derivation of the corresponding bootstrap solutions of $\text{LFT}_{b \in \mathbb{R}}$ [5]. The analytic continuation $b \rightarrow i\beta$ with $\beta \in \mathbb{R}$ of the $\text{LFT}_{b \in \mathbb{R}}$ bootstrap solution is not possible [6]. This reflects the fact that the $\text{LFT}_{i\beta \in \mathbb{R}}$ theory, which we will refer to as imaginary LFT (also called “timelike Liouville” by string theorists), is inherently ill-defined as a quantum field theory.

The existence of the imaginary Liouville theory had long been suggested by quantum gravity models [6–11] and by geometric statistical models, such as the $O(n)$ loop models and the related Potts models. Indeed these latter models have provided a statistical interpretation of the 3-point function given by the imaginary Dorn, Otto, Zamolodchikov,

and Zamolodchikov (DOZZ) constant [12,13], postulated to be the 3-point function of $\text{LFT}_{i\beta}$ [7,14]. The long-standing question remains of whether and how the probabilistic formulation of the imaginary LFT can be achieved, thereby placing the theory on solid footing and resolving the inherent ill-definedness.

A probabilistic construction of a compactified version of the $\text{LFT}_{i\beta}$ [15,16], named compactified imaginary Liouville theory (CILT), has recently been achieved. This theory, based on a GFF taking values on a circle, is strictly related to the Coulomb gas formalism for minimal models [17,18], and has discrete spectrum. In particular, the central charge $c_{\text{charge}} < 1$ can only take discrete values and the correlation functions vanish unless they satisfy a selection rule, known as the neutrality condition. These selection rules impose strong constraints, making the theory too limited to describe observables in geometric statistical models or for applications in quantum gravity.

Here, we propose a probabilistic construction of $\text{LFT}_{i\beta}$ based on a real (noncompactified) GFF and the theory of imaginary Gaussian multiplicative chaos (GMC), with a continuous family of scaling exponents. We show in particular that our theory is the first explicit Lagrangian field theory that generates the so-called imaginary DOZZ structure constant without requiring a neutrality constraint. The imaginary DOZZ constant governs amplitudes in geometric statistical models, such as the 3-point connectivity of the random Potts model, and emerges as a fundamental component of new bootstrap solutions. Our approach establishes a natural extension of compactified imaginary LFT and offers a new perspective on CFTs with central charge $c_{\text{charge}} \leq 1$, with potential implications for statistical physics and 2D quantum gravity.

Consider a surface Σ with boundary $\partial\Sigma$ and equipped with a fixed background metric. We denote by dv and $d\ell$

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

the corresponding area and length measures on Σ and $\partial\Sigma$, and by K and k , respectively, the Ricci and geodesic curvature of Σ and $\partial\Sigma$. The $\text{LFT}_{i\beta}$ action takes the form

$$\begin{aligned} S_L^{\text{im}}(\Sigma, \phi) = & \frac{1}{4\pi} \int_{\Sigma} (|d\phi|^2 + iQK\phi + \mu e^{i\beta\phi}) dv + \\ & + \frac{1}{2\pi} \int_{\partial\Sigma} (iQk\phi + \mu_B e^{i\frac{\beta}{2}\phi}) d\ell, \end{aligned} \quad (1)$$

where $\phi(x)$ is the fundamental field, $\mu, \mu_B \in \mathbb{R}$ are usually known as, respectively, the bulk and the boundary cosmological constants. We take the parameter $\beta \in \mathbb{R}$ and one sets $Q = (\beta/2) - (2/\beta)$, ensuring that the resulting quantum field theory is conformally invariant with central charge $c_{\text{charge}} = 1 - 6Q^2 \leq 1$.

Let us discuss first the simplest case of the imaginary $\text{LFT}_{i\beta}$ on a circle. The “circle Liouville theory” corresponds to the $\text{LFT}_{i\beta}$ on the flat unit disk, $\Sigma = \mathbb{D}$ with the Euclidean metric and $\mu = 0$. We split the Liouville field $\phi(x)$ as $\phi(x) = X(x) + c$, where c is a constant and X is the (real) GFF on the disk with covariance $\mathbb{E}[X(x)X(y)] = -\log|x-y||1-x\bar{y}|$ (Neumann boundary condition). As we show below, the important difference with respect to previous works is that we allow the constant c to take general complex values, $c \in \mathbb{C}$.

The basic objects in LFT are defined through an ultraviolet regularization of the GFF X_ϵ at scale $\epsilon > 0$. The primary field of the imaginary LFT are vertex field V_α with scaling dimension $\Delta = (\alpha/2)[(\alpha/2) - Q]$, with $\alpha \in \mathbb{R}$. They are defined by

$$V_\alpha(x) := \lim_{\epsilon \rightarrow 0} \epsilon^{-\frac{\alpha^2}{2}} e^{i\alpha(X_\epsilon(x) + c)}, \quad x \in \mathbb{D}. \quad (2)$$

The simplest nontrivial correlation function of the theory is the 1-point function:

$$\langle V_\alpha \rangle = \int V_\alpha(0) e^{-S_L^{\text{im}}(\mathbb{D}, \phi)} \mathcal{D}\phi, \quad (3)$$

where $\int \mathcal{D}\phi$ denotes the path integral over ϕ . The probabilistic construction of the above correlation function involves the GMC on the circle $\mathbb{S}$:

$$M_\beta^\mathbb{S} := \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \int_0^{2\pi} \left(e^{-\frac{\beta^2}{8}} e^{i\frac{\beta}{2} X_\epsilon(e^{i\theta})} \right) d\theta, \quad (4)$$

and the corresponding moment generating function

$$\mathcal{G}_\beta^\mathbb{S}(\mu) = \mathbb{E}[e^{-\mu M_\beta^\mathbb{S}}]. \quad (5)$$

We restrict ourselves to the range of values $\beta \in (0, \sqrt{2})$. In that case, the limit in (4) is nontrivial and defines a random variable with finite L^2 moments, $\mathbb{E}[|M_\beta^\mathbb{S}|^2] < \infty$. Outside of this range, the limit cannot be given sense as a random

variable. The full phase diagram of the imaginary GMC was studied in [19]. Furthermore, for $\beta \in (0, \sqrt{2})$, the random variable has exponential moments so that the expectation (5) is well defined [15].

Splitting the integration $\int \mathcal{D}\phi \dots = \int dc \mathbb{E}[\dots]$, and using the definition (1), the 1-point correlation function (3) can be expressed as

$$\langle V_\alpha \rangle = \int_{\mathcal{C}} e^{i(\alpha-Q)c} \mathcal{G}_\beta^\mathbb{S}(\mu_B e^{i\frac{\beta}{2}c}) dc. \quad (6)$$

The crucial point here is the choice of the contour $\mathcal{C}$ in the c complex plane. As proposed in [6], the definition of the $\text{LFT}_{i\beta}$ path integral could, in principle, be understood by the use of Morse theory. This would involve deforming the integration path into a sum of Lefschetz thimbles, which are the infinite-dimensional generalization of steepest descent contours [20]. Building from this idea, the semiclassical limit of the imaginary DOZZ function was studied and found to exhibit an analytic behavior similar to that of the inverse Γ function [6]. Note that, in the minisuperspace approximation where only the constant mode c is integrated over, $M_\beta^\mathbb{S} = 1$ and $\mathcal{G}_\beta^\mathbb{S}(\mu_B e^{i(\beta/2)c}) \rightarrow \exp(\mu_B e^{i(\beta/2)c})$, the integrand in Eq. (6) is the same as the Γ function. Taking $\mathcal{C}$ as a U-shaped (Hankel type) contour provides the integral representation of the inverse Γ function. It was showed in [21] that U-shaped integrals are key to performing the analytic continuation of the lattice LFT_b when b approaches $i\beta$. Inspired by these results, we set $\mathcal{C} = \mathcal{U}$, where $\mathcal{U}$ consists in the lines $[0, -i\infty)$, $[0, (4\pi/\beta)]$, and $[(4\pi/\beta), (4\pi/\beta) - i\infty)$, see Fig. 1. It turns out that this

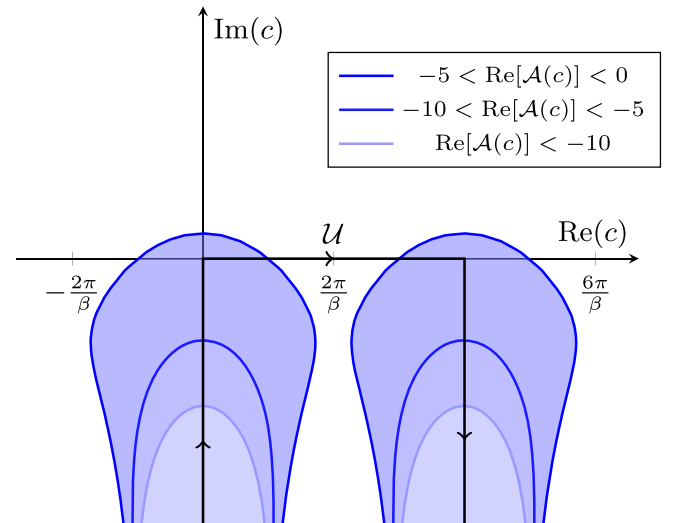


FIG. 1. In the c complex plane, the regions where $\text{Re}[\mathcal{A}(c)]$ takes smaller and smaller negative values, where $\mathcal{A}(c) = i\beta s c/2 + \log \mathcal{G}_\beta^\mathbb{S}(\mu e^{i\beta/2c})$, are shaded in different blue tones. One can see that on the $\mathcal{U}$ contour (shown in black) the one-point function (6) is well defined. The parameters used in the above picture are $\beta = 0.6$, $s = 0.37$, and $\mu_B = 1$.

choice of contour is perfectly consistent with the behavior of the Laplace transform $\mathcal{G}_\beta^\mathbb{S}(\mu)$ as we show rigorously that

$$\mathcal{G}_\beta^\mathbb{S}(\mu) \stackrel{\mu \rightarrow +\infty}{\sim} \Gamma\left(1 + \frac{\beta^2}{4}\right)^{\frac{4}{\beta^2}} \Gamma\left(\frac{4}{\beta^2} + 1\right) \mu^{-\frac{4}{\beta^2}}. \quad (7)$$

A short proof of Eq. (7) is provided in the End Matter. By taking $\mathcal{C} = \mathcal{U}$, Eq. (6) then perfectly makes sense because the integral converges on the contour $\mathcal{U}$. It yields a non-trivial quantity for $s = -[2/\beta](\alpha - Q)$:

$$\begin{aligned} \langle V_\alpha \rangle^{\text{LFT}_{i\beta}} &= \int_{\mathcal{U}} e^{i(\alpha-Q)c} \mathcal{G}_\beta^\mathbb{S}(\mu_B e^{i\frac{\beta}{2}c}) dc \\ &= \frac{4\pi}{\beta} \frac{e^{-i\pi s} \mu_B^s}{\Gamma(1+s)} \frac{\Gamma\left(1 + s \frac{\beta^2}{4}\right)}{\Gamma\left(1 + \frac{\beta^2}{4}\right)^s}. \end{aligned} \quad (8)$$

If $s \in \mathbb{N}$, the two vertical branches of the contour cancel each other and we are left with the horizontal contribution of the contour. This expression is computable with straightforward Fourier analysis and yields

$$\begin{aligned} \langle V_\alpha \rangle^{\text{LFT}_{i\beta}} &= \int_0^{\frac{4\pi}{\beta}} e^{-i\frac{\beta}{2}sc} \mathbb{E}\left[e^{-\mu_B e^{i\frac{\beta}{2}c} M_\beta^\mathbb{S}}\right] dc \\ &= \frac{4\pi}{\beta} \frac{(-\mu_B)^s}{s!} \mathbb{E}\left[(M_\beta^\mathbb{S})^s\right] \\ &= \frac{4\pi}{\beta} \frac{(-\mu_B)^s}{s!} \frac{\Gamma\left(1 + s \frac{\beta^2}{4}\right)}{\Gamma\left(1 + \frac{\beta^2}{4}\right)^s}. \end{aligned} \quad (9)$$

The expression (9) can be seen as the circular version of the Coulomb gas (CG) integral and it is an evident extension of (8) for real values of s . Moreover, we also observe below that, up to some normalization-dependent multipliers, the $\langle V_\alpha(0) \rangle^{\text{LFT}_{i\beta}}$ satisfies the same shift relations as those satisfied by the real LFT_b on the circle. This is not surprising: the correlation functions of LFT theories are known to obey shift relations stemming from the insertion of degenerate states, which are assumed to be present in the theory. The behavior of these states depends only on the structure of the Virasoro algebra and is therefore expected to remain invariant under the analytic continuation of the parameters b and α . The 1-point function of the real LFT_b , $\langle V_\alpha(0) \rangle^{\text{LFT}_b}$, can be computed as the solutions of these shift relations or via the standard probabilistic expression

$$\langle V_\alpha \rangle^{\text{LFT}_b} = \frac{2}{b} \mu^{-\tilde{s}} \Gamma\left(1 - \frac{b^2}{4}\right)^{\tilde{s}} \Gamma(\tilde{s}) \Gamma\left(1 + \tilde{s} \frac{b^2}{4}\right), \quad (10)$$

with $\tilde{s} = [2/b](\alpha - Q)$. There is a simple relation between the 1-point function of the standard theory $\langle V_\alpha \rangle^{\text{LFT}_b}$, which can be formally continued to $b = i\beta$, and the 1-point function of our theory:

$$\langle V_{i\alpha} \rangle^{\text{LFT}_{b \rightarrow i\beta}} = \frac{i e^{i\pi s}}{2 \sin \pi s} \langle V_\alpha \rangle^{\text{LFT}_{i\beta}}. \quad (11)$$

This last expression echoes the relation between the real DOZZ formula and the imaginary DOZZ formula [22], the coefficient $(i e^{i\pi s} / 2 \sin \pi s)$ playing the role of the step function in [23]. We observe also that $\langle V_{i\alpha} \rangle^{\text{LFT}_{b \rightarrow i\beta}}$ can be obtained directly from (6) if one chooses as a contour $\mathcal{C} = i\mathbb{R}$: by standard contour deformation, one has $(1 - e^{2i\pi s}) \int_{i\mathbb{R}} dc \dots = \int_{\mathcal{U}} dc \dots$, which yields the relation (11). Notice that the $\langle V_{i\alpha} \rangle^{\text{LFT}_{b \rightarrow i\beta}}$ for $\beta = \sqrt{2}$, i.e., $c_{\text{charge}} = 1$, was also derived in [8], while a study of the solutions of the shift relation for $c_{\text{charge}} \leq 1$ and for non-vanishing bulk cosmological constant, was considered recently in [24].

The 1-point function of the imaginary LFT reveals that, depending on the choice of contour for integrating the constant mode, one obtains two strictly related theories. The first one corresponds to the analytic continuation of the real LFT_b under $b \rightarrow i\beta$. A characteristic feature of this solution is that it includes the CG integrals as its residues when $s \in \mathbb{N}$. The second, the $\text{LFT}_{i\beta}$, is defined using the contour $\mathcal{U}$ and constitutes a direct extension of the s -dimensional CG integrals to $s \in \mathbb{R}$. We have also analyzed and exactly solved a special bulk-boundary 2-point function. In particular, we checked that thanks to the asymptotic behavior (A8) the integral over the $\mathcal{U}$ contour is well defined in this case as well, see [25] which cites Refs. [26–32].

The results discussed so far pertain to the simplest correlation function and rely on the behavior of the imaginary chaos on the circle. To gain a deeper understanding, it is necessary to investigate more complex correlation functions that depend on the imaginary GMC on a two-dimensional manifold. Among these, the 3-point correlation function on the sphere $\mathbb{S}^2$ is the simplest and most significant. As mentioned in the introduction, it is well known that, unlike the 1-point function, the 3-point function of the LFT_b does not admit an analytic continuation to $b \rightarrow i\beta$ when $\beta^2 \notin \mathbb{Q}$. However, we show that it is possible to define a path integral formulation of $\text{LFT}_{i\beta}$ which extends the CG results and agrees with the alternative bootstrap solutions defined for $c_{\text{charge}} \leq 1$, namely, the imaginary DOZZ structure constant.

We identify $\mathbb{S}^2$ with the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ that we equip with the background metric $|z|_+^{-4} |dz|^2$, where $|z|_+ = \max(|z|, 1)$. The area form dv and the scalar curvature K are, respectively, given by $dv = |z|_+^{-4} dx dy$ for $z = x + iy$ and $K = 4\delta_{|z|=1}$. We split $\phi(x)$ as $\phi(x) = X(x) + c$, where c is a complex constant and $X(x)$ is a real centered GFF on $\hat{\mathbb{C}}$ with covariance $\mathbb{E}[X(x)X(y)] = -\log|x-y| + \log|x|_+|y|_+$. The vertex operator $V_\alpha(x)$ is still defined as in (2), where one replaces the disc GFF by X . The n -point correlation function is

expressed as

$$\left\langle \prod_{i=1}^n V_{\alpha_i}(x_i) \right\rangle = \int \prod_i V_{\alpha_i}(x_i) e^{-S_L^{\text{im}}(\hat{C}, \phi)} \mathcal{D}\phi. \quad (12)$$

In this Letter, we restrict our focus to the 3-point functions $C_\alpha = \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle$, $\alpha = \{\alpha_i\}$, which are the structure constants of the theory. The main object entering the definition of the functions C_α is the imaginary chaos with insertion of charges:

$$M_{\beta, \alpha}^{\hat{C}} := \frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \int_{\hat{C}} \frac{|z|^{\beta\alpha_1} |z-1|^{\beta\alpha_2}}{|z|_+^{\beta\bar{\alpha}}} e^{-\frac{\beta^2}{2} e^{i\beta X_\epsilon(x)}} dv, \quad (13)$$

where $\bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$ and $X_{g_0, \epsilon}$ is an ultraviolet regularization of the GFF at scale ϵ .

The imaginary chaos $M_{\beta, \alpha}^{\hat{C}}$ is well defined and nontrivial [19] when $\alpha_i > Q, \beta^2 < 2$. Both conditions are needed to make the random variable probabilistically well defined, with finite L^2 moments [15]. Again, these conditions also ensure that $M_{\beta, \alpha}^{\hat{C}}$ has finite exponential moments in such a way that the corresponding generating function

$$\mathcal{G}_{\beta, \alpha}^{\hat{C}}(\mu) = \mathbb{E}[e^{-\mu M_{\beta, \alpha}^{\hat{C}}}] \quad (14)$$

is well defined [15]. Analogously to the circle case, we define the $\text{LFT}_{i\beta}$ by integrating the constant mode c over an U-type contour, $\mathcal{U}$, consisting in the lines $[0, -i\infty)$, $[0, (2\pi/\beta)]$, and $[(2\pi/\beta), (2\pi/\beta) - i\infty)$ —see Fig. 2. The structure constants of the theory are

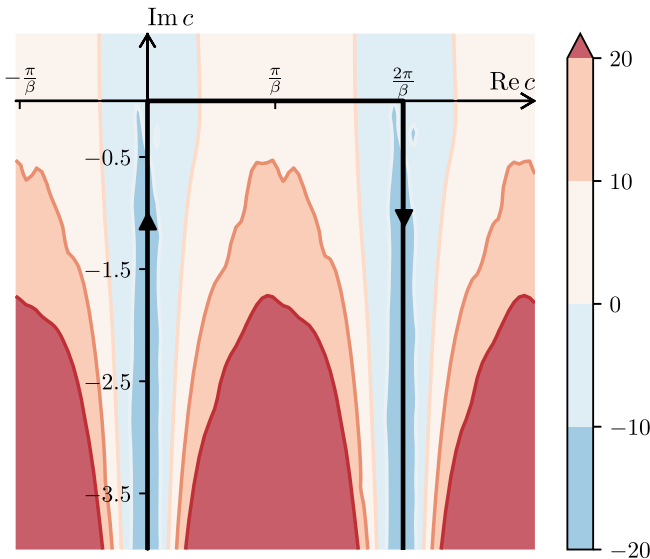


FIG. 2. Contour lines of the real part of $i(\bar{\alpha} - Q)c + \log[\mathcal{G}_{\beta, \alpha}^{\hat{C}}(e^{i\beta c})]$, as from numerical simulations with $N = 12\,000$ independent samples of the chaos, $\beta = 0.4$, $\alpha_1 = \alpha_2 = -4.0$, $\alpha_3 = -2.4$.

$$C_\alpha = \int_{\mathcal{U}} e^{i(\bar{\alpha} - 2Q)c} \mathcal{G}_{\beta, \alpha}^{\hat{C}}(\mu e^{i\beta c}) dc. \quad (15)$$

We conjecture that the $\mathcal{G}_{\beta, \alpha}^{\hat{C}}(\mu)$ behaves similarly to the circle case $\mathcal{G}_{\beta}^{\mathbb{S}}(\mu)$. We computed numerically the function $\mathcal{G}_{\beta, \alpha}^{\hat{C}}(\mu)$, for different values of β and the α_i 's, satisfying the conditions above.

In Fig. 2, we show the contour lines of $i(\bar{\alpha} - Q)c + \log \mathcal{G}_{\beta, \alpha}^{\hat{C}}(e^{i\beta c})$, supporting our conjecture that the definition (15) perfectly makes sense.

Let us rewrite Eq. (15) in a more suggestive form:

$$C_\alpha = (-1 + e^{-2i\pi s}) \int_0^{-i\infty} e^{-i\beta s c} \mathcal{G}_{\beta, \alpha}^{\hat{C}}(\mu e^{i\beta c}) dc + \int_0^{\frac{2\pi}{\beta}} e^{-i\beta s c} \mathcal{G}_{\beta, \alpha}^{\hat{C}}(\mu e^{i\beta c}) dc, \quad (16)$$

with $s = (2Q - \bar{\alpha}/\beta)$. When $s \in \mathbb{N}$, the definition (16) includes as a special case the diagonal sector of the CILT for which the structure constants are given by the Dotsenko-Fateev integrals given below (29). The $\text{LFT}_{i\beta}$, as defined above, represents an extension of the Coulomb gas correlation functions—and hence of the CILT—to real values of the parameter s . Fortunately, we can compare numerically the 3-point function (16) to the imaginary DOZZ structure constant C_α^{ImDOZZ} . This constant was introduced in [7, 14, 22, 33] as the unique solution to a pair of shift equations at $c_{\text{charge}} \leq 1$. Moreover, it can be derived as the unique analytic continuation of the Dotsenko-Fateev integrals with 3 electric charges [12, 14, 33] consistent with the duality $\beta \rightarrow -(4/\beta)$. This is precisely why our theory should satisfy

$$C_\alpha = C_\alpha^{\text{ImDOZZ}}. \quad (17)$$

Our numerical simulations exhibit excellent agreement with Eq. (17). The quality of that agreement is illustrated in Fig. 3. Further details about numerical simulations are given in [25], including a discussion of their range of validity.

In summary, we have proposed a probabilistic definition of the imaginary $\text{LFT}_{i\beta}$ correlation functions based on the imaginary GMC, giving sense to a path integral in which the zero mode is integrated along a Hankel-type contour. In the case of the 3-point function, this leads to the expression (15). The validity of our construction relies on the asymptotic behavior of the moment generating function of the imaginary GMC, which we confirmed in the case of imaginary chaos on the circle, where exact results are available. Furthermore, we did test our theory through a highly nontrivial comparison between numerical simulations and the imaginary DOZZ structure constant. Our results not only provide a direct probabilistic definition

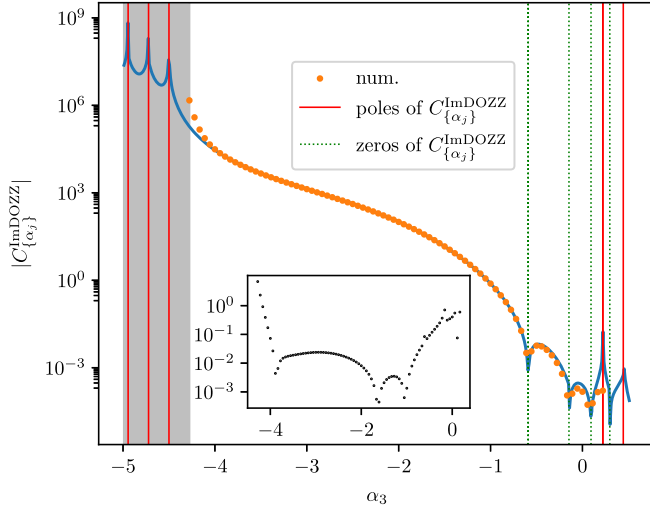


FIG. 3. Overview of the match between the imaginary DOZZ formula and the numerical simulations of $\text{LFT}_{i\beta}$ three-point function. The relative deviation is shown in the inset. The area $\alpha_3 \leq Q$ is shaded in gray, and the poles and zeros of $C^{\text{ImDOZZ}}_{\{\alpha_j\}}$ are marked by vertical lines. See [25] for further details.

of the imaginary DOZZ structure constant, but also highlight the existence of a CFT that arises as the analytic continuation of CILT. An important open question raised by our results concerns the factorization properties of the 4-point function. Notably, the only known noncompact bootstrap solution featuring C^{ImDOZZ} as structure constants, introduced in [23], is not an extension of the Coulomb gas integrals [25]. We leave this question for future investigation.

Acknowledgments—We thank Xiangyu Cao for participating in the initial part of the work and for numerous subsequent discussions. We thank as well Laurent Chevillard and Vincent Vargas for fruitful discussions and preliminary numerics in the early stages of this work. We acknowledge valuable discussions with Juhan Aru and Sylvain Ribault. R. U. is supported by the CNRS 80 prime project. R. R. is partially supported by the Institut Universitaire de France (IUF) and by the ANR-21-CE40-0003. This work was performed using computational resources from the Mésocentre computing center of Université Paris-Saclay, CentraleSupélec, and École Normale Supérieure Paris-Saclay supported by CNRS and Région Île-de-France.

Data availability—The data supporting this study's findings are available within the article.

- [1] S. Ribault, [arXiv:1406.4290](#).
- [2] C. Guillarmou, A. Kupiainen, and R. Rhodes, [arXiv:2403.12780](#).
- [3] J. Teschner, *Classical Quantum Gravity* **18**, R153 (2001).
- [4] P. Nolin, W. Qian, X. Sun, and Z. Zhuang, *Phys. Rev. Lett.* **134**, 117101 (2025).
- [5] C. Guillarmou, A. Kupiainen, R. Rhodes, and V. Vargas, [arXiv:2112.14859](#).
- [6] D. Harlow, J. Maltz, and E. Witten, *J. High Energy Phys.* **12** (2011) 071.
- [7] V. Schomerus, *J. High Energy Phys.* **11** (2003) 043.
- [8] M. Gutperle and A. Strominger, *Phys. Rev. D* **67**, 126002 (2003).
- [9] G. Giribet, *Phys. Rev. D* **85**, 086009 (2012).
- [10] T. Bautista, A. Dabholkar, and H. Erbin, *J. High Energy Phys.* **10** (2019) 284.
- [11] S. Collier, L. Eberhardt, B. Mühlmann, and V. A. Rodriguez, *SciPost Phys.* **16**, 057 (2024).
- [12] M. Picco, R. Santachiara, J. Viti, and G. Delfino, *Nucl. Phys. B* **875**, 719 (2013).
- [13] Y. Ikhlef, J. L. Jacobsen, and H. Saleur, *Phys. Rev. Lett.* **116**, 130601 (2016).
- [14] I. Kostov and V. Petkova, *Theor. Math. Phys.* **146**, 108 (2006).
- [15] C. Guillarmou, A. Kupiainen, and R. Rhodes, *Commun. Am. Math. Soc.* **5**, 571 (2025).
- [16] S. Chatterjee, [arXiv:2504.02348](#).
- [17] V. Dotsenko and V. Fateev, *Nucl. Phys. B* **312**, 691 (1984).
- [18] B. Nienhuis, *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (1987), Vol. 11.
- [19] H. Lacoïn, R. Rhodes, and V. Vargas, *Commun. Math. Phys.* **337**, 569 (2015).
- [20] E. Witten, [arXiv:1001.2933](#).
- [21] X. Cao, R. Santachiara, and R. Usciatì, *J. High Energy Phys.* **03** (2023) 061.
- [22] A. B. Zamolodchikov, *Theor. Math. Phys.* **142**, 183 (2005).
- [23] S. Ribault and R. Santachiara, *J. High Energy Phys.* **08** (2015) 109.
- [24] T. Bautista and A. Bawane, *Phys. Rev. D* **106**, 126011 (2022).
- [25] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/d723-gk4m> for a detailed discussion of Liouville theory on a circle, numerical simulations on the sphere, and comparison with previous bootstrap solutions.
- [26] Y. Huang, R. Rhodes, and V. Vargas, [arXiv:1502.04343](#).
- [27] S. Sheffield, [arXiv:math/0312099](#).
- [28] G. Remy and T. Zhu, *Commun. Math. Phys.* **395**, 179 (2022).
- [29] Y. V. Fyodorov and J.-P. Bouchaud, *J. Phys. A* **41**, 372001 (2008).
- [30] G. Remy, *Duke Math. J.* **169**, 177 (2020).
- [31] P. J. Forrester and S. O. Warnaar, [arXiv:0710.3981](#).
- [32] C. Guillarmou, R. Rhodes, and V. Vargas, *Publ. Math. IHES* **130**, 111.
- [33] V. S. Dotsenko, *Nucl. Phys. B* **907**, 208 (2016).

End Matter

Imaginary GMC on the circle: Exact results—We give a derivation of Eq. (7). We use the following property of the GFF under consideration:

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-\frac{\beta^2}{4}} \mathbb{E} \left[e^{i\frac{\beta}{2} X_\epsilon(e^{i\theta_1})} e^{i\frac{\beta}{2} X_\epsilon(e^{i\theta_2})} \right] = |e^{i\theta_1} - e^{i\theta_2}|^{\frac{\beta^2}{2}}. \quad (\text{A1})$$

An explicit implementation of the UV regularization ϵ is provided in Supplemental Material [25]. By Wick's theorem, the n th moment of the imaginary chaos $\mathbb{E}[(M_\beta^\mathbb{S})^n]$, $n \in \mathbb{N}$ is expressed in terms of the integral:

$$\mathbb{E}[(M_\beta^\mathbb{S})^n] = \int_{[0, 2\pi]^n} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^{\frac{\beta^2}{2}} \prod_{i=1}^n \frac{d\theta_i}{2\pi}. \quad (\text{A2})$$

This is a Selberg-type integral whose value is given by [31]

$$\mathbb{E}[(M_\beta^\mathbb{S})^n] = \Gamma\left(1 + \frac{n\beta^2}{4}\right) \Gamma\left(1 + \frac{\beta^2}{4}\right)^{-n}. \quad (\text{A3})$$

This allows us to give an integral representation of the moment-generating function $\mathcal{G}_\beta^\mathbb{S}(\mu)$:

$$\begin{aligned} \mathcal{G}_\beta^\mathbb{S}(\mu) &= \sum_{n=0}^{\infty} \frac{(-\mu)^n}{n! \Gamma\left(1 + \frac{\beta^2}{4}\right)^n} \Gamma\left(1 + \frac{n\beta^2}{4}\right) = \\ &= \sum_{n=0}^{\infty} \frac{(-\mu)^n}{n! \Gamma\left(1 + \frac{\beta^2}{4}\right)^n} \int_0^{+\infty} t^{\frac{n\beta^2}{4}} e^{-t} dt = \\ &= \int_0^{+\infty} e^{-\bar{\mu} t^{\frac{\beta^2}{4}} - t} dt \end{aligned} \quad (\text{A4})$$

for $\bar{\mu} = \mu / \Gamma[1 + (\beta^2/4)]$, where we use the integral expression of the Γ function.

In the limit $\mu \rightarrow -\infty$, the above integral is dominated by its saddle-point: $\mathcal{G}_\beta^\mathbb{S}$ grows exponentially in this μ direction, see [25]. In the limit $\mu \rightarrow \infty$, the integral has no real saddle point, but a change of variable produces

$$\mathcal{G}_\beta^\mathbb{S}(\mu) = \bar{\mu}^{-\frac{4}{\beta^2}} \int_0^\infty e^{-t^{\frac{\beta^2}{4}} - \bar{\mu}^{-\frac{4}{\beta^2}} t} dt \quad (\text{A5})$$

and expanding the term $\exp(-\bar{\mu}^{-\frac{4}{\beta^2}} t)$ in power series leads to the asymptotics (7). We have also solved the case where one inserts in the correlation function, besides the vertex $V_\alpha(0)$ at the origin, a vertex $V_\eta(1)$ at the boundary, with $\eta = \beta/2$. In the bulk-boundary correlation function the main role is played by

$$\mathcal{G}_{\beta, \eta}^\mathbb{S}(\mu) = \mathbb{E}[e^{-\mu M_{\beta, \eta}^\mathbb{S}}], \quad (\text{A6})$$

where

$$M_{\beta, \eta}^\mathbb{S} = \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta} - 1|^{\beta\eta} e^{i\frac{\beta}{2} X_{g_0}(e^{i\theta})} d\theta. \quad (\text{A7})$$

Setting $\eta = \beta/2$, we have when $\mu \rightarrow \infty$:

$$\mathcal{G}_{\beta, \eta}^\mathbb{S}(\mu) \sim \frac{4}{\beta^2} \Gamma\left(\frac{4}{\beta^2} + 1\right) \Gamma\left(1 + \frac{\beta^2}{4}\right)^{\frac{4}{\beta^2}} \mu^{-\frac{4}{\beta^2} - 1}. \quad (\text{A8})$$

Imaginary DOZZ structure constant—The C_α^{ImDOZZ} is expressed in terms of functions $\Upsilon_{(\beta/2)}(x)$, which are strictly related to the double Γ function, see Sec. 7 of [15]. Using the same field normalization as in [15], $C_{\{\alpha_i\}}^{\text{ImDOZZ}}$ is

$$\begin{aligned} C_\alpha^{\text{ImDOZZ}} &= \frac{2\pi}{\beta} \left(\frac{-\mu}{\gamma\left(\frac{\beta^2}{4}\right)} \right)^s \left(\frac{\beta}{2} \right)^{Q\beta s} \frac{\Upsilon_{\frac{\beta}{2}}\left(Q + \frac{\beta}{2} - \frac{\bar{\alpha}}{2}\right)}{\Upsilon_{\frac{\beta}{2}}\left(\frac{\beta}{2}\right)} \times \\ &\times \frac{\prod_{i=1}^3 \Upsilon_{\frac{\beta}{2}}\left(\frac{\beta}{2} - \frac{\bar{\alpha}}{2} + \alpha_i\right)}{\prod_{i=1}^3 \Upsilon_{\frac{\beta}{2}}\left(\frac{\beta}{2} + \alpha_i\right)}, \end{aligned} \quad (\text{B1})$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. Using the recursion properties of the $\Upsilon_{(\beta/2)}(z)$ functions:

$$\Upsilon_{\frac{\beta}{2}}\left(z + \frac{\beta}{2}\right) = \gamma\left(\frac{\beta}{2}z\right) \left(\frac{\beta}{2}\right)^{1-\beta z} \Upsilon_{\frac{\beta}{2}}(x), \quad (\text{B2})$$

$$\Upsilon_{\frac{\beta}{2}}\left(z + \frac{2}{\beta}\right) = \gamma\left(\frac{2}{\beta}z\right) \left(\frac{\beta}{2}\right)^{\frac{4}{\beta}z-1} \Upsilon_{\frac{\beta}{2}}(x), \quad (\text{B3})$$

one can show that when $s \in \mathbb{N}$, Eq. (B1) yields the value of the Coulomb gas integral with three electric charges:

$$\begin{aligned} C_\alpha^{\text{CG}} &\stackrel{s \in \mathbb{N}}{=} \int_0^{\frac{2\pi}{\beta}} dc e^{-i\beta s} \mathcal{G}_{\beta, \alpha}^\mathbb{C} = \frac{2\pi (-\mu)^s}{\beta s!} \mathbb{E}[(M_{\beta, \alpha}^\mathbb{C})^s] = \\ &\stackrel{s \in \mathbb{N}}{=} \frac{2\pi (-\mu)^s}{\beta s!} \int_{\mathbb{C}^s} \prod_{i=1}^s d\xi_i \prod_{i=1}^s |\xi_i|^{\alpha_1 \beta} |\xi_i - 1|^{\alpha_2 \beta} \\ &\times \prod_{i < j=1}^s |\xi_i - \xi_j|^{\beta^2} = \\ &= \frac{2\pi}{\beta} \left(\frac{-\mu}{\gamma\left(\frac{\beta^2}{4}\right)} \right)^s \frac{\prod_{j=1}^s \gamma\left(j\frac{\beta^2}{4}\right)}{\prod_{j=1}^{s-1} \prod_{i=1}^3 \gamma\left(-\frac{\beta\alpha_i}{2} - j\frac{\beta^2}{4}\right)}. \end{aligned}$$